

SINGULARITIES OF ONE-PARAMETER PEDAL UNFOLDINGS OF
 SPHERICAL PEDAL CURVES

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ABSTRACT. In this paper, we present the concept of one-parameter pedal unfoldings of a pedal curve in the unit sphere S^2 , and we classify their generic singularities with respect to \mathcal{A} -equivalence.

1. INTRODUCTION

Let I be an open interval containing zero, and let S^2 be the unit sphere in Euclidean space \mathbb{R}^3 . A C^∞ map $\mathbf{r} : I \rightarrow S^2$ is called a *spherical unit speed curve* if $\|\frac{d\mathbf{r}}{ds}(s)\|$ is 1 for any $s \in I$. For a given spherical unit speed curve $\mathbf{r} : I \rightarrow S^2$, we put

$$\mathbf{t}(s) = \frac{d\mathbf{r}}{ds}(s), \quad \mathbf{n}(s) = \mathbf{r}(s) \times \mathbf{t}(s),$$

where $\mathbf{r}(s) \times \mathbf{t}(s)$ denotes the vector product of $\mathbf{r}(s)$ and $\mathbf{t}(s)$. The construction clearly shows that the vector $\mathbf{t}(s)$ is perpendicular to the vector $\mathbf{r}(s)$ and that the vector $\mathbf{n}(s)$ is perpendicular to both $\mathbf{r}(s)$ and $\mathbf{t}(s)$. The map $\mathbf{n} : I \rightarrow S^2$ is called the *spherical dual* of \mathbf{r} ; the singularities of spherical dual curves are Legendrian singularities that are relatively well investigated [1, 2, 3, 4, 5, 21].

For a point $P \in S^2$, let E_P denote the set $\{X \in S^2 \mid P \cdot X = 0\}$, where $P \cdot X$ denotes the scalar product of P and X . For a given spherical unit speed curve $\mathbf{r} : I \rightarrow S^2$, consider a point P of $S^2 - \{\pm\mathbf{n}(s) \mid s \in I\}$, where \mathbf{n} is the spherical dual of \mathbf{r} . The *spherical pedal curve relative to the point P* for a given spherical unit speed curve $\mathbf{r} : I \rightarrow S^2$ is a curve obtained by mapping $s \in I$ to the nearest point in $E_{\mathbf{n}(s)}$ from P . The pedal curve relative to P for \mathbf{r} is denoted by $ped_{\mathbf{r},P}$, and the point P is called the *pedal point* of the pedal curve $ped_{\mathbf{r},P}$. Note that all points in $E_{\mathbf{n}(s)}$ are equidistant from $\pm\mathbf{n}(s)$; hence, the point P must lie outside $\{\pm\mathbf{n}(s) \mid s \in I\}$ to satisfy the definition of $ped_{\mathbf{r},P}$. The classification of singularities of spherical pedal curves can be found in literature [17, 18, 19].

Suppose that the location of the pedal point P moves smoothly, depending on one-parameter $\lambda \in J$, where J is an open interval containing zero in \mathbb{R} . In other words, suppose that there exist an open interval J containing zero and a C^∞ immersion $P : J \rightarrow S^2$. Then, the *pedal unfolding of the pedal curve $ped_{\mathbf{r},P(0)}$* can be defined as the map $Un-ped_{\mathbf{r},P} : I \times J \rightarrow S^2 \times J$, given by

$$Un-ped_{\mathbf{r},P}(s, \lambda) = (ped_{\mathbf{r},P(\lambda)}(s), \lambda).$$

Two C^∞ map-germs $f, g : (\mathbb{R}^n, 0) \rightarrow (\mathbb{R}^p, 0)$ are said to be \mathcal{A} -equivalent if there exist germs of C^∞ -diffeomorphisms $h_1 : (\mathbb{R}^n, 0) \rightarrow (\mathbb{R}^n, 0)$ and $h_2 : (\mathbb{R}^p, 0) \rightarrow (\mathbb{R}^p, 0)$ such that $f \circ h_1 = h_2 \circ g$. For a spherical unit speed curve germ $\mathbf{r} : (I, 0) \rightarrow S^2$, we put $\kappa(s) = \mathbf{n}(s) \cdot \mathbf{t}'(s)$, where \mathbf{t}' denotes

