EXTENDED LIE GEOMETRY, EXTENDED PARABOLIC
LIE GEOMETRY, EXTENDED EQUIFORM LAGUERRE
GEOMETRY AND EXTENDED LAGUERRE
GEOMETRY AND THEIR REALIZATIONS
IN THE DIFFERENTIABLE
MANIFOLDS. I

By

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(Received March 1, 1961)

The parameters of the transformation groups hitherto considered have been exclusively so-called variable constants, so that the geometries in the sense of the "Erlanger Programm" of F. Klein (1872) have been recognized as theories of invariants exclusively under such transformation groups, when the transformations are expressed in terms of coordinates.

The tangent spaces for the geometries of connections have hitherto been recognized with respect to such transformation groups.

The groups of structure of the principal fibre bundles as well as the group manifolds of the geometry of groups in the sense of E. Cartan (1) have also been such group manifolds.

The present author (11) has, however, discovered that there exist infinitely many linear transformations, of which the group parameters are functions of coordinates. Thus in one of his previous papers (18), he showed that the "Erlanger Programm" may be amplified by extending the classical transformation groups $\mathfrak{C}$ by the corresponding transformation groups $\mathfrak{H}$ with functions of coordinates as group parameters to extended transformation groups $\mathfrak{G} = \mathfrak{C} \mathfrak{H} = \mathfrak{H} \mathfrak{C}$.

He has started in (8), (20), (21), (24), (25), (26), (27), (28), (29), (30) to extend all the branches arising in the following system by extending respective transformation group parameters to functions of coordinates:
All the results are realized (18), (19) in the differentiable Manifolds (Atlas in the sense of (6)) in the sense of S. S. Chern (5) and C. Ehresmann (4). It should however be noticed that the structure groups are different from those in their senses, those in the present author's sense being the extended ones as was explained above.

In this paper, an extended Lie's higher sphere geometry will be developed and then an extended parabolic Lie geometry, an extended equiform Laguerre geometry, an extended conformal geometry, an extended dual conformal geometry, and an extended Laguerre geometry will be derived from it.

The present author has already introduced them briefly elsewhere (6), (9), (10), (12), (13), (15), (16), (18), (19), (24), (25). But, here a detailed systematic exposition is aimed at.

These extended geometries are situated among others in the positions, which is indicated by * in the following system:
When the following project of the present author is performed, the whole global theory of differentiable manifolds will become quite clear, so that the most important problem of geometry gets thus solved:

(i) Global theory of connections with the structure groups obtained by extending the group parameters to functions of coordinates.
(ii) Lie's principal fibre bundle geometry,
(iii) parabolic Lie principal fibre bundle geometry,
(iv) equiform Laguerre principal fibre bundle geometry,
(v) Laguerre principal fibre bundle geometry,

even the theories of respective local connections having hitherto been unsuccessful even for J. A. Schouten and E. Cartan.

So we shall succeed to introduce into differentiable manifolds all the extended geometries corresponding to all the branches given in the first table of the system, justifying the following joint notion of E. Cartan ([(32), p. 701, Foot-note) and the present author:

"Les connexions affines que j'ai introduites rentrent dans les connexions encore plus générales que à M. Schouten (math. Zeitschr., 13 (1922), 56-81); mais le point de vue de M. Schouten est différent du mien. Pour lui le transport parallèle (lineare Übertragung) est la notion géométrique essentielle; pour moi, elle n'est qu'un moyen qui tient aux propriétés de l'espace affine et qui ne peut plus s'utiliser, au moins directement, pour établir la notion d'espace à connexion projective (ou conforme, etc.)."

The present author has the same notion as E. Cartan had, because a choice of a linear connection for one and the same differentiable manifold corresponding to a structural group means a choice of the paths as tangents to curves and subvarieties under consideration.

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**) Part II will appear afterwards.
formal Geometry from that in the Extended Lie Geometry.


§ 1. Preliminaries.

1. Extended Stereographic Projection. II-Geodesic (n + 2) - Hyperspherical Point Coords. The present author has [28] established an extended conformal geometry, which is obtained by extending* group parameters of the conformal geometry to functions of coordinates, in the extended Euclidean space ([18], [20], [21]). Here it will briefly be recapitulated. For our purpose, we have first to convert the projective-geometrical extended Euclidean space of ([28], Art. 5) into a sphere-geometrical extended Euclidean space. This is executed by extending the ordinary stereographic projection.

Consider in an extended Euclidean space \(E^{n+1}\), a II-geodesic hypersphere ([20]) with II-geodesic radius \(i\) and with center \((0, 0, \ldots, 0)\) together with the
points
\[ S: (0, 0, \ldots, 0, -1), \quad P': (\xi^1), \quad P: (\xi^1, \xi^2, \ldots, \xi^n, t), \]
where \( \xi^1 \) and \( \xi^2 \) are II-geodesic rectangular Cartesian coordinates.

The II-geodesic curve \( SP' \) has the equations
\[
(1.1) \quad \frac{\xi^1 - \xi^1}{\xi^1 - 0} = \frac{\xi^{n+1} - \xi^{n+1}}{\xi^{n+1} + t}, \quad (l = 1, 2, \ldots, n),
\]
where \( \xi \)'s are the current coordinates. Hence
\[
\frac{\xi^1 - \xi^1}{\xi^1} = \frac{\xi^{n+1} - \xi^{n+1}}{\xi^{n+1} + t}, \quad \left( \xi^1 \xi^1 + \xi^{n+1} \xi^{n+1} = \left( \frac{1}{2} \right)^2 \right),
\]
whence follows:
\[
(1.2) \quad \xi^1 = \frac{\xi^1}{\xi^{n+1} + t}.
\]
\[ \xi^1 \xi^1 + \xi^{n+1} \xi^{n+1} = (\xi^1)^2, \quad (1 + \xi^{n+1}) (1 - \xi^{n+1}) = \xi^1 \xi^1 = (\xi^{n+1} + t)^2 (\xi^1 \xi^1), \]
whence follows:
\[
(1.3) \quad \xi^{n+1} = \frac{1}{2} \frac{\xi^1 \xi^1 - 1}{\xi^1 \xi^1 + 1}, \quad \xi^{n+1} + 1 = \frac{1}{2 (\xi^1 \xi^1 + 1)},
\]
\[
(1.4) \quad \xi^1 = \frac{\xi^1}{2 (\xi^1 \xi^1 + 1)}.
\]
The point (1.2) is the extended stereographic projection of the point \( (\xi^1, \xi^{n+1}) \) on the II-geodesic hypersphere
\[ \xi^1 \xi^1 + \xi^{n+1} \xi^{n+1} = \left( \frac{1}{2} \right)^2. \]

Set
\[
(1.5) \quad \begin{cases} \rho \cdot \xi^k = (\xi^1 \xi^1 + 1) \xi^k = \xi^k, & (l, k = 1, 2, \ldots, n), \\ \rho \cdot \xi^{n+1} = (\xi^1 \xi^1 + 1) \xi^{n+1} = -\frac{1}{2} (\xi^1 \xi^1 - 1), \\ (\rho = 0) \cdot \xi^{n+2} = \frac{1}{2} (\xi^1 \xi^1 + 1) it = \frac{i}{2} (\xi^1 \xi^1 + 1), \end{cases}
\]
where
\[ t = 1 \quad \text{at finiteness}, \]
\[ t = 0 \quad \text{at infinity}, \]
then we have
\[
(1.6) \quad \rho \xi^k \xi^k + \xi^{n+1} \xi^{n+1} + \xi^{n+2} \xi^{n+2} = 0.
\]

*) There are four kinds of extensions of groups: (i) The extension problem ("Erweiterungs-
problem"), which was proposed and solved by Otto Schreier [2], [3], p. 89) runs: Given two abstract
groups \( \mathcal{G} \) and \( \mathcal{R} \), it is required to exhaust the groups \( \mathcal{G} \), which contain \( \mathcal{R} \) as normal subgroup, so that \( \mathcal{G}/\mathcal{R} = \mathcal{R} \).

(ii) Paul Dedekind [17] beyond access for the present author considered extensions of the structural
group of an abstract fibred space. (Cf. [8], [11], [16], [19].) (iii) Another extension of groups is
done from the view-point of factorization. (iv) The "extension" in the present author's sense is
done by extending the group parameters to functions of coordinates giving rise to "enlargement" of the
"Erlanger Programm."
We will call (1.5) accompanied by the identity \([1.6]\) the \(\text{II-geodesic } (n+2)\)-hyperspherical point coordinates of the point \((\xi^l)\).

From (1.5), we see the following correspondence of the fundamental systems, of the coordinates:

\[
\begin{align*}
\varphi^k &= 0 & \xi^k &= 0 & \xi^k &= 0 \\
\varphi^{n+1} &= 0 & \xi^{n+1} &= 0 & \xi^l \xi^l &= 1 \\
\varphi^{n+2} &= 0 & t &= 0 & \xi^l \xi^l &= -1
\end{align*}
\]

These \((n+2)\) figures are II-geodesic hyperspheres, which are mutually orthogonal. (Extended Poincaré-Klein representation !)

2. **II-Geodesic \((n+2)\)-Hyperspherical Hypersphere Coordinates, Linear Equation to a II-Geodesic Hypersphere.**

We set

\[
(2.1) \quad \begin{cases}
\eta_l = \frac{\alpha^l}{\epsilon R}, & (l = 1, 2, \ldots, n; \epsilon = \pm 1), \\
\eta_{n+1} = \frac{-1}{2\epsilon R} - (\alpha^l \alpha^l - R^2 - 1), \\
\eta_{n+2} = \frac{i}{2\epsilon R} - (\alpha^l \alpha^l - R^2 + 1).
\end{cases}
\]

Then we have the identity:

\[
(2.2) \quad \eta_l \eta_l + \eta_{n+1} \eta_{n+1} + \eta_{n+2} \eta_{n+2} = 1.
\]

It will readily be seen that the equation

\[
(2.3) \quad (\xi^l - \alpha^l \xi^l - \alpha^l l^l) = R^2
\]

to a II-geodesic hypersphere becomes

\[
(2.4) \quad \eta_{l} \xi^l + \eta_{n+1} \xi^{n+1} + \eta_{n+2} \xi^{n+2} = 0.
\]

We will call \((\eta_l, \eta_{n+1}, \eta_{n+2})\) the II-geodesic \((n+2)\)-hyperspherical hypersphere coordinates.

In the limit \(R \to 0\), the (2.1) becomes to (1.5).

3. **Extended Conformal Transformation Group. Extended Conformal Geometry.**

**Theorem 1**: ([28], Theorem 5, p. 132) Extended conformal transformation group coincides with the group of extended \((n+2)\)-ary orthogonal transformation group with unit determinant in II-geodesic \((n+2)\)-hyperspherical coordinates:

\[
(3.1) \quad \xi^l = \alpha^l_{m}(\xi^p) \xi^m, \quad (l, m, p = 1, 2, \ldots, n+2),
\]

where \(\{\alpha^l_{m}(\xi^p)\}\) is an orthogonal matrix with unit determinant.

**Theorem 2**: ([28], Theorem 1, p. 120) The most general extended conformal
transformation consists in $n$-dimensional equiform transformations and extended inversions in doubly oriented II-geodesic $(n-1)$-dimensional hypersphere.

**Theorem 3**: ([28], Theorem 2, p. 121) The most general extended conformal transformation in $n$-dimensional space is an extended equiform transformation.

**Theorem 4**: ([28], Theorem 3, p. 122) The extended conformal transformation group is identical with the extended equiform transformation group.

