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ESSENTIALLY NORMALOID AND ESSENTIALLY CONVEXOID OPERATORS*

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ABSTRACT. In this note some elementry facts on essentially normaloid and essentially convexoid operators are established. It is shown that (i) a normaloid operator need not be essentially normaloid and vice versa, (ii) a paranormal operator is not essentially convexoid, (iii) the class of essentially normaloid operators is a closed nowhere dense subset of the algebra of operators, and (iv) for an essentially convexoid (essentially normaloid, essentially spectraloid) operator T, there exists a compact operator K such that $T+K$ is convexoid (normaloid, spectraloid).

In what follows, an operator on a complex Hilbert space H will be meant a bounded linear transformation on H. Let $B(H)$, $K(H)$ denote the Banach algebra of operators on H, the two sided ideal of compact operators on H and the Calkin algebra of H. The notation $\pi(T)$ is used for the canonical image of T in the Calkin algebra. The essential spectrum, the essential numerical range, the essential spectral radius and the essential unmerical radius of T will be denoted by $\sigma_e(T)$, $W_e(T)$, $r_e(T)$ and $|W_e(T)|$ respectively.

An operator T is called essentially normaloid if $r_e(T)=\|\pi(T)\|$; essentially convexoid if con $\sigma_e(T)=W_e(T)$ and essentially spectraloid if $r_e(T)=|W_e(T)|$.

The author [4] has shown that if T is reduction-spectraloid (reduction-normaloid) then T is essentially spectraloid (essentially normaloid). In particular, every reduction-convexoid operator is essentially convexoid. However the convexoidity alone is not sufficient to guarantee the essential convexoidity (see Luecke [3, Theorem 7]). Similarly, as we shall show in Theorem 2, the normaloidity does not always imply the essential normaloidity. First we establish the following theorem which is analogous to Theorem 5 of [3].

Theorem 1. *If*

$$T=\begin{pmatrix} A & C \\ 0 & B \end{pmatrix}$$

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on $H\oplus H$, where C is an (essentially) invertible operator, then T is not (essentially) normaloid.

Proof. (I). Since C is invertible, there exists $M>0$ such that $\|Cx\|\geq M\|x\|$ and $\|C^*x\|\geq M\|x\|$ for all x in H. For a unit vector x in H, $T\begin{pmatrix}0\\x\end{pmatrix}=\begin{pmatrix}Cx\\Bx\end{pmatrix}$ and hence $\|T\|^2\geq\|Cx\|^2+\|Bx\|^2\geq M^2+\|Bx\|^2$. From this we get

(1) $$\|T\|>\|B\| .$$

Similarly, as $T^*\begin{pmatrix}x\\0\end{pmatrix}=\begin{pmatrix}A^*x\\C^*x\end{pmatrix}$, we obtain

(2) $$\|T\|>\|A\| .$$

Since $\sigma(T)\subseteq\sigma(A)\cup\sigma(B)$, we have

(3) $$r(T)\leq\max\{r(A), r(B)\} .$$

From (1), (2) and (3), we infer that $r(T)\leq\max\{r(A), r(B)\}\leq\max\{\|A\|, \|B\|\}<\|T\|$, showing that T is not normaloid.

(II). The proof is analogous to given in Part II of Theorem 5 [3].

Theorem 2. *There exists a normaloid operator which is not essentially normaloid and vice versa.*

Proof. Let $H=H_1\oplus H_2\oplus H_3$, where each H_i is infinite dimensional. Let

$$A=\begin{pmatrix}0 & I\\0 & 0\end{pmatrix}$$

be an operator on $H_1\oplus H_2$. Let $N=\text{diag}\{1, 1/2, 1/3\cdots\}$ be an operator on H_3. If $T=A\oplus N$, then $r(T)=r(N)=1$ and $\|T\|=\max\{\|A\|, \|N\|\}=1$; thus T is normaloid. Since by Theorem 1, A is not essentially normaloid, T is not essentially normaloid. To show the existence of an essentially normaloid but non-normaloid operator, take any non-normaloid compact operator.

Ando [1] has constructed a paranormal operator which is not convexoid. This raises the following question: Does there exist a paranormal operator which is not essentially convexoid? Our next result answers this question in negative. As an immediate consequence of this, we get that the class of essentially hyponormal operators is a proper subset of the class of essentially paranormal operators (and hence of essentially normaloid operators).

Theorem 3. *There exists a paranormal operator which is not essentially convexoid.*

Proof. Let U be the unilateral shift of infinite multiplicity. Let $A=U+1$. Then by [1], the operator

$$T=\begin{pmatrix} A & (A^*A-AA^*)^{1/2} \\ 0 & 0 \end{pmatrix}$$

defined on $H\oplus H$ is paranormal. We claim that T is not essentially convexoid. First observe that Bdry $\sigma_e(A)\subseteq\sigma_e(T)\subseteq\sigma_e(A)$ and so con $\sigma_e(A)=$con $\sigma_e(T)$.

