YOKOHAMA MATHEMATICAL
JOURNAL, VOL. 26, 1978

LOCALLY CONVEX SPACES WITH THE PROPERTY (Σ)

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(Received May 9, 1977)

1. Introduction

In [3], Kluvánek has introduced the concept of the property (Σ) of topological vector space as the generalization of metrizable topological vector space. Drewnowski [2] has discussed the relation between the existence of control measure for vector measure and the property (Σ) of locally convex space which is the range of vector measure. Recently Kluvánek and Knowles [5] have proved the following theorems.

Let T be a set, S a σ-algebra of subsets of T, X a quasi-complete, Hausdorff locally convex space and $m: S \rightarrow X$ a vector measure.

(1) If X is metrizable and m is non-atomic, then the weak closure of range $R(m)$ of m coincides with $\overline{co}\ R(m)$ (Theorem V.6.1.).

(2) If X is metrizable, then every vector measure m is closed (Theorem IV.7.1.).

In this paper we shall extend these results in the case X has the property (Σ). For this object, in § 2 we shall consider the properties of X with the property (Σ). In § 3, we shall consider the applications of it.

2. Locally convex spaces with the property (Σ)

Let X be a Hausdorff locally convex space.

Definition 2.1. We say that X has the property (Σ) if every family $\{x_i\}_{i \in I}$ of non-zero elements of X such that every countable subfamily $\{x_j\}_{j \in J}$ ($J \subset I$) is summable is at most countable.

If X is metrizable, then X has the property (Σ). Further, it is known that the class of spaces with the property (Σ) is effectively larger than the class of metrizable spaces.

Let T be a set, R a ring of subsets of T, X a locally convex space and $m: R \rightarrow X$ a countably additive vector measure.

Definition 2.2. We say that m satisfies the countable chain condition (C.C.C.) if each family of pairwise disjoint sets of the non-zero measure is at most countable.

Proposition 2.1. *The following statements are equivalent.*

(1) *X has the property (Σ).*

(2) *For any set T and any σ-ring φ of subsets of T, any vector measure m: $\varphi \to X$ satisfies (C.C.C).*

(3) *For any set T, any σ-ring φ of subsets of T and any vector measure m: $\varphi \to X$ there exists a set $Q \in \varphi$ such that for any set $E \in \varphi$ we have $m(E-Q)=0$.*

Proof. (1)$\Rightarrow$(2). It is obvious by Zorn's Lemma.

(2)$\Rightarrow$(3). By hypothesis there exists a countable maximal family $\{E_n\}_{n \in N}$ of pairwise disjoint sets with $m(E_n) \neq 0$ for all n. Put $Q = \bigcup_{n \in N} E_n$. Then Q has the required property.

(3)$\Rightarrow$(1). See Kluvânek [3] Theorem 3.2.

Proposition 2.2. *If any singleton set in X is G_δ-set, then X has the property (Σ).*

Proof. Let $\{U_n\{_{n \in N}$ be a sequence of neighborhoods of $0 \in X$ with $\bigcap_{n \in N} U_n = \{0\}$. and let $\{x_i\}_{i \in I}$ a family of non-zero elements of X such that every countable subfamily $\{x_j\}_{j \in J}$ $(J \subset I)$ is summable. Put $I_n = \{i \in I: x_i \notin U_n\}$ for any $n \in N$. Then I_n is a finite set. Put $J = \bigcup_{n \in N} I_n$. Then J is a countable set. Since $\bigcap_{n \in N} U_n = \{0\}$, we have $I=J$.

Proposition 2.3. *Let H be a closed subspace of X. If H and the quotient space X/H have the property (Σ), then X has the property (Σ).*

Proof. Let $\{x_i\}_{i \in I}$ be a family of non-zero elements of X such that every countable subfamily $\{x_j\}_{j \in J}$ $(J \subset I)$ is summable and φ: $X \to X/H$ the canonical mapping. Then $\{\varphi(x_j)\}_{j \in J}$ $(J \subset I)$ is summable. Since X/H has the property (Σ), $\{\varphi(x_i)\}_{i \in I}$ is countable. Put $A = \{x_i: \varphi(x_i) = \dot{a} \in X/H, \dot{a} \neq \dot{0}\}$. Then A is a finite set. Further, put $B = \{x_i: \varphi(x_i) = \dot{0}\}$. Since H has the property (Σ), B is a countable set. Therefore we have the assertion.

The following theorem is an extension of Musial [6] Theorem 2 and S. Ohba [7] Theorem 1.

Let $\mathfrak{J}$ be a δ-ring (that is, a ring closed under countable intersection) of subsets of T and m: $\mathfrak{J} \to X$ a countably additive vector measure. Put $N(m) = \{E \in \mathfrak{J}: F \in \mathfrak{J} \to m(E \cap F) = 0\}$.

Theorem 2.1. *If $\mathfrak{J}$ is a δ-ring and m: $\mathfrak{J} \to X$ satisfies C.C.C., then there exists a finite, non-negative measure ν on $\mathfrak{J}$ such that $N(\nu) = N(m)$. In particular, if $\mathfrak{J}$ is a σ-ring, then the converse is true.*

Proof. By Zorns Lemma and C.C.C. there exists a countable maximal family

$\{E_n\}_{n\in N}$ of pairwise disjoint sets with $m(E_n)\neq 0$ for all $n\in N$. Put $\varphi_n=\mathfrak{F}\cap E_n$ $(n\in N)$. Then φ_n is a σ-ring.