4. **II-Geodesic $(n+2)$-Hyperspherical Point Coordinates and II-Geodesic $(n+1)$-Hyperspherical Hypersphere Coordinates in an Extended Non-Euclidean Space.** Let us consider the extended conformal geometry realized in the extended non-Euclidean space ([26]; [27], [28], Art. 15). For it we introduce double orientation:

- Direct orientation by the distinction of the double sign: $\epsilon R, (\epsilon = \pm 1)$,
- Correlative orientation by the distinction of the double sign: $\epsilon \psi, (\epsilon = \pm 1)$,

\[ \frac{\pi}{2} = \frac{r}{k} + \psi, \]  

(4.1)

where

- $\epsilon R$ is the II-geodesic radius of the II-geodesic hypersphere.
- $\epsilon \psi$ is a II-geodesic $(n-1)$-flat.

A point is oriented correlative by the distinction of the double sign: $\pm \xi^l, \pm u_l$.

We start with the equation:

\[ \cos \frac{R}{k} = \frac{a^p \xi^p}{\sqrt{a^p a^p} \sqrt{\xi^p \xi^p}}, \]

\[ \cos \phi = \frac{a_p u_p}{\sqrt{a_p a_p} \sqrt{u_p u_p}}, \]

(\(p = 1, 2, \ldots, n, n+1\))

or

\[ a^p \xi^p = \cos \frac{R}{k} (a^p a^p) \xi^p, \]

\[ a_p u_p = \cos \phi (a_p a_p) (u_p u_p), \]

(4.2)

\[ a^p a^p = \xi^p \xi^p = k^2, \]

(4.3)

\[ a^p = ka_p. \]
It represents a doubly oriented II-geodesic hypersphere.
The (4.2) may be rewritten as follows:

\[
\csc \left( \frac{R}{k} \right) \left( a^p \xi^p + (i \csc \frac{R}{k}) (a^p a^p) (i \xi^p \xi^p) \right) + (i \cos \phi) (a^p u_p) (u^p u_p) = 0.
\]

If we set

\[
\eta^p = a^p \csc \frac{R}{k}, \quad \zeta_p = a^p \csc \phi,
\]

(4.5)

\[
\eta^{n+2} = i \cot \frac{R}{k}, \quad \zeta_{n+2} = i \cot \phi,
\]

(\(\phi = 1, 2, \ldots, n+1\)),

then we have the identity:

\[
\eta^p \eta^p + \eta^{n+2} \eta^{n+2} = 1.
\]

(4.6)

We will call the

(\(\eta^p, \eta^{n+2}\))

the II-geodesic \((n+2)\)-hyperspherical hypersphere coordinates of the doubly oriented II-geodesic hypersphere (4.4).

If we set

\[
\rho \cdot \xi^p = \xi^p,
\]

(4.7)

\[
\rho \cdot \xi^{n+2} = i (\xi^p \xi^p)^{\frac{1}{2}},
\]

\(\rho \cdot u_p = u_p,\)

(\(\rho \cdot u^{n+2} = i (u^p u_p)^{\frac{1}{2}},\))

(\(\rho = 1, 2, \ldots, n+1; 0 < \rho: \text{otherwise arbitrary})\),

then we have the identity:

\[
\xi^p \xi^p + \xi^{n+2} \xi^{n+2} = 0.
\]

(4.8)

We will call the \((4.7)\) the II-geodesic \((n+2)\)-hyperspherical point

coordinates of the (doubly) oriented
point $(x^p)$.  

5. **Linear Equation to the Doubly Oriented II-Geodesic Hypersphere.** Owing to (4.5) and (4.7), the equation (4.4) becomes now to the linear equation:

\[
(5.1) \quad \eta^p x^p + \eta^{n+2} x^{n+2} = 0.
\]

The Absolute

\[
(5.2) \quad \eta^{n+2} x^{n+2} = 0
\]

may be oriented by the distinction of the double sign:

\[
(5.3) \quad (\eta) = 0, 0, \ldots, 0, \pm 1.
\]

The minimal II-geodesic

\[
(5.4) \quad \eta^l x^l = 0, \quad (\eta^l \eta^l = 0)
\]

is reckoned in the doubly-oriented II-geodesic $(n+1)$-hypersphere.

The law of the left-hand side is the same as that of the extended conformal geometry, which was stated in Art. 3. The law of the right-hand side is correlative to it and is the same as that of the extended conformal geometry in the abstract sense. But in concrete sense it will be called the extended dual conformal geometry or extended non-Euclidean Laguerre geometry.

6. **Angle and II-Geodesic Tangential Distance.**

I. When two doubly oriented II-geodesic hyperspheres

\[
(\bar{\eta}) \text{ and } (\bar{\eta}') \quad \text{ and } (\bar{\xi}) \text{ and } (\bar{\xi}')
\]

are given,

a **coaxal system** $(\lambda \bar{\eta} + \mu \bar{\eta}')$  

an **homocentric system** $(\lambda \bar{\xi} + \mu \bar{\xi}')$

is determined.

When, in particular, the two doubly oriented II-geodesic hyperspheres are orthogonally incline:

\[
(\bar{\eta} \bar{\eta}') = 0, \quad (\bar{\xi} \bar{\xi}') = 0,
\]

\[
(\eta^p x^p + \eta^{n+2} x^{n+2} = 0).
\]

\[
(5.1) \quad \zeta_{n+1} u_{n+1} = 0.
\]

\[
(5.2) \quad \eta^{n+2} x^{n+2} = 0
\]

may be oriented by the distinction of the double sign:

\[
(5.3) \quad (\eta) = 0, 0, \ldots, 0, \pm 1.
\]

\[
(5.4) \quad \eta^l x^l = 0, \quad (\eta^l \eta^l = 0)
\]

is reckoned in the doubly-oriented II-geodesic $(n+1)$-hypersphere.

The law of the left-hand side is the same as that of the extended conformal geometry, which was stated in Art. 3. The law of the right-hand side is correlative to it and is the same as that of the extended conformal geometry in the abstract sense. But in concrete sense it will be called the extended dual conformal geometry or extended non-Euclidean Laguerre geometry.

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When, in particular, the two doubly oriented II-geodesic hyperspheres are orthogonally incline:

\[
(\bar{\eta} \bar{\eta}') = 0, \quad (\bar{\xi} \bar{\xi}') = 0,
\]
any other doubly oriented II-geodesic hypersphere

\begin{align*}
\eta &= \lambda \xi + \mu \xi', \\
\xi &= \lambda \xi + \mu \xi',
\end{align*}

is given by

\( (\lambda^2 + \mu^2 = 1) \).

When, in particular,

\begin{align*}
(\eta) &\quad | \quad (\xi)
\end{align*}

is a doubly oriented

null II-geodesic hypersphere, we have

\begin{align*}
(\eta \bar{\eta}) &= \lambda^2 (\bar{\eta} \eta) + \mu^2 (\bar{\eta}' \eta') \\
(\xi \bar{\xi}) &= \lambda^2 (\bar{\xi} \xi) + \mu^2 (\bar{\xi}' \xi')
\end{align*}

\( = \lambda^2 + \mu^2 = 0 \),

so that

\begin{align*}
\eta &= \bar{\eta} \pm i \bar{\eta}', \\
\xi &= \bar{\xi} \pm i \bar{\xi}'.
\end{align*}

Setting

\begin{align*}
\xi' &= \bar{\eta} + i \bar{\eta}', \\
\xi' &= \bar{\xi} - i \bar{\xi}',
\end{align*}

we call

(\xi) and (\bar{\xi}) the foci of the coaxal system.

\( u = \xi + i \xi', \quad \bar{u} = \xi - i \xi' \),

we call

(\eta) and (\bar{\eta}) the directrices of the homocentric system.

\begin{align*}
(\eta, \eta') &\quad | \quad (\xi, \xi')
\end{align*}

The angle

\( (\eta, \eta') = \frac{1}{2i} \log(\eta \bar{\eta}, \eta' \bar{\eta}') \).

The II-geodesic tangential distance

\( (\xi, \xi') = \frac{1}{2ik} \log(u \bar{u}, \xi' \xi') \).

between two doubly oriented II-geodesic hyperspheres

(\eta), (\eta')

belonging to this

coaixal homocentric system is given by

\begin{align*}
(6.1) \quad (\eta, \eta') &= \frac{1}{2i} \log(\xi \bar{\xi}, \eta' \bar{\eta}'), \\
(\xi, \xi') &= \frac{1}{2ik} \log(u \bar{u}, \xi' \xi').
\end{align*}
II. From (6.1), we can deduce the formula:

\[
\cos (\eta, \eta') = \frac{(\eta \eta')}{\sqrt{(\eta \eta)} \sqrt{(\eta' \eta')}}.
\]

(6.2)

\[
\cos \frac{(\zeta \zeta')}{k} = \frac{(\zeta \zeta')}{\sqrt{(\zeta \zeta)} \sqrt{(\zeta' \zeta')}}.
\]


Theorem 1:\ The most general extended conformal transformation in \(n\)-dimensional space is composed of extended equiform transformations and extended inversions in doubly oriented II-geodesic \((n-1)\)-dimensional hypersphere.

Theorem 2:\ The most general conformal transformation in \(n\)-dimensional space is an extended equiform transformation. (Cf. Theorem 4,° Art. 3.).

Theorem 3:\ Any extended \((n+2)\)-ary orthogonal transformation with unit determinant

\[
\xi^l = \alpha^l_\mu (\xi^p) \xi^m, \quad (l, m, p = 1, 2, \cdots, n+2)
\]

is decomposable into extended inversion and extended non-Euclidean transformation and is thus an extended conformal transformation.

8. Some Formulas concerning the Extended Conformal Transformations.

1°. Extended conformal transformation

\[
\xi^l = \alpha^l_\mu (\xi^p) \xi^m, \quad (l, m, p = 1, 2, \cdots, n+2),
\]

where \(\alpha^l_\mu (\xi^p)\) is an orthogonal matrix with unit determinant, is accompanied by a transformation

\[
d\xi^l = \alpha^l_\mu (\xi^p)d\xi^m
\]

within the \(l\)-th II-geodesic coaxal system:

\[
\xi^l = a^l \cos \sigma + c^l \sin \sigma, \quad (\sigma = \text{parameter})
\]

of doubly oriented II-geodesic hyperspheres (cf. (8.11)), where

\[
a^la^l = c^lc^l = 1, \quad a^lc^l = 0,
\]

so that

\[
d\sigma^2 = d\xi^l d\xi^l = d\xi^m d\xi^m
\]

is an invariant.

2°. The \(\xi^l\) of the II-geodesic coaxal system (8.3) are solutions of

\[
\frac{d^2\xi^l}{d\sigma^2} + \xi^l = 0.
\]
By (8.2), we have

\[ \frac{d\xi^l}{d\sigma} = \alpha^l_m(\xi^p) \frac{d\xi^m}{d\sigma}, \]  

(8.7)

\[ \frac{d^2\xi^l}{d\sigma^2} = \frac{d}{d\sigma} \alpha^l_m(\xi^p) \frac{d\xi^m}{d\sigma} + \alpha^l_m(\xi^p) \frac{d^2\xi^m}{d\sigma^2}, \]  

(8.8)

so that

\[ \frac{d^2\xi^l}{d\sigma^2} + \xi^l = \frac{d}{d\sigma} \alpha^l_m(\xi^p) \frac{d\xi^m}{d\sigma} + \alpha^l_m(\xi^p) \left( \frac{d^2\xi^m}{d\sigma^2} + \xi^m \right). \]  

(8.9)

In order that the II-geodesic coaxal system (8.6) must be transformed into the II-geodesic coaxal system

\[ \frac{d^2\xi^l}{d\sigma^2} + \xi^l = 0, \]

it is necessary and sufficient that

\[ \frac{d}{d\sigma} \alpha^l_m(\xi^p) \frac{d\xi^m}{d\sigma} = 0 \]

within the II-geodesic coaxal system (8.6). Thus we have

\[ \frac{d^2\xi^l}{d\sigma^2} + \xi^l = \alpha^l_m(\xi^p) \left( \frac{d^2\xi^m}{d\sigma^2} + \xi^m \right), \]  

(8.11)

\[ \frac{d^2\xi^l}{d\sigma^2} = \alpha^l_m(\xi^p) \frac{d^2\xi^m}{d\sigma^2}. \]  

(8.12)

From (8.1), we obtain

\[ \frac{d\xi^l}{d\sigma} = \frac{d}{d\sigma} \alpha^l_m(\xi^p) \xi^m + \alpha^l_m(\xi^p) \frac{d\xi^m}{d\sigma}, \]  

(8.13)

[8.7] and (8.13) give

\[ \frac{d}{d\sigma} \alpha^l_m(\xi^p) \xi^m = 0 \]

within the II-geodesic coaxal system.