Suppose to the contrary that T is essentially convexoid. Then $\|(\pi(T)-\lambda)^{-1}\|\leq 1/d(\lambda, \text{con}\,\sigma_e(T))=1/d(\lambda, \text{con}\,\sigma_e(A))$ for all $\lambda\notin \text{con}\,\sigma_e(A)=\text{con}\,\sigma_e(T)$. In particular, $\|(\pi(T)+1)^{-1}\|\leq 1/d(1, \text{con}\,\sigma_e(T))=1$. Let $P=\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$ be the orthogonal projection on $H\oplus 0$. Since $B(H)/K(H)$ is a C^*-algebra, there exists a Hilbert space H_0 such that $B(H)/K(H)$ is isometrically isomorphic to a closed, selfadjoint subalgebra of $B(H_0)$. Let v be this isometric isomorphism. Then $v\cdot\pi(P)$ is an orthogonal projection in $B(H_0)$. Let $H_0=M\oplus M^{\perp}$, where M is the range of $v\cdot\pi(P)$. Relative to this decomposition of H_0,

$$v\cdot\pi(T)=\begin{pmatrix} A_1 & C_1 \\ 0 & 0 \end{pmatrix}$$

Now

$$(v\cdot\pi(T)+1)^{-1}=\begin{pmatrix} (A_1+1)^{-1} & -(A_1+1)^{-1}C_1 \\ 0 & 1 \end{pmatrix}.$$

Therefore

$$\|(\pi(T)+1)^{-1}\|=\|(v\cdot\pi(T)+1)^{-1}\|>1\,,$$

a contradiction. Thus T is not essentially convexoid.

Theorem 4. *The class of essentially normaloid operators is a closed nowhere dense subset of* $B(H)$.

Proof. First we note that $\|\pi(T)\|=\|v\cdot\pi(T)\|$ and $\sigma_e(T)=\sigma(v\cdot\pi(T))$, where v is the isometric (algebra) embedding of the Calkin algebra in $B(H_0)$, H_0 being a Hilbert space. Therefore T is essentially normaloid if and only if $v\cdot\pi(T)$ is normaloid. From this we conclude that the class of essentially normaloid operators is closed. To complete the proof, it now suffices to show that for an essentially normaloid operator T, there exists a sequence $\{T_n\}$ of non-essentially normaloid operators converging to T.

Since $\pi(T)$ is normaloid, there exists a complex number λ in $\sigma_e(T)$ such that $|\lambda|=\|\pi(T)\|$. Let $\{x_n\}$ be a sequence of orthonormal vectors in H for which $\|(T-\lambda)x_n\|+\|(T-\lambda)^*x_n\|\to 0$. By Stampfli's corollary to Theorem 2 [5], T is unitarily equivalent, via unitary operator U, to $T\oplus\lambda+K$ on $H\oplus H_1$, where H_1 is a separable

Hilbert space and K is a compact operator on $H\oplus H_1$. Let $B=\begin{pmatrix}0 & 1\\ 0 & 0\end{pmatrix}$ be an operator on H_1 and let $S_n=T\oplus(\lambda-B/n)+K$ be an operator on $H\oplus H_1$. We claim that S_n is not essentially normaloid. To this end, observe that $\|\pi(\lambda-B/n)\|>|\lambda|=\|\pi(T)\|=r_e(\lambda-B/n)$. Therefore

$$(1)\qquad \|\pi(S_n)\|=\max\{\|\pi(T)\|,\|\pi(\lambda-B/n)\|\}=\|\pi(\lambda-B/n)\|$$

Also

$$\begin{aligned} r_e(S_n)&=\max\{r_e(T), r_e(\lambda I-B/n)\}\\ &=\max\{\|\pi(T)\|, r_e(\lambda-B/n)\}\\ &=r_e(\lambda-B/n)\,. \end{aligned}$$

Thus

$$(2)\qquad r_e(S_n)<\|\pi(\lambda-B/n)\|\,.$$

From (1) and (2), we conclude that S_n is not essentially normaloid. Also $\|T-US_nU^*\|=\|S-S_n\|=1/n\|B\|=1/n\to 0$ as $n\to\infty$. Thus $\{T_n\}=[US_nU^*\}$ is the desired sequence of operators.

Our final result establishes a conjecture due to Lueck [3, Conjecture 2].

Theorem 5. (i) *If T is an essentially convexoid operator, then there exists a compact operator K such that $T+K$ is convexoid.*

(ii) *If T is essentially normaloid (essentially spectraloid), then there exists a compact operator K such that $T+K$ is normaloid (spectraloid).*

Proof. (i) To prove the result, we note that for any operator T, there exists a compact operator K such that

$$(1)\qquad \|T+K+\lambda\|=\|\pi(T+\lambda)\|$$

for all complex numbers [2]. Moreover

$$(2)\qquad W_e(T)=\bigcap_{\lambda\in C}\{\mu\colon |\mu-\lambda|\le\|\pi(T-\lambda)\|\}$$

Combining (1) and (2), we get

$$\begin{aligned} W_e(T)&=\bigcap_{\lambda\in C}\{\mu\colon |\mu-\lambda|\le\|T+K-\lambda\|\}\\ &=\overline{W(T+K)}\,. \end{aligned}$$

Therefore if T is essentially convexoid, then $\operatorname{con}\sigma(T+K)\supseteq\operatorname{con}\sigma_e(T)=W_e(T)=\overline{W(T+K)}$, which shows that $T+K$ is convexoid. Parts (i) and (ii) are now obvious.

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