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TABLE 1. Normal forms of \mathcal{A} -simple monogerm $(\mathbb{R}^2, 0) \rightarrow (\mathbb{R}^3, 0)$ ([15])

<i>Germ</i>	<i>Name</i>
$f(s, \lambda) = (s, s^2, \lambda)$	<i>Immersion</i>
$f(s, \lambda) = (s^3 + \lambda s, s^2, \lambda)$	<i>Cross-cap</i> (S_0)
$f(s, \lambda) = (s^3 \pm \lambda^{k+1} s, s^2, \lambda), (k \geq 1)$	S_k^\pm
$f(s, \lambda) = (\lambda^2 s \pm s^{2k+1}, s^2, \lambda), (k \geq 2)$	B_k^\pm
$f(s, \lambda) = (\lambda s^3 \pm \lambda^k s, s^2, \lambda), (k \geq 3)$	C_k^\pm
$f(s, \lambda) = (\lambda^3 s + s^5, s^2, \lambda)$	F_4
$f(s, \lambda) = (\lambda s + s^{3k-1}, s^3, \lambda), (k \geq 2)$	H_k

the derivative of \mathbf{t} . Then, the point $\mathbf{r}(0)$ is called the *inflection point* (resp., *ordinary inflection point*) if $\kappa(0) = 0$ holds (resp., $\kappa(0) = 0$ and $\kappa'(0) \neq 0$ hold). For any $k \geq 0$, a C^∞ immersed curve germ $P : (J, 0) \rightarrow S^2$ is said to *have* $(k+1)$ -*point contact with* $\mathbf{r} : (I, 0) \rightarrow S^2$ at $P(0) = \mathbf{r}(0)$ if $P(0) = \mathbf{r}(0)$, $F \circ P(0) = (F \circ P)'(0) = \dots = (F \circ P)^{(k)}(0) = 0$, and $(F \circ P)^{(k+1)}(0) \neq 0$ hold for any neighbourhood U of $\mathbf{r}(0)$ and any non-singular C^∞ function $F : U \rightarrow \mathbb{R}$ such that $F \circ \mathbf{r}(s) = 0$ (for details on $(k+1)$ -point contact, see [5]). It can be clearly seen that a C^∞ immersed curve germ $P : (J, 0) \rightarrow S^2$ has 1-point contact with $\mathbf{r} : (I, 0) \rightarrow S^2$ at $P(0) = \mathbf{r}(0)$ if and only if P and \mathbf{r} are transverse at $P(0) = \mathbf{r}(0)$.

Theorem 1. *Let I, J be open intervals containing $0 \in \mathbb{R}$, and let $\mathbf{r} : I \rightarrow S^2$ be a spherical unit speed curve such that $\mathbf{r}(0)$ is not an inflection point. Furthermore, let $P : J \rightarrow S^2$ be a C^∞ immersion. Then, the following hold:*

- (1) *The germ of pedal unfolding $Un\text{-}ped_{\mathbf{r}, P} : (I \times J, (0, 0)) \rightarrow S^2 \times J$ is immersive if and only if $P(0) \neq \mathbf{r}(0)$.*
- (2) *The germ of pedal unfolding $Un\text{-}ped_{\mathbf{r}, P} : (I \times J, (0, 0)) \rightarrow S^2 \times J$ is \mathcal{A} -equivalent to the cross-cap in Table 1 if and only if $P(0) = \mathbf{r}(0)$ and P, \mathbf{r} are transverse at $P(0) = \mathbf{r}(0)$.*
- (3) *The germ of pedal unfolding $Un\text{-}ped_{\mathbf{r}, P} : (I \times J, (0, 0)) \rightarrow S^2 \times J$ is \mathcal{A} -equivalent to S_k^\pm in Table 1 if and only if $P(0) = \mathbf{r}(0)$ and P has $(k+1)$ -point contact with \mathbf{