If we introduce $\Omega^l_m(\xi^p)$ by the demand

\[ \Omega^l_m(\xi^p)\alpha^l_m(\xi^p)=\delta^l_n, \quad \Omega^l_m(\xi^p)\alpha^l_m(\xi^p)=\delta^l_n, \]

(8.15)
and set

\[(8.16) \quad \Lambda_{\alpha}^{\beta} = \Omega_{\beta}^{\gamma} \frac{\partial \alpha_{\gamma}}{\partial \xi_{\alpha}},\]

then

\[(8.17) \quad \frac{d^{2} \xi^{\ell}}{d \sigma^{2}} + \xi^{\ell} = \frac{d}{d \sigma} \left\{ \alpha_{m}^{l} (\xi^{r}) \frac{d \xi^{m}}{d \sigma} \right\} + \alpha_{m}^{l} (\xi^{r}) \xi^{r} = \alpha_{m}^{l} (\xi^{r}) \left( \frac{d^{2} \xi^{m}}{d \sigma^{2}} + \xi^{r} \right),\]

so that

\[(8.18) \quad \frac{d}{d \sigma} \alpha_{m}^{l} (\xi^{r}) \frac{d \xi^{m}}{d \sigma} = \alpha_{m}^{l} (\xi^{r}) \Lambda_{\alpha}^{\beta} \frac{d \xi^{\alpha}}{d \sigma} \frac{d \xi^{\beta}}{d \sigma} = 0\]

within the II-geodesic coaxal system.

3. The II-geodesic coaxal system \((8.3)\) constitute the reference system of the II-geodesic \((n+2)\)-hyperspherical coordinates \((\xi^{r})\).

§ 2. Extended Laguerre Geometry and Extended Equiform Laguerre Geometry.

9. Laguerre Geometry and Equiform Laguerre Geometry. Consider an oriented hypersphere with center \((x^{i})\), \((i=1, 2, \ldots, n)\) and with radius \(r\) in the \(n\)-dimensional rectangular Cartesian space. Then it is enveloped by the oriented hyperplanes

\[(9.1) \quad l_{i} x^{i} - p = r, \quad (l_{i} l_{i} = 1),\]

where the left-hand side is the Hesse's normal form.

If we put

\[(9.2) \quad \xi^{i} = x^{i}, \quad \xi^{n+1} = i r,\]

\[(9.3) \quad u_{i} = l_{i}, \quad u_{n+1} = i,\]

then \((9.1)\) becomes

\[(9.4) \quad u_{i} \xi^{i} - p = 0, \quad (l=1, 2, \ldots, n+1),\]

where

\[(9.5) \quad u_{i} u_{i} = 0.\]
(9.4) is the Hesse’s normal form for an oriented minimal (in the sense that \( u_iu_i = 0, \neq 1 \)) hyperplane in the \((n+1)\)-dimensional Cartesian space. Thus we obtain the following correspondence:

<table>
<thead>
<tr>
<th>Cartesian Space ( \mathbb{R}^{n+1} )</th>
<th>Laguerre Space ( \mathbb{L}^n )</th>
</tr>
</thead>
<tbody>
<tr>
<td>Point ((\xi^l))</td>
<td>Oriented hypersphere ((\xi^l))</td>
</tr>
<tr>
<td>Oriented minimal hyperplane ((u_i, \rho), (u_iu_i = 0))</td>
<td>Oriented hyperplane ((u_i, \rho), (u_iu_i = 1, i = 1, 2, \ldots, n))</td>
</tr>
<tr>
<td>Oriented hyperplane ((9.6))</td>
<td>Oriented hyperplane pair enveloped by the oriented hyperspheres ((U_i, \rho), (U_iU_i = 1))</td>
</tr>
<tr>
<td>((9.7)) ((U_i, \rho), (U_iU_i = 1))</td>
<td>Homocentric system of oriented hyperspheres ((\lambda \xi^l + \mu \eta^l)/(\lambda + \mu))</td>
</tr>
<tr>
<td>Straight line ((\lambda \xi^l + \mu \eta^l)/(\lambda + \mu))</td>
<td></td>
</tr>
</tbody>
</table>

Fundamental transformation group:

Congruent transformation group | Laguerre group

\(=\text{\((n+1)\)-ary orthogonal transformation group with unit determinant.}\)

Fundamental invariants:

Squared distance between two points: | Squared tangential distance between two oriented hyperspheres:

\(\xi^l\) and \(\eta^l\): | \((\xi^l - \eta^l)(\xi^l - \eta^l) = d^2\).

Cosine of the angle \(\Theta\) between two oriented hyperplanes: | Cosine of the Laguerre hyperplane pair enveloped by oriented hyperspheres

\((U_i), (V_i): \cos \Theta = U_iV_i.\)

This correspondence enables us to formulate the \(n\)-dimensional Laguerre geometry of \(\mathbb{L}^n\) realized in the \(n\)-dimensional Cartesian space as follows:

The \(n\)-dimensional Laguerre geometry is a theory of invariants under the Laguerre group, the space-elements being
Thereby the oriented linear hypercomplex of oriented hyperspheres is the totality of the oriented hyperspheres, which are tangentially equidistant \( R \) from a fixed oriented hypersphere, which can eventually be an (oriented) point. There is no distinction in the Laguerre geometry between an (oriented) point and an oriented hypersphere, the former belonging to the category of the latter.

**The law of the \( n \)-dimensional Laguerre geometry is the same as the law of the \( (n+1) \)-
dimensional Euclidean geometry in abstract sense.**

**Minimal Projection.** The correspondence mentioned above may be established directly by the so-called *minimal projection* as follows:

The equation to a minimal hypercone with vertex \( a^i \) in the \( (n+1) \)-dimensional Cartesian space may be written in the form:

\[
(\xi^i-a^i)(\xi^i-a^i)+(\xi^{n+1}-a^{n+1})^2 = 0, \quad (i=1, 2, \ldots, n).
\]

Consider the case

\[ a^{n+1} = ir. \]

If we cut this minimal hypercone with the hyperplane \((n\text{-dimensional Cartesian space}) \ \xi^{n+1} = 0\), then we obtain

\[
(\xi^i-a^i)(\xi^i-a^i)-r^2 = 0,
\]

which represent an hypersphere with center \( a^i \) and with radius in the \( n \)-dimensional Cartesian space \( \xi^{n+1} = 0 \), in which the Laguerre space is established. Thus the following correspondence is established geometrically:

<table>
<thead>
<tr>
<th>( (n+1) )-dimensional Cartesian Space</th>
<th>( n )-dimensional Laguerre Space</th>
</tr>
</thead>
<tbody>
<tr>
<td>Point ((a^i, a^{n+1}=ir))</td>
<td>Oriented hypersphere with center ((a^i)) and with radius ( r )</td>
</tr>
</tbody>
</table>

Correspondingly we obtain the following correspondence:

<table>
<thead>
<tr>
<th>( (n+1) )-dimensional Equiform Space</th>
<th>( n )-dimensional Equiform Laguerre Space</th>
</tr>
</thead>
</table>

10. **Extended Laguerre Geometry.** As in the last Art., we have the following correspondence:
### Extended Euclidean Space $\mathbb{E}^{n+1}$

| II-geodesic rectangular coordinates $(\xi^l)$ | Oriented II-geodesic hypersphere with center $(\xi^l)$ and with II-geodesic radius $r = -\xi^{n+1}$ |
| (II-geodesic distance)$^2$ | (II-geodesic tangential distance)$^2$ |

between two points $(\xi^l), (\eta^l)$: $(\xi^l - \eta^l) (\xi^l - \eta^l)$

II-geodesic curve homocentric system of oriented II-geodesic hyperspheres

$$\xi^l = a^l s + c^l, \quad (a'a^l = 1)$$

Minimal II-geodesic flat oriented II-geodesic flats pair

$$(u_l): \quad (u_l \xi^l = 0, \quad (u_l u_l = 0).$$

---

**Extended Laguerre Transformation Group.** Given an orthogonal Matrix $(a_{m}^{l}(\xi^{p}))$ with unit determinant

\begin{equation}
|a_{m}^{l}(\xi^{p})| = 1, \tag{10.1}
\end{equation}

where $(\xi^l), \,(l = 1, 2, \cdots, n+1)$ are Laguerre coordinates of an oriented hypersphere, we set

\begin{equation}
\omega^{l} = a_{m}^{l}(\xi^{p}) d \xi^{m}, \tag{10.2}
\end{equation}

which is not exact in general.

A straight forward calculation shows that

\begin{equation}
\frac{d}{ds} \frac{\omega^{l}}{ds} = a_{m}^{l} \left\{ \frac{d^{2} \xi^{m}}{ds^{2}} + \Lambda_{m}^{l} \frac{d \xi^{k}}{ds} \frac{d \xi^{k}}{ds} \right\}, \tag{10.3}
\end{equation}

where $ds$ is the common tangential segment between two consecutive oriented hyperspheres $\xi^l(s)$ and $\xi^l(s + ds)$:

\begin{equation}
ds^2 = d\xi^l d\xi^l \tag{10.4}
\end{equation}

and

\begin{equation}
\Lambda_{m}^{l} \overset{\text{def}}{=} \Omega_{m}^{\ell}(\xi^{p}) \frac{\partial a_{k}^{l}}{\partial \xi^{k}} = - a_{k}^{l} \frac{\partial \Omega_{m}^{\ell}}{\partial \xi^{k}}, \tag{10.5}
\end{equation}
where $\Omega_{\ell}^{m}(\xi^{p})$ are defined by

$$\Omega_{\ell}^{k}(\xi^{p}) = \delta_{\ell}^{k}, \quad \Omega_{l}^{m}(\xi^{p}) a_{k}^{\ell}(\xi^{p}) = \delta_{k}^{l}. \quad (10.6)$$

The differential equation of the type

$$\frac{d^{2}\xi^{m}}{ds^{2}} + \Lambda_{u}^{n} \frac{d\xi^{h}}{ds} \frac{d\xi^{k}}{ds} = 0 \quad (10.7)$$

is well known. We will call the solutions $\xi^{m}(s)$ of (10.7) homocentric system of II-geodesic hyperspheres. Formerly, only local properties of $\xi^{m}(s)$ had been known as paths of teleparallelism. Now (10.3) and (10.7) tell us that

$$\frac{d\omega^{l}}{ds} = 0 \quad \Leftrightarrow \quad \frac{d^{2}\xi^{m}}{ds^{2}} + \Lambda_{u}^{n} \frac{d\xi^{h}}{ds} \frac{d\xi^{k}}{ds} = 0 \quad (10.8)$$

and the differential equation

$$\frac{d}{ds} \frac{\omega^{l}}{ds} = 0 \quad (10.9)$$

may be integrated readily:

$$\omega^{l} = a^{l} ds, \quad (a^{l} = \text{const.}), \quad (10.10)$$

where

$$a^{l} a^{l} = 1 \quad (10.11)$$

owing to

$$ds^{2} = d\xi^{l} d\xi^{l} = \omega^{l} \omega^{l}, \quad (10.12)$$

which follows from (10.2) and (10.4).

We set

$$d\xi^{l} = a^{l} ds = \omega^{l}, \quad (10.13)$$

what is admissible, notwithstanding $\omega^{l}$ is not exact in general.

Integrating (10.13) once more, we have

$$\xi^{l} = a^{l} s + c^{l}, \quad (c^{l} = \text{const.}), \quad (10.14)$$

The definition of $\xi^{l}$ by (10.14) means that we adopt the l-th homocentric system of II-geodesic hyperspheres as a generalization of coordinate axis.