Since m on φ_n satisfies C.C.C., there exists a finite, non-negative measure ν_n on φ_n such that $N(\nu_n)=N(m|\varphi_n)$ $(n\in N)$ by Musial [6] Theorem 2. For any set $E\in\mathfrak{F}$ put $\nu(E)=\sum_{n\in N}(1/2^n)\cdot\nu_n(E\cap E_n)/(1+\sup\{\nu_n(A)\colon A\in\varphi_n\})$ (since φ_n is a σ-ring and ν_n is finite, we have $\sup\{\nu_n(A)\colon A\in\varphi_n\}<\infty$) $N(m)\subset N(\nu)$ is obvious.

The proof of $N(\nu)\subset N(m)$. Let E be a set of $N(\nu)$. Then we have $E\cap E_n\in N(m|\varphi_n)$ $(n\in N)$. Since $m(E\cap\bigcup_{n\in N}E_n)=\sum_{n\in N}m(E\cap E_n)$, we have $E\cap\bigcup_{n\in N}E_n\in N(m)$. $E-\bigcup_{n\in N}E_n\in N(m)$ is obvious. Therefore we have $E\in N(m)$. If $\mathfrak{F}$ is a σ-ring, the converse is obvious by Musial [6] Theorem 2.

Proposition 2.4. *If $\tilde{X}$ has the property (Σ), then for any s-bounded vector measure $m\colon\mathfrak{F}\to X$ ($\mathfrak{F}$ is a δ-ring) there exists a finite, non-negative measure ν on $\mathfrak{F}$ such that $N(\nu)=N(m)$ where $\tilde{X}$ is the completion of X.*

The proof is obvious.

3. Applications

Let T be a set, S a σ-algebra of subsets of T, X a Hausdorff locally convex space assumed quasi-complete, X' its dual and $m\colon S\to X$ a countably additive vector measure. Set $R(m)=\{m(E)\colon E\in S\}$, $R(m, E)=\{m(F)\colon F\subset E, F\in S\}$ for every set $E\in S$ and $N(m)=\{E\in S\colon R(m, E)=\{0\}\}$.

Definition 3.1. A vector measure m is called absolutely continuous, if there exists a finite non-negative measure ν on S such that $N(\nu)\subseteqq N(m)$. It is well known that if m is absolutely continuous, then there exists a finite non-negative measure ν on S such that $N(\nu)=N(m)$.

Definition 3.2. A set $A\in S$ is called an atom of m if $A\notin N(m)$ and if $B\in S$ implies that either $A\cap B\in N(m)$ or $A-B\in N(m)$. If there are no atom of m then m is called non-atomic.

Let Y be a locally convex space and $\Phi\colon X\to Y$ a continuous linear map. Since m is a vector measure, $\Phi\circ m(E)=\Phi(m(E))$ is also a vector measure on S.

Proposition 3.1. *If m is an absolutely continuous, non-atomic vector measure, then also $\Phi\circ m$ is a non-atomic vector measure.*

Proof. Let ν be a finite non-negative measure on S such that $N(\nu)=N(m)$. Since m is non-atomic, ν is non-atomic. Then $\Phi\circ m$ is non-atomic by Kluvânek and Knowles ([5] Lemma V.6.3).

Corollary. *If m is an absolutely continunus, non-atomic vector measure, then for every $x' \in X'$ $x' \circ m$ is non-atomic scalar measure.*

Theorem 3.1. *If m is non-atomic vector measure and X has the property (Σ), then the weak closure of $R(m)$ coincides with $\overline{co}\ R(m)$ where $\overline{co}\ R(m)$ is the closed convex hull of $R(m)$.*

Proof. Since X has the property (Σ), m is absolutely continuous. It is obvious by the above Corollary and Kluvânek and Knowles ([5] Lemma V.6.5).

Corollary. *If m is non-atomic and X is metrizable, then the weak closure of $R(m)$ coincides with $\overline{co}\ R(m)$.* (Kluvânek and Knowles [5] Lemma V.6.5).

Remark. Since $\overline{\text{co}}\ R(m)$ is weakly compact set in X (Kluvânek and Knowles' Theorem IV.6.1 ([5])), we have the following.

If m is non-atomic vector measure and X has the property (Σ) then the weak closure of $R(m)$ is weakly compact convex set in X.

A set $E \in S$ is called m-null if $E \in N(m)$. Two set $E, F \in S$ are m-equivalent if $E \Delta F = (E-F) \cup (F-E)$ is m-null. If $E \in S$, then $[E]_m$ is the class of all sets $F \in S$ which are m-equivalent to E. We put $S(m) = \{[E]_m : E \in S\}$.

On the set $S(m)$ we define a uniform structure $\tau(m)$ in the following way. Let P be a family of semi-norms defining the topology of X. For each $p \in P$ and $E \in S$ we put $p(m)(E) = \sup\{p(x) : x \in \text{co}\, R(m, E)\}$ where $\text{co}\, R(m, E)$ is the convex hull of $R(m, E)$. Further, define the semi-distance d_p on $S(m)$ by putting $d_p([E]_m, [F]_m) = p(m)(E \Delta F)$, $E, F \in S$.

The family $\{d_p : p \in P\}$ gives the nniform structure $\tau(m)$ on $S(m)$.

Definition 3.3. A vector measure m is called closed if $S(m)$ is $\tau(m)$-complete.

Theorem 3.2. *If X has the property (Σ), then every vector measure m is closed.*

Proof. Since X has the property (Σ), there exists a finite non-negative measure ν on S such that $N(\nu) = N(m)$. Then we can prove in the same way as the proof of Kluvânek and Knowles ([5] Theorem IV.7.1)).

Corollary. *If X is metrizable, then every vector measure m is closed.* (Kluvânek and Knowles ([5] Theorem IV.7.1)).

The authors wish to express their appreciation to Professor I. Kluvânek for his encouragement.

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