The expression (10.14) shows us that the homocentric system behave as for "meet" and "join" like straight lines just as ordinary homocentric systems do.

Now (10.2) may be rewritten:

$$d\xi^{l} = a^{l}_{\mu}(\xi^{p}) d\xi^{m} = a^{l} ds. \quad (10.15)$$

This may be integrated once more as (10.14):

$$\xi^{l} = a^{l}_{\mu}(\xi^{p}) d\xi^{m} = a^{l}_{\mu}(\xi^{p}) \xi^{m} + a^{l}_{\mu}, \quad (10.16)$$
where

\[
(10.17) \quad a_o^{d_{-\int \xi^{m}d\alpha_\mu^{(\xi^{p})}}}.
\]

Now we have

\[
(10.18) \quad \frac{d^2\xi^l}{ds^2} = a^\tau_n^{(\xi^{p})}\left\{ \frac{d^2\xi^m}{ds^2} + \Lambda^{\mu}_{\nu} \frac{d\xi^h}{ds} \frac{d\xi^k}{ds} \right\}
\]

quite as in the case of (8.17), where \( l, p, h, k = 1, 2, \ldots, n + 1 \) and for ordinary homocentric system, we have

\[
(10.19) \quad \frac{d^2\xi^m}{ds^2} = 0,
\]

while

\[
(10.20) \quad \frac{d^2\xi^l}{ds^2} = \frac{d^2\omega^l}{dsds} = 0
\]

along the homocentric system (10.7). Hence along the homocentric system (10.7), we must have

\[
(10.21) \quad \Lambda^{\mu}_{\nu} \frac{d\xi^h}{ds} \frac{d\xi^k}{ds} = 0.
\]

Now, quite as in the case of (8.18), we have

\[
(10.22) \quad \frac{d}{ds} \frac{d^2\xi^l}{ds} = a^\tau_n^{(\xi^{p})} \frac{d\xi^m}{ds} = a^\tau_n^{(\xi^{p})} \Lambda^{\mu}_{\nu} \frac{d\xi^h}{ds} \frac{d\xi^k}{ds} = 0
\]

along the homocentric system (10.7). The same result may be deduced as follows.

If we differentiate (10.15):

\[
\frac{d\xi^l}{ds} = a^\tau_n^{(\xi^{p})} \frac{d\xi^m}{ds},
\]

then we have

\[
\frac{d^2\xi^l}{ds^2} = \frac{d\alpha_\mu^{(\xi^{p})} d\xi^m}{ds} + a^\tau_n^{(\xi^{p})} \frac{d^2\xi^m}{ds^2},
\]

whence follows (10.22).

Now (10.17) becomes

\[
-a_o^{d_{-\int \xi^{m}d\alpha_\mu^{(\xi^{p})}}} = \int \int \{ da_\nu^{(\xi^{p})} d\xi^m \} = 0
\]

by (10.22), the condition for that the repeated integral may be converted into the double integral being evidently fulfilled. Hence
By virtue of (10.16), any transformation of the type
\[ \tilde{\xi}^l = a_n^l (\xi^p) \xi^m + a_o^l, \]
where \( a_n^l (\xi^p) \) is an orthogonal matrix with unit determinant, will be called an extended Laguerre transformation.

By (10.16), an ordinary homocentric system
\[ \xi^l = a^l s + c^l, \quad (a^l a^l = 1) \]
is transformed into an homocentric system of II-geodesic hyperspheres
\[ \xi^l = \delta s + \eta^l, \quad (\delta a^l = 1). \]

The totality of the extended Laguerre transformations (10.24) forms a group, which we will call the extended Laguerre group and the geometry belonging to it an extended Laguerre geometry.

The common tangential II-geodesic distance is invariant under the extended Laguerre group.

For, if we consider two II-geodesic hyperspheres \( (\xi^l) \) and \( (\eta^l) \) corresponding to \( a_n^l (\xi^p) \), then setting
\[ a_o^l = \eta^l - b_o^l (\xi^p), \]
we have e. g.
\[ \xi^l - a_o^l = \tilde{\xi}^l - \delta o^l \quad a_o^l = a_n^l (\xi^p) \xi^m \]
i. e.
\[ \xi^l - a_n^l (\xi^p) \Omega^m (\xi^p) a_o^l = a_n^l (\xi^p) \xi^m, \]
\[ \xi^l - \delta o^l \quad \eta^q = a_n^l (\xi^p) \{ \xi^m - \Omega^m (\xi^p) b_o^q (\xi^p) \} \]
or
\[ \xi^l - \eta^l = a_n^l (\xi^p) (\xi^m - \eta^m), \quad (\eta^m \overset{\text{def}}{=} \Omega^m (\xi^p) b_o^q (\xi^p)), \]
so that
\[ (\xi^l - \eta^l) (\xi^l - \eta^l) = (\xi^m - \eta^m) (\xi^m - \eta^m), \]
where
\[ \eta^m = \Omega^m (\xi^p) b_o^q (\xi^p) = \Omega^m (\xi^p) \eta^q - a_o^q, \]
\[ (10.26) \quad \eta^q = a_o^q (\xi^p) \eta^m + a_o^q. \]
Any linear II-geodesic hypersphere complex
\[
\bar{u}_l \xi^l + \bar{u}_{n+1} = 0, \quad (\bar{u}_l \bar{u}_l = 1)
\]
is transformed by an extended Laguerre transformation (10.24) into a linear II-
geodesic hypersphere complex
\[
(10.27) \quad u_l \xi^l + u_{n+1} = 0, \quad (u_l u_l = 1),
\]
where
\[
(10.28) \quad u_l \overset{\text{def}}{=} \Omega_l^{h} \hat{u}_h, \quad u_{n+1} \overset{\text{def}}{=} \hat{u}_{n+1} = -\hat{u}_l \xi^l = -u_l \overline{\xi}^l,
\]
the \( (\Omega_l^b(\xi'^p)) \) being an orthogonal matrix with unit determinant.

The extended Laguerre angle \( \Theta \) between two linear II-geodesic hypersphere complexes \((u_p)\) and \((u'_p)\) is given by
\[
(10.29) \quad \cos \Theta = \frac{u_l u'_l}{\sqrt{u_l u_l} \sqrt{u'_l u'_l}}
\]
and is invariant under the extended Laguerre group. For, e.g. we have
\[
(10.30) \quad \hat{u}_l \hat{u}'_l = \hat{u}_l \delta^h_l \hat{u}_h' = \Omega_l^j u_j \left[ \Omega_l^h a^j_h \right] \hat{u}_h' = \delta^{lh} u_j a^h_l \hat{u}_h' = u_h u'_h,
\]
where
\[
(10.31) \quad u_h' \overset{\text{def}}{=} a^h_l \hat{u}_h'.
\]

**Theories and Theorems.** The theories and theorems of the ordinary Laguerre
geometry are all invariant in the extended Laguerre geometry. Especially a detailed theory of the
Laguerre’s differential geometry of the author ([4], Bd. II) will be recommended.

**11. Extended Equiform Laguerre Geometry.** Quite as in Art. 10, we
have the following correspondence:

<table>
<thead>
<tr>
<th>Extended Equiform Space ( \mathfrak{E}^{n+1} )</th>
<th>Extended Equiform Laguerre Space ( \mathfrak{E}^n )</th>
</tr>
</thead>
</table>

Hence we may introduce the results of the extended equiform geometry \( \mathfrak{E}^{n+1} \) ([21],
Art. 25) into our extended equiform Laguerre geometry \( \mathfrak{E}^n \). The ordinary equiform
transformations are of the form
\[
\xi'^n = \rho \cdot a^m_n \xi^n + a'^n_b,
\]
where \( \xi^n \) and \( \xi'^n \) are rectangular Cartesian coordinates, \( (a^m_n) \) an orthogonal matrix
with unit determinant and \( \rho, a^m_n \) and \( a'^n_b \) are all constants. They may be extended
in two ways. Correspondingly the extended equiform Laguerre transformations may
be grasped in two steps.
I. Consider the transformation

\[ d\overline{\xi} = \rho(\xi^p)\alpha_m^\ell(\xi^p)d\xi^m, \]

where \( \xi^m \) are the ordinary Laguerre coordinates of an oriented hypersphere and \( (\alpha_m^\ell(\xi^p)) \) is an orthogonal matrix with unit determinant.

If we put

\[ d\overline{s}^2 = d\overline{\xi}^l d\overline{\xi}^l, \quad ds^2 = d\xi^l d\xi^l, \]

then we have

\[ d\overline{s} = \rho(\xi^p)ds, \]

so that (11.1) becomes

\[ \frac{d\overline{\xi}^l}{d\overline{s}} = \alpha_m^\ell(\xi^p)\frac{d\xi^m}{ds}, \]

whence follows:

\[ \frac{d^2\overline{\xi}^l}{d\overline{s}^2} = \frac{1}{\rho} \left( \frac{d\alpha_m^\ell(\xi^p)}{ds} \frac{d\xi^m}{ds} + \alpha_m^\ell(\xi^p) \frac{d^2\xi^m}{ds^2} \right). \]

Now for \( \xi^m \), we have

\[ \frac{d^2\xi^m}{ds^2} = 0, \quad \frac{d\xi^m}{ds} = a^m, \quad (a^m a^m = 1) \]

and the differential equations

\[ \frac{d\alpha_m^\ell(\xi^p)}{ds} \frac{d\xi^m}{ds} = 0 \]

for the homocentric system of \( \Pi \)-geodesic hyperspheres quite as in the case of (10.22). Hence from (11.5), we have

\[ \frac{d^2\xi^l}{d\overline{s}^2} = 0; \quad \frac{d\xi^l}{d\overline{s}} = \overline{a}^\ell, \quad (\overline{a}^\ell \overline{a}^\ell = 1), \]

so that

\[ \xi^l' = \overline{a}^l s + \overline{c}^l, \quad (\overline{a}^l \overline{a}^l = 1). \]

The equations (11.9) represent an homocentric system of \( \Pi \)-geodesic hyperspheres.

From (11.6) and (11.7), it follows that

\[ a^m \frac{d\alpha_m^\ell(\xi^p)}{ds} = 0 \]

or

\[ a^m \alpha_m^\ell(\xi^p) + k^l = 0, \quad (k^l = \text{const.}) \]

as a consequence of the equation

\[ \xi^m = a^m s + c^m \]
of the homocentric system of II-geodesic hyperspheres.

If we multiply (11.11) with $s$ and introduce $a^m s = \xi^m$, then it results that
\begin{equation}
\xi^m \alpha^l_n(\xi^p) + k l = 0,
\end{equation}
what tells us that $\xi^m(s) \alpha^l_n(\xi^p(s))$ is a linear function of $s$.

From (11.4) and (11.6), it follows that
\begin{equation}
\frac{d\xi^l}{d\tilde{s}} = \alpha^l_n(\xi^p) a^m
\end{equation}
and from (11.8) that
\begin{equation}
\dot{a}^l = \alpha^l_n(\xi^p) a^m.
\end{equation}
Thus the direction consines ($a^m$) undergo the extended orthogonal transformation (11.15).

The integral of (11.14) is (11.9):
\begin{equation}
\Xi^l = (\alpha^l_n(\xi^p) a^m \tilde{s} + \xi^l = \dot{a}^l \tilde{s} + \xi^l, (\xi^l = \text{const.}).
\end{equation}

From (11.1), we have
\begin{equation}
\Xi^l = \rho(\xi^p) \alpha^l_n(\xi^p) \xi^m - \int \xi^m d\{\rho(\xi^p) \alpha^l_n(\xi^p)\}
\end{equation}
\begin{equation*}
de\Xi^l = \rho(\xi^p) \alpha^l_n(\xi^p) \xi^m + \xi^l(\xi^p).
\end{equation*}

II. Now we come to an extended equiform Laguerre transformation in narrow sense.

When namely
\begin{equation}
\rho = \text{const.},
\end{equation}
(11.17) becomes
\begin{equation}
\Xi^l = \rho \cdot \alpha^l_n(\xi^p) \xi^m + \xi^l_o,
\end{equation}
where $\xi^l_o$ is a constant. For,
\begin{equation}
\int \xi^m d\{\rho \alpha^l_n(\xi^p)\} = \rho \int d\alpha^l_n(\xi^p) \int \xi^m = \rho \int \{d\alpha^l_n(\xi^p) d\xi^m\}
\end{equation}
\begin{equation*}
= -\xi^l_o = \text{const.}
\end{equation*}
by (11.7).

When conversely
\begin{equation}
\int \xi^m d\{\rho(\xi^p) \alpha^l_n(\xi^p)\} = \int d\{\rho(\xi^p) \alpha^l_n(\xi^p)\} \int \xi^m
\end{equation}
\begin{equation*}
= \int \{d\{\rho(\xi^p) \alpha^l_n(\xi^p)\} d\xi^m\} = -\xi^l_o = \text{const.},
\end{equation*}
we have

\begin{equation}
(11.20) \quad \int \rho (\xi^p) \alpha^l_n (\xi^p) \, d\xi^m = 0
\end{equation}

or

\begin{equation}
(11.21) \quad \frac{d\rho}{\rho} = - \frac{\alpha^l_n (\xi^p) d\xi^m}{\alpha^l_n (\xi^p) d\xi^m}.
\end{equation}

Now by (11.5),

\begin{equation}
(11.22) \quad \frac{d^2 \xi^l}{ds^2} = 0 \text{ gives }
\end{equation}

\begin{equation}
\frac{d\alpha^l_n (\xi^p) ds}{ds} = \alpha^l_n (\xi^p) \frac{d^2 \xi^m}{ds^2}.
\end{equation}

Hence (11.21) becomes

\begin{equation}
(11.23) \quad \frac{d\rho}{\rho} = \frac{\alpha^l_n (\xi^p) d^2 \xi^m}{\alpha^l_n (\xi^p) d\xi^m} ds,
\end{equation}

whence follows:

\begin{equation}
(11.24) \quad \frac{d\rho}{\rho} = \frac{\Omega_l^q (\xi^p) \alpha^l_n (\xi^p) d\xi^m}{\Omega_l^q (\xi^p) \alpha^l_n (\xi^p) d\xi^m} ds = \frac{d\xi^q}{d\xi^q},
\end{equation}

where

\begin{equation}
(11.25) \quad \Omega_l^q (\xi^p) \alpha^l_n (\xi^p) = \delta^q_m, \quad \Omega_l^q (\xi^p) \alpha^m_n (\xi^p) = \delta^q_l.
\end{equation}

Integrating (11.24), we obtain

\begin{equation}
\rho \frac{d\xi^q}{ds} = 1 | c^q = \text{const.}
\end{equation}

or

\begin{equation}
(11.26) \quad d\xi^q = c^q \rho ds = c^q ds.
\end{equation}

Now (11.6) tells us:

\begin{equation}
(11.27) \quad d\xi^q = a^q ds.
\end{equation}

Thus (11.26) and (11.27) give

\begin{equation}
(11.28) \quad c^q \rho = a^q.
\end{equation}

Hence \( \rho \) must be a constant.

Thus we have (11.19), when and only when \( \rho = \text{const.} \).

The group consisting of the transformations of the type

\begin{equation}
(11.17)
\end{equation}

\begin{equation}
(11.19)
\end{equation}
will be called the *extended equiform Laguerre transformation group* in the
wider
sense.

**Theories and Theorems.** *The theories and the theorems of the ordinary equiform
Laguerre geometry are all invariant in the extended equiform Laguerre geometry.*

§ 3. **Extended Lie Geometry.**

**12. Extended Lie Geometry.** In

Art. 2, | Art. 4,
---|---
we have had the II-geodesic \((n+2)\)-hyperspherical hypersphere coordinates

\[(2.1)\]  
\[(4.5)\]

fulfilling the identity

\[(2.2)\]  
\[(4.6)\].

Now we renormalize the coordinates

\[(2.1)\]  
\[(4.5)\]  
\[(4.5)\]

as follows:

\[\eta_{i} = \alpha_{i},\]  
\[\eta_{m} = \alpha_{m},\]  
\[\zeta_{n+1} = i \cos \frac{r}{k},\]  
\[\zeta_{m} = \alpha_{m},\]  
\[\zeta_{n+2} = i \cos \phi,\]  
\[\zeta_{n+3} = i \sin \frac{r}{k},\]

(i = 1, 2, ..., n),  
(m = 1, 2, ..., n + 1),

and set

\[\eta_{n+3} = \epsilon R,\]  
\[\eta_{n+3} = i \sin \frac{r}{k},\]  
\[\zeta_{n+3} = i \cos \phi,\]  

the identity

\[(12.2)\]  
\[(4.6)\]  
\[(4.6)\]

becomes

\[(12.3)\]  
[\[\eta_{i} \eta_{i} + \eta_{n+1} \eta_{n+1} + \eta_{n+2} \eta_{n+2} + \eta_{n+3} \eta_{n+3} = 0.\]  
[\[\zeta_{i} \zeta_{i} + \zeta_{n+1} \zeta_{n+1} + \zeta_{n+2} \zeta_{n+2} + \zeta_{n+3} \zeta_{n+3} = 0.\]
Comparing the two sides, we see that

\[(12.4)\quad \eta_m - \zeta_m, \quad \eta_{n+2} - \zeta_{n+3}, \quad \eta_{n+3} = \zeta_{n+2}\]

These coordinates correspond to a doubly oriented II-geodesic hypersphere.

\[(12.1)\] accompanied by \[(12.4)\] will be called the II-geodesic \((n+3)\)-hyperspherical hypersphere coordinates.

An equation of the form

\[(12.5)\quad A^m \eta_m + A^{n+2} \eta_{n+2} + A^{n+3} \eta_{n+3} = 0, \quad B^m \zeta_m + B^{n+2} \zeta_{n+2} + B^{n+3} \zeta_{n+3} = 0,\]

accompanied by the condition

\[(12.6)\quad A^m A^m + A^{n+2} A^{n+2} + A^{n+3} A^{n+3} = 1, \quad B^m B^m + B^{n+2} B^{n+2} + B^{n+3} B^{n+3} = 1,\]

represents a linear II-geodesic hypersphere hypercomplex, which is defined by the totality of doubly oriented II-geodesic hyperspheres, which are equally inclined to

\[\text{II-geodesic tangentially distant from a fixed doubly oriented II-geodesic hypersphere:}\]

\[\star \quad A^m, A^{n+2} \quad | \quad B^m, B^{n+3}.\]

Indeed

\[(12.7)\quad \cos(A, \eta) = \frac{(A_\eta)_{n+2}}{\sqrt{(AA)_{n+2}}} , \quad \cos(B, \zeta) = \frac{(B_\zeta)_{n+1} + B^{n+3} \zeta_{n+3}}{\sqrt{(BB)_{n+1} + (B^{n+3})^2}} .\]

where

\[(A_\eta)_k = A^1 \eta_1 + \cdots \cdots + \eta_{k}, \quad (B_\zeta)_k = B^1 \zeta_1 + \cdots \cdots + \zeta_{k},\]

and thus

\[(12.8)\quad \sin^2 \frac{(A, \eta)}{2} = \frac{(A_\eta)_{n+2} + i \sqrt{(AA)_{n+2}} \eta_{n+3}}{2i \sqrt{(AA)_{n+2}} \eta_{n+3}}, \quad \sin^2 \frac{(B, \zeta)}{2k} = \frac{(B_\zeta)_{n+1} + B^{n+3} \zeta_{n+1} + i \sqrt{(BB)_{n+1} + (B^{n+3})^2} \zeta_{n+1}}{2i \sqrt{(BB)_{n+1} + (B^{n+3})^2} \zeta_{n+1}} .\]

Thus the doubly oriented II-geodesic hypersphere

\[(\eta^\alpha), \quad (\zeta^\alpha), \quad (\alpha = 1, 2, \ldots, n+2)\]

is equally inclined to

\[\text{II-geodesic tangentially distant from}\]
the fixed doubly oriented II-geodesic hypersphere

\[(A^\alpha), \quad (B^\alpha),\]

which will be called the fundamental doubly oriented II-geodesic hypersphere.

Under the linear transformations of the form

\[
(12.9) \quad \tilde{A}^\alpha = A^\alpha A^\beta A^\gamma, \quad \tilde{B}^\alpha = B^\alpha B^\beta B^\gamma,
\]

where

\[
(A_\beta^\cdot), \quad (B_\beta^\cdot)
\]
is an orthogonal matrix such that

\[(12.10) \quad |A_\beta^\cdot| = 1, \quad |B_\beta^\cdot| = 1,
\]

the identities (12.3), (12.6) and (12.7) are invariant. The totality of such transformations forms a \((n+3)^2-(n+3)-n+3\) -parametric group, which will be called extended Lie's \(n(n+2)\) -parametric group and the geometry belonging to it the extended Lie geometry (or the extended Lie's higher II-geodesic hypersphere geometry).

The condition for that the doubly oriented II-geodesic hyperspheres

\[\eta, \quad \overline{\eta}, \quad \zeta, \quad \overline{\zeta}\]

may touch is given by

\[(12.11) \quad (\eta \overline{\eta})_{n+3} = 0, \quad (\zeta \overline{\zeta})_{n+3} = 0.
\]

For,

\[\cos (\eta, \overline{\eta}) = \frac{(\eta \overline{\eta})_{n+2}}{\sqrt{(\eta \overline{\eta})_{n+2}} \sqrt{\overline{(\eta \overline{\eta})_{n+2}}}} = \epsilon, \quad (\epsilon = \pm 1),\]

and

\[\pm i \sqrt{(\eta \overline{\eta})_{n+2}} = \eta_{n+3} \quad \epsilon i \sqrt{\overline{(\eta \overline{\eta})_{n+2}}} = \overline{\eta}_{n+3}, \quad (\epsilon = \pm 1).
\]

**Fundamental Invariant of Two Linear II-geodesic Hypersphere Hyperscomplexes** \((A) \text{ and (B)}\) is given by

\[(12.12) \quad I_{AB}^{def} = \frac{(AB)_{n+3}}{\sqrt{\sqrt{AA_{n+3}} \sqrt{BB_{n+3}}}}.
\]

The quantity

\[W(AB) \quad \text{and} \quad D(AB)
\]
defined by

\[(12.13) \quad I_{AB} = \cos W(AB) \quad \text{and} \quad I_{AB} = \cos \frac{D(AB)}{k}.
\]
will be called
\( H \)-angle. \hspace{1cm} \text{II-geodesic } H \text{-distance.}

It has the following interpretation:

\[
(12.14) \quad I_{AB} = \frac{\cos \phi_{AB} - \cos \phi_A \cos \phi_B}{\sin \phi_A \sin \phi_B}, \quad I_{AB} = \frac{\cos \frac{S_{AB}}{k} - \cos \frac{S_A}{k} \cos \frac{S_B}{k}}{\sin \frac{S_A}{k} \sin \frac{S_B}{k}},
\]

where
\( \phi_A \quad \phi_B \quad S_A \quad S_B \)
is the fundamental angle \hspace{1cm} \text{II-geodesic tangential distance}
considered on (12.8) of the linear II-geodesic hypersphere hypercomplexes

\[
A \quad B \\
A \quad B
\]

and
\( \psi_{AB} \quad S_{AB} \)
the angle \hspace{1cm} \text{the II-geodesic tangential distance}
between the fundamental doubly oriented II-geodesic hyperspheres (A) and (B).

**Proof.** \( \cos \phi_A = \frac{(A^n)_{n+2}}{\sqrt{(AA)_{n+2}} \sqrt{(AA)_{n+2}}} = \frac{-A^{n+3}}{i \sqrt{(AA)_{n+2}}} \), etc.,

\( \sin \phi_A = \frac{\sqrt{(AA)_{n+3}}}{\sqrt{(AA)_{n+2}}} \), etc.,

\( \cos \phi_{AB} = \frac{(AB)_{n+2}}{\sqrt{(AA)_{n+2}} \sqrt{(BB)_{n+2}}} \).

\[
\frac{\cos \phi_{AB} - \cos \phi_A \cos \phi_B}{\sin \phi_A \sin \phi_B} = \left( \frac{A^{n+2}}{i \sqrt{(AA)_{n+2}}} \frac{B_{n+2}}{i \sqrt{(BB)_{n+2}}} - \frac{(AB)_{n+2}}{\sqrt{(AA)_{n+2}} \sqrt{(BB)_{n+2}}} \right)
\]

\[
= \left( \frac{\sqrt{(AA)_{n+3}} \sqrt{(BB)_{n+3}}}{\sqrt{(AA)_{n+2}} \sqrt{(BB)_{n+2}}} \right) - \frac{(AB)_{n+3}}{\sqrt{(AA)_{n+2}} \sqrt{(BB)_{n+2}}} = -I_{AB}.
\]

When \( I_{AB} = 0 \), the two linear II-geodesic hypersphere hypercomplexes (A) and (B) are said to be in involution.
13. **Some Further Formulas concerning the Extended Lie Transformations.** It is known that the following correspondence holds:

<table>
<thead>
<tr>
<th>$(n+1)$-dimensional Conformal Space</th>
<th>$n$-dimensional Lie Space</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td></td>
</tr>
</tbody>
</table>

Now when $A_j(A')$ reduce to constants, the extended Lie transformations reduce to the ordinary Lie transformations. Correspondingly we obtain the following correspondence:

<table>
<thead>
<tr>
<th>Extended $(n+1)$-dimensional Conformal Space</th>
<th>Extended $n$-dimensional Lie Space</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td></td>
</tr>
</tbody>
</table>

In the following lines, we will deduce some further formulas concerning the extended Lie group.

**1.** Equations of the type

(13.1) $A^{**}=A^\alpha \cos \theta + C^\alpha \sin \theta,$

(13.2) $A^\alpha A^\alpha = C^\alpha C^\alpha = 1\), $A^\alpha C^\alpha = 0,$

($\theta$ = parameter)

represent a **pencil of linear II-geodesic hypersphere hypercomplexes**, which is analogous to the coaxal system

of doubly oriented hyperspheres.

From (13.1), (13.2), we see that

(13.3) $d\theta^2 = dA^\alpha d\overline{A}^\alpha = dA^\alpha dA^\alpha$

$k^2 d\sigma^2 = dB^\alpha dB^\alpha$

is an invariant under the extended Lie transformation (12.9).

**2.** The $A^\alpha$ of the pencil (13.1) are solutions of

(13.4) $\frac{d^2 A^\alpha}{d\theta^2} + A^\alpha = 0.$

$\frac{d^2 B^\alpha}{d\sigma^2} + B^\alpha = 0.$

The extended Lie transformation (12.9), where

$A_j(A')$ | $B_j(B')$

is an orthogonal matrix with unit determinant, is accompanied by the condition

(13.5) $d\overline{A}^\alpha = A_j(A') dA^\alpha$ | $d\overline{B}^\alpha = B_j(B') dB^\alpha$
within the $\alpha$-th pencil (13.1) owing to (13.3).

By (13.5), we have

\begin{align*}
(13.6) \quad \frac{d\bar{A}^\alpha}{d\theta} &= \bar{A}^\alpha_r (A^r) \frac{dA^\beta}{d\theta}, \\
(13.7) \quad \frac{d^2 \bar{A}^\alpha}{d\theta^2} &= \frac{d}{d\theta} \bar{A}^\alpha_r (A^r) \frac{dA^\beta}{d\theta} + \bar{A}^\alpha_r (A^r) \frac{d^2 A^\beta}{d\theta^2},
\end{align*}

so that

\begin{align*}
(13.8) \quad \frac{d^2 \bar{A}^\alpha}{d\theta^2} + \bar{A}^\alpha &= \frac{d}{d\theta} \bar{A}^\alpha_r (A^r) \frac{dA^\beta}{d\theta} + \bar{A}^\alpha_r (A^r) \left( \frac{d^2 A^\beta}{d\theta^2} + A^\beta \right).
\end{align*}

In order that the pencil (13.4) may be transformed into the pencil

\begin{align*}
\frac{d^2 \bar{A}^\alpha}{d\sigma^2} + \bar{A}^\alpha &= 0, \\
\frac{d^2 \bar{B}^\alpha}{d\sigma^2} + \bar{B}^\alpha &= 0,
\end{align*}

it is necessary and sufficient that

\begin{align*}
(13.9) \quad \frac{d\bar{A}^\alpha(A^r)}{d\theta} \frac{dA^\beta}{d\theta} &= 0, \\
(13.10) \quad \frac{d^2 \bar{A}^\alpha}{d\sigma^2} &= \bar{A}^\alpha_r (A^r) \left( \frac{d^2 A^\beta}{d\sigma^2} + A^\beta \right), \\
(13.11) \quad \frac{d^2 \bar{B}^\alpha}{d\sigma^2} &= \bar{B}^\alpha_r (B^r) \left( \frac{d^2 B^\beta}{d\sigma^2} + k^2 B^\beta \right).
\end{align*}

Within the pencil (13.1). Thus we have

\begin{align*}
(13.12) \quad \frac{d\bar{A}^\alpha}{d\theta} &= \bar{A}^\alpha_r (A^r) \frac{dA^\beta}{d\theta}, \\
(13.13) \quad \frac{d\bar{B}^\alpha}{d\sigma} &= \bar{B}^\alpha_r (B^r) \frac{dB^\beta}{d\sigma}.
\end{align*}

From (12.9), we obtain

\begin{align*}
(13.6) \text{ and } (13.12) \text{ give } \\
&\quad \frac{d\bar{A}^\alpha(A^r)}{d\theta} \frac{dA^\beta}{d\theta} = 0
\end{align*}

and

\begin{align*}
(13.10) \text{ and } (13.12) \text{ give } \\
&\quad \frac{d\bar{B}^\alpha(A^r)}{d\sigma} \frac{dB^\beta}{d\sigma} = 0.
\end{align*}
within the pencil (13.1).

If we introduce

\[ \Omega^\alpha_\gamma(A') \quad \Omega^\alpha_\gamma(B') \]

by the demand

\[
\begin{align*}
\Omega^\gamma_\delta(A') A^\gamma_\delta(A') &= \delta^\gamma_\delta, \\
\Omega^\gamma_\delta(B') A^\gamma_\delta(B') &= \delta^\gamma_\delta
\end{align*}
\]

and set

\[
\begin{align*}
\Lambda^\nu_\mu &= \frac{\partial A^\nu_\mu}{\partial A^\mu} \\
\Lambda^\nu_\mu &= \frac{\partial B^\nu_\mu}{\partial B^\mu}
\end{align*}
\]

then

\[
\frac{d^2 A^\nu}{d\theta^2} = \frac{d}{d\theta} \left\{ A^\nu_\alpha(A') \frac{d A^\beta}{d\theta} \right\} = A^\nu_\alpha(A') \left\{ \frac{d^2 A^\beta}{d\sigma^2} + \frac{\partial A^\beta}{\partial \sigma} \frac{d^2 A^\gamma}{d\sigma^2} \right\}
\]

\[
\begin{align*}
+ A^\nu_\alpha \frac{\partial A^\beta}{\partial \theta} \frac{d A^\beta}{d\theta}
\end{align*}
\]

so that

\[
\begin{align*}
\frac{d}{d\theta} A^\nu_\alpha(A') \frac{d A^\beta}{d\theta} &= 0 \\
\frac{d}{d\sigma} B^\nu_\beta(B') \frac{d B^\beta}{d\sigma} &= 0
\end{align*}
\]

within the pencil.


14. Parabolic Lie Geometry. The Laguerre-geometrical coordinates of an oriented hypersphere with center \((\xi^i), (i=1, 2, \cdots, n)\) and with radius \(r\) are \((\xi^i, \xi^{n+1})\), where \(\xi^{n+1} = i \cdot r\).

The square of the common tangential segment of two oriented hyperspheres
(ξ) and (α) is given by

\[(\xi^i - a^i)(\xi^i - a^i) + (\xi^{n+1} - a^{n+1})^2.\]

The linear hypersphere hypercomplex is the totality of the oriented hyperspheres (ξ, ξ^{n+1}), whose common tangential segment from inclination to a fixed oriented hypersphere (α, α^{n+1}) is constant:

\[R \alpha \]

and is given by

\[(14.2) \quad f(ξ, t) \equiv (ξ^i - a^i)(ξ^i - a^i) + (ξ^{n+1} - a^{n+1})^2 - R^2 = 0.\]

If we put

\[(14.3) \quad \begin{cases} u_i \overset{\text{def}}{=} \frac{1}{2R} \frac{\partial f}{\partial \xi^i} = \frac{ξ^i - a^i}{R}, \\ u_{n+1} \overset{\text{def}}{=} \frac{1}{2R} \frac{\partial f}{\partial ξ^{n+1}} = \frac{ξ^{n+1} - a^{n+1}}{R}, \end{cases}\]

then we have

\[(14.4) \quad u_i ξ^i + u_{n+1} ξ^{n+1} - p - R = 0,\]

where

\[(14.5) \quad p = u_i a^i + u_{n+1} a^{n+1},\]

\[(14.6) \quad u_i u_i + u_{n+1} u_{n+1} = 1.\]

The equation (14.4) accompanied by (14.5) and (14.6) is the tangential equation of the linear hypercomplex (14.2) of oriented hyperspheres (ξ, ξ^{n+1}).

If we put

\[(14.7) \quad u_i = u_i, \quad u_{n+1} = u_{n+1}, \quad u_{n+2} = i, \quad ξ^{n+2} = iR,\]

then (14.4) becomes

\[(14.9) \quad u_i ξ^i - p = 0, \quad (l = 1, 2, \ldots, n+2),\]

where

\[(14.10) \quad u_i u_i = 0.\]

The system (ξ, 1) constitute the coordinates of the linear hypercomplex (14.4) of oriented hyperspheres and obeys the same law as the (n+2)-dimensional Cartesian coordinates.

The system (u_i, p) constitutes the coordinates of the pair of oriented hyperplanes "tangent" to the linear hypercomplex (14.9) of the (n+2)-dimensional minimal (in the sense of (14.10)!) hyperplanes in the (n+2)-dimensional Cartesian space.
The geometry of (14.9) in this sense was called \textit{parabolic Lie geometry} ([13]). It belongs to the "Erlanger Programm" of F. Klein, the group parameters being constants.

\section{15. Extended Parabolic Lie Geometry.} Since the law of the parabolic Lie geometry is in abstract sense the same as that of the \((n+2)\)-dimensional Euclidean geometry, there exists a more general geometry, which obeys in abstract sense the same law as that of the extended Euclidean geometry. The geometry so extended will be called the \textit{extended parabolic Lie geometry}.

Given an \textit{orthogonal matrix} \((a_{m}^{l}(\xi^{p}))\), \((l, m, p, \ldots = 1, 2, \ldots, n+2)\) with unit determinant

\begin{equation}
|a_{m}^{l}(\xi^{p})|=1,
\end{equation}

where \((\xi^{l}), l = 1, 2, \ldots, n+2\) are the parabolic Lie coordinates spoken of in the last Art. of the linear hypercomplex, we set

\begin{equation}
\omega^{l}= a_{m}^{l}(\xi^{p})d\xi^{m},
\end{equation}

which is anholonomic in general.

A straightforward calculation shows that

\begin{equation}
\frac{d}{ds} \frac{\omega^{l}}{ds} = a_{m}^{l} \left( \frac{d^{2}\xi^{m}}{ds^{2}} + \Lambda_{u}^{m} \frac{d\xi^{h}}{ds} \frac{d\xi^{k}}{ds} \right),
\end{equation}

where \(ds\) is the "common tangential" segment between two consecutive linear hypercomplexes \(\xi(s)\) and \(\xi(s+ds)\):

\begin{equation}
ds^{2} = d\xi^{l}d\xi^{l}\n\end{equation}

and

\begin{equation}
\Lambda_{u}^{m} = \Omega^{r}(\xi^{p}) \frac{\partial a_{h}^{l}(\xi^{p})}{\partial\xi^{k}} = a_{h}^{l} \frac{\partial\Omega^{r}(\xi^{p})}{\partial\xi^{k}},
\end{equation}

where \(\Omega^{r}(\xi^{p})\) are defined by

\begin{equation}
\Omega_{h}^{l}(\xi^{p})a_{h}^{l}(\xi^{p}) = \delta_{h}^{h}, \quad \Omega^{r}(\xi^{p}) a_{h}^{l}(\xi^{p}) = \delta_{h}^{l}.
\end{equation}

The differential equation of the type

\begin{equation}
\frac{d^{2}\xi^{m}}{ds^{2}} + \Lambda_{u}^{m} \frac{d\xi^{h}}{ds} \frac{d\xi^{k}}{ds} = 0
\end{equation}

has been well studied. We will call the solution of \textit{homocentric system of the linear hypercomplexes of the II.-geodesic hyperspheres}. Formerly, only local coordinates of
\( \xi^m(s) \) had been known as paths of teleparallelism. Now (15.3) and (15.4) tells us that

\[
(15.8) \quad \frac{d}{ds} \frac{\omega^l}{ds} = 0 \iff \frac{d^2 \xi^m}{ds^2} + \Lambda_n^u \frac{d \xi^k}{ds} = 0
\]

and the differential equation

\[
(15.9) \quad \frac{d}{ds} \frac{\omega^l}{ds} = 0
\]

may be integrated readily:

\[
(15.10) \quad \omega^l = a^l ds, \quad (a^l = \text{const.}),
\]

where

\[
(15.11) \quad a^l a^l = 1
\]

owing to

\[
(15.12) \quad ds^2 = d\xi^l d\xi^l = \omega^l \omega^l,
\]

which follows from (15.2) and (15.4).

We set

\[
(15.13) \quad d\xi^l \overset{\text{def}}{=} a^l ds = \omega^l,
\]

what is admissible, notwithstanding \( \omega^l \) is anholonomic in general.

Integrating (15.13) once more, we have

\[
(15.14) \quad \xi^l = a^l t + c^l, \quad (c^l = \text{const.}).
\]

The definition of \( \xi^l \) by (15.13) means that we adopt the \( l \)-th "homocentric system" as a generalization of coordinate axis.

The expression (15.14) shows us that the "homocentric system" behave as for "meet" and "join" like straight lines just as ordinary homocentric systems of oriented hyperspheres do.

Now (15.2) may be rewritten:

\[
(15.15) \quad d\xi^l = a^l (\xi^m) d\xi^m = a^l ds.
\]

This may be integrated once more as (15.14):

\[
(15.16) \quad \xi^l = \int a^l (\xi^m) d\xi^m = a^l (\xi^m) + a^l,
\]

where

\[
(15.17) \quad a^l \overset{\text{def}}{=} -\int \xi^m da^m (\xi^p).
\]
Now we have

\[ \frac{d^2 \xi^l}{ds^2} = a_n^l(\xi^p) \left( \frac{d^2 \xi^m}{ds^2} + \Lambda_{u}^{n} \frac{d \xi^h}{ds} \frac{d \xi^k}{ds} \right) \]

quite as in the case of (10.18), where \( l, p, h, k = 1, 2, \ldots, n+2 \) and for ordinary "homocentric system", we have

\[ \frac{d^2 \xi^m}{ds^2} = 0, \]

while

\[ \frac{d^2 \xi^l}{ds^2} = \frac{d}{ds} \frac{\omega^l}{ds} = 0 \]

along the "homocentric system" (15.7). Hence along the "homocentric system" (15.7) we must have

\[ \Lambda_{\mu}^{n} \frac{d \xi^h}{ds} \frac{d \xi^k}{ds} = 0. \]

Now, quite as in the case of (10.22), we have

\[ \frac{d}{ds} a_n^l(\xi^p) \frac{d \xi^m}{ds} = a_n^l(\xi^p) \Lambda_{u}^{n} \frac{d \xi^h}{ds} \frac{d \xi^k}{ds} = 0 \]

along the "homocentric system" (15.7).

The same result may be deduced as follows.

If we differentiate (15.15):

\[ \frac{d \xi^l}{ds} = a_n^l(\xi^p) \frac{d \xi^m}{ds} = a^l, \]

then we have

\[ \frac{d^2 \xi^l}{ds^2} = \frac{d a_n^l(\xi^p)}{ds} \frac{d \xi^m}{ds} + a_n^l(\xi^p) \frac{d^2 \xi^m}{ds^2}, \]

whence follows (15.22).

Now (15.17) becomes

\[ -a_0^l = \int da_n^l(\xi^p) d \xi^m = \iint \left\{ da_n^l(\xi^p) d \xi^m \right\} = 0 \]

by (15.22), the condition for that the repeated integral may be converted into the double integral being evidently fulfilled. Hence
By virtue of (15.16), any transformation of the type

\[ \xi^l = a^l s + c^l, \quad (a^l a^l = 1) \]

is transformed into an "homocentric system" of II-geodesic hypercomplexes

\[ \dot{\xi}^l = \dot{a} s + \dot{c}^l, \quad (a\dot{a} t = 1). \]

The totality of the extended parabolic Lie transformations (15.24) forms a group ((30)Appendix), which we will call the extended parabolic Lie group and the extended parabolic Lie geometry belongs to it.

As in the case of (13.13), we can deduce

\[ da_l^m(\xi^p)\xi^m = 0. \]

The null-differential form

\[ d\xi_i d\xi_l = 0, \quad (l = 1, 2, \ldots, n+2) \]

i.e., the relation

\[ dR^2 = dr^2 - d\xi^i d\xi_i, \quad (i = 1, 2, \ldots, n) \]

is invariant under the extended parabolic Lie group. Thereby \( r \) is the II-geodesic radius of the oriented II-geodesic hypersphere as an element of the linear hypercomplex and \( R \) the generalized II-geodesic radius of the linear hypercomplex.

The common "tangential II-geodesic distance" is an invariant under the extended parabolic Lie group.

For, if we consider two II-geodesic hypersphere hypercomplexes \((\xi^l)\) and \((\overline{\eta}^l)\) corresponding to \( a^l_m(\xi^p) \), then setting

\[ a^l_b = \overline{\eta}^l - b^l(\xi^p), \]

we have e.g.

\[ \xi^l - a^l_b = \xi^l - \partial^i_l a^o_b = a^l_m(\xi^p)\xi^m \quad \text{i.e.} \quad \xi^l - a^l_m(\xi^p)\Omega^m_l(\xi^p)\eta^o = a^l_m(\xi^p)\xi^m, \]

\[ \xi^l - \partial^i_l \xi^q = a^l_m(\xi^p) \left\{ \xi^m - \Omega^m_l(\xi^p)\dot{\xi}^q(\xi^p) \right\} \]
or
\[ \xi^l - \eta^l = a^l_m (\xi^p \xi^m - \eta^m), \quad (\eta^m = \Omega^m_0 b^m_0 (\xi^p)), \]
so that
\[ (\xi^l - \eta^l) (\xi^l - \eta^l) = (\xi^m - \eta^m) (\xi^m - \eta^m), \]
where
\[ \eta^m = \Omega^m_0 (\xi^p) b^m_0 (\xi^p) = \Omega^m_0 (\xi^p) (\eta^p - a^m_0), \]

\[ \eta^q = a^q_m (\xi^p) \eta^m + a^q_0. \]

Any linear hypercomplex of linear II-geodesic hypersphere hypercomplexes
\[ \bar{u}_l \xi^l + \bar{u}_{n+3} = 0, \quad (\bar{u}_l \bar{u}_l = 1) \]
is transformed by an extended parabolic Lie transformation (15.24) into a linear hypercomplex of linear II-geodesic hypersphere hypercomplexes
\[ \bar{u}_l \xi^l + \bar{u}_{n+3} = 0, \quad (\bar{u}_l \bar{u}_l = 1), \]
where
\[ u^l_0 \Omega^l_0 \bar{u}_l = \bar{u}_n + (\bar{u}_l \xi^l), \]
the \((\Omega^l_0 (\xi^p))\) being an orthogonal matrix with unit determinant.

The extended parabolic Lie angle between two linear hypercomplexes of linear II-geodesic hypersphere hypercomplexes \((u_0)\) and \((u_0')\) is given by
\[ \cos \Theta = \frac{u_0 u_0'}{\sqrt{u_0 u_0} \sqrt{u_0' u_0'}} \]
and is invariant under the extended parabolic Lie group. For, e.g. we have
\[ \bar{a}_l' \bar{a}_l' = \bar{a}_l \delta^l_k \bar{a}_k' = \Omega_0^l u_0 \left\{ \Omega_0^l a^k_h \right\} \bar{a}_k' \]
\[ = \delta^l_k u_0 a^k_h \bar{a}_k' = u_h u_k', \]
where
\[ u^l_h = a^l_h \bar{a}_k'. \]

**Theories and Theorems.** The theories and theorems of the ordinary parabolic Lie geometry are all invariant in the extended parabolic Lie geometry.
§ 5. Deduction of the Extended Conformal Geometry, the Extended Equiform Laguerre Geometry and the Extended Laguerre Geometry from the Extended Lie Geometry.


If we adjoin a linear II-geodesic hypersphere hypercomplex to the space of the extended Lie geometry, then an extended conformal space arises. (Thus an extended conformal space is a special linear II-geodesic hypersphere hypercomplex.)

If the linear II-geodesic hypersphere hypercomplex, which is adjoined, be

\[
(0, 0, \ldots, 0, 1),
\]

then the totality of the linear II-geodesic hypersphere hypercomplexes \(A^a\), which are in involution with \((0, 0, \ldots, 0, 1)\), satisfies the condition

\[
0.A^1 + 0.A^2 + \cdots + 0.A^{n+2} + 1.A^{n+3} = 0
\]

i.e.,

\[
A^{n+3} = 0.
\]

As a consequence, we have

\[
A^aA^a = A^1A^1 + \cdots + A^{n+2}A^{n+2} = 1, \quad (\alpha = 1, 2, \ldots, n+3),
\]

\[
\eta_aA^a = \eta_1A^1 + \cdots + \eta_{n+2}A^{n+2} = 0.
\]

If we write \(\eta_m\) in place of \(\eta_m\),

\[
\eta_m : (i\eta_{n+3}), \quad (m = 1, 2, \ldots, n+2),
\]

then, from \(\eta_a\eta_a = 0\), we obtain

\[
\eta_m\eta_m = 1, \quad (m = 1, 2, \ldots, n+2).
\]

For \(A^{n+3} = 0\), (12.8) gives

\[
\sin^2 \left( \frac{(A, \eta)}{2} \right) = \frac{1}{2},
\]

so that

\[
\angle (A, \eta) = \frac{\pi}{2}.
\]

Thus the \(A^a\) reduces to a doubly oriented II-geodesic hypersphere \((A^mA^m) = 1\) and the doubly oriented II-geodesic hyperspheres as elements of \(A^a\) meet the doubly oriented II-geodesic hyperspheres \(A^m\) orthogonally. Thus an extended conformal geometry is obtained.
If we consider \( (\eta_l) \) thereby to belong to the linear II-geodesic hypersphère hypercomplex \((0, 0, \ldots, 1, 0)\), then
\[
0 \cdot \eta_1 + 0 \cdot \eta_2 + \cdots + 0 \cdot \eta_{n+2} + 1 \cdot \eta_{n+3} = 0,
\]
i.e.
\[
\eta_{n+3} = i \sin \frac{r}{k} = 0.
\]
Thus we are led to consider II-geodesic null-hyperspheres only. So we come to an extended conformal space.

17. Deduction of an Extended Dualconformal Geometry from the Extended Lie Geometry. If we adjoin the linear II-geodesic hypersphere hypercomplex \((0, 0, \ldots, 1, 0)\), then for the linear II-geodesic hypersphere hypercomplex \((A^n)\), we have
\[
A^{n+2} = 0,
\]
so that
\[
\eta_\alpha A^\alpha = \eta_1 A^1 + \cdots + \eta_{n+1} A^{n+1} + \eta_{n+3} A^{n+3} = 0, \quad (\alpha = 1, 2, \ldots, n+3).
\]
If we write \( \eta_m \) in place of
\[
\eta_m: (i\eta_{n+2}), \quad (m = 1, 2, \ldots, n+1, n+3),
\]
then we obtain
\[
\eta_1 \eta_1 + \cdots + \eta_{n+1} \eta_{n+1} + \eta_{n+3} \eta_{n+3} = 1
\]
and (12.8) becomes
\[
\sin^2 \frac{(A, \zeta)}{2k} = \frac{1}{2}
\]
so that
\[
(A, \zeta) = -\frac{\pi}{2} - k.
\]
Thus the doubly oriented II-geodesic hyperspheres \((\eta_l)\) are situated to \((A^1, A^1, \ldots, A^{n+1}, A^{n+1})\) at an orthogonal II-geodesic distance.

In this way we obtain the extended dual conformal space.

If hereby \((\eta_n)\) also belongs to \((0, 0, \ldots, 1, 0)\), then
\[
0 \cdot \eta_1 + 0 \cdot \eta_2 + \cdots + 0 \cdot \eta_{n+1} + 1 \cdot \eta_{n+2} + 0 \cdot \eta_{n+3} = 0
\]
i.e.
\[
\eta_{n+2} = i \sin \phi = i \cos \frac{r}{k} = 0.
\]
Thus we are led to oriented II-geodesic \((n-1)\)-flats only of the extended non-Euclidean space. So we are led to the extended dual conformal space.
18. Deduction of the Extended Equiform Laguerre Geometry from the Extended Lie Geometry. If we adjoin the special linear II-geodesic hypersphere hypercomplex \((0, 0, \ldots, 1, i, 0)\), which is a doubly oriented II-geodesic hypersphere, then for the linear II-geodesic hypersphere hypercomplex \(A^\alpha\), which is in involution with it, we have

\[
0\cdot A^1 + 0\cdot A^2 + \cdots + 0\cdot A^n + 1\cdot A^{n+1} + iA^{n+2} + 0\cdot A^{n+3} = 0
\]

i. e.

\[
A^{n+1} + iA^{n+2} = 0
\]

so that

\[
A^\alpha A^\alpha = A^1 A^1 + A^n A^n + A^{n+3} A^{n+3} \neq 0, \quad (\alpha = 1, 2, \ldots, n+3).
\]

If we normalize \((\eta_\alpha, \eta_\alpha \eta_\alpha = 0)\), as

\[
\begin{cases}
\rho \cdot \eta_m = \alpha_m, & (m = 1, 2, \ldots, n), \\
\rho \cdot \eta_{n+1} = -\frac{1}{2}(\alpha_m \alpha_m - R^2 - 1), \\
\rho \cdot \eta_{n+2} = -\frac{i}{2}(\alpha_m \alpha_m - R^2 + 1), \\
\rho \cdot \eta_{n+3} = iR,
\end{cases}
\]

then, since

\[
\rho \cdot (\eta_{n+1} + i\eta_{n+2}) = 1,
\]

taking \(\rho = 1\), we have

\[
\begin{align*}
A^\alpha \eta_\alpha &= A^1 \eta_1 + \cdots + A^n \eta_n + A^{n+3} \eta_{n+3} + A^{n+1} (\eta_{n+1} + i\eta_{n+2}), \quad (\alpha = 1, \ldots, n+3) \\
&= A^1 \eta_1 + \cdots + A^n \eta_n + A^{n+3} \eta_{n+3} + A^{n+1} = 0,
\end{align*}
\]

\[
A^1 A^1 + \cdots + A^n A^n + A^{n+3} A^{n+3} \neq 0.
\]

This is of the same form as the Hesse's normal form of an oriented \(n\)-flat in the \((n+1)\)-dimensional extended Euclidean space. Thus we are led to an extended equiform Laguerre space.

19. Deduction of the Extended Laguerre Geometry from the Extended Lie Geometry. For this purpose, it suffices to replace both (18.2) and (18.6) by

\[
A^1 A^1 + \cdots + A^n A^n + A^{n+3} A^{n+3} = 1.
\]

If we adjoin an oriented II-geodesic hypersphere \((0, 0, \ldots, 1, i, 0)\) to the extended Lie space, then for the totality of the oriented II-geodesic hyperspheres, \(\tau\), we have

\[
\eta_{n+1} + i\eta_{n+2} = 0,
\]

so that

\[
\eta_\alpha \eta_\alpha = \eta_1 \eta_1 + \cdots + \eta_n \eta_n + \eta_{n+3} \eta_{n+3} = 0.
\]

In this case, we have

\[
\eta_\alpha A^\alpha = \eta_m A^m + \eta_{n+1} (A^{n+1} + iA^{n+2}) + \eta_{n+3} A^{n+3} = 0, \quad (m = 1, 2, \ldots, m),
\]

\[
(A^1 A^1 + \cdots + A^n A^n + A^{n+3} A^{n+3} = 1).
\]
For (18.3), we have

$$\eta_{n+3}^{-1}(\eta_{n+1} + i\eta_{n+2}) = \frac{1}{R} = 0.$$  

Thus $\eta$ becomes a II-geodesic $(n-1)$-flat and for the oriented II-geodesic hypersphere, which is adjoined, from the $(n+3)$-th component, we obtain the II-geodesic radius $= 0$.

what shows that the oriented II-geodesic hypersphere reduces to a II-geodesic null-hypersphere. Since $(0, 0, \ldots, 0, 1, i, 0)$ corresponds to a point $(\xi^1 = \xi^2 = \ldots = \xi^n = 0, \xi^{n+1} = -i)$ on the II-geodesic hypersphere

$$\xi^1 \xi^1 + \ldots + \xi^{n+1} \xi^{n+1} = (1)^2,$$

projecting this II-geodesic hypersphere from that point upon the tangent II-geodesic $(n-1)$-flat at $(\xi^1 = \xi^2 = \ldots = \xi^n = 0, \xi^{n+1} = i)$ extended-stereographically, we obtain a point at infinity.

The relation

$$0 . \eta_1 + 0 . \eta_2 + \ldots + 0 . \eta_n + 1 . \eta_{n+1} + i . \eta_{n+2} + 0 . \eta_{n+3} = 0$$

means that we are considering the oriented II-geodesic hyperspheres passing through the point at infinity. If we recognize this fact on the II-geodesic hypersphere

$$\xi^1 \xi^1 + \ldots + \xi^{n+1} \xi^{n+1} = (1)^2,$$

then we see that we are considering the totality of the oriented II-geodesic hyperspheres passing through the point $(0, 0, \ldots, 0, -i)$. Thus we obtain an extended Poincaré-Klein representation of the extended Euclidean space.

In the following lines, we shall show that $\eta$ become the coordinates of the oriented II-geodesic $n$-flat, when $\eta$ becomes an oriented II-geodesic $(n-1)$-flat. We consider an oriented II-geodesic hypersphere with center $(a)$ and with II-geodesic radius $R$ and let the center $(a)$ recede to infinity. For,

$$\rho \cdot \eta_m = \frac{a_m}{R} \quad (m = 1, 2, \ldots, n),$$

$$\rho \cdot \eta_{n+1} = -\frac{1}{2R} (a_m a_m - R^2 - 1),$$

$$\rho \cdot \eta_{n+2} = -\frac{i}{2R} (a_m a_m - R^2 + 1),$$

$$\rho \cdot \eta_{n+3} = \epsilon i, \quad (\rho = 1),$$

let $C(a_m, 0, 0, \ldots, 0)$, $OC = \vec{R}$ (II-geodesic length) and let the direction consines of $\overrightarrow{OC}$ be $\{i\}$. Then
\[
\frac{a_m}{R} = \frac{a_m}{\overline{R}} \left( \frac{\overline{R}}{R} \right) = \overline{l} m^{\overline{R}/R}.
\]

Let \( A \) and \( B \) be the points of intersection of the \( II \)-geodesic curve \( OC \) with the oriented \( II \)-geodesic hypersphere \((a_m, R)\) such that \( OA < OC \). Let the \( II \)-geodesic \( n \)-flat into which \((a_m, R)\) reduces on receding of \( C \) and \( B \) to infinity, be

\[
l_i \xi - p = 0.
\]

Then

\[
\frac{a_m}{R} = \overline{l} m^{\overline{R}/R} \rightarrow l_m.
\]

Hence

\[
\eta_m = \frac{a_m}{R} \rightarrow l_m,
\]

\[
\eta_{n+1} = -\frac{1}{2R} (a_m a_m - R^2 - 1) \rightarrow -\frac{1}{2R} (a_m a_m - R^2)
\]

\[
\rightarrow -\frac{1}{2R} (II \text{- geodesic power of } (0, 0, \ldots, 0)) \rightarrow (-p),
\]

since

\[
\frac{"II \text{- geodesic power"}}{R} = \frac{OA \cdot OB}{R} = \frac{OA(OA + 2R)}{R}
\]

\[
\rightarrow 2OA = 2p.
\]

Thus, in

\[
\eta \eta = \eta_m \eta_m + \eta_{n+1} (\eta^{n+1} + i \eta^{n+2}) + \eta_{n+3} \eta^{n+3} = 0,
\]

the \( \eta \) becomes an oriented \( II \)-geodesic \((n-1)\)-flat, that is to say, we are treating the totality of the oriented \( II \)-geodesic \((n-1)\)-flats \( \eta \) meeting a fixed oriented \( II \)-geodesic hypersphere at a fixed angle.

When, in particular, the linear \( II \)-geodesic hypersphere hypercomplex \((A)\) be an oriented \( II \)-geodesic hypersphere \((A)\), \((A^l A^l = 0)\), and the oriented \( II \)-geodesic hyperspheres \( \eta \) touch it, so that

\[
\sin^2 \left( \frac{\eta_{l} A^l}{2} \right) = -\frac{A^{n+3} + \epsilon i \sqrt{A^l A^l + \ldots + A^{n+2} A^{n+2}}}{2\epsilon \sqrt{A^l A^l + \ldots + A^{n+2} A^{n+2}}},
\]

(\( \eta \)) reduces to \( II \)-geodesic \((n-1)\)-flat as was mentioned above and \((A)\) is so normalized that

\[
A^{n+1} + i A^{n+2} = 1,
\]

in

\[
A^\ast \eta = A_m \eta_m + \eta_{n+1} (A^{n+1} + i A^{n+2}) + \eta_{n+3} A^{n+3} = 0,
\]
we have
\[ \eta_{n+1} + i\eta_{n+2} = 0, \quad \eta_m\eta_m + \eta_{n+3}\eta_{n+3} = 0, \]
so that
\[ \eta_{n+3} = i \sqrt{|l_m|^2} = i. \]
Thus finally we arrive at
\[ \eta_{\alpha} A^\alpha = l_m A^m - p + i \cdot i R = 0. \]
This represents that the tangent II-geodesic \((n-1)\)-fat of an oriented II-geodesic hypersphere \((A^m, R)\) is
\[ \eta_m = l_m, \quad \eta_{n+1} = -p, \quad \eta_{n+3} = i. \]

REFERENCES


[30] T. Takasu, Extended Projective Geometry obtained by Extending the Group Parameters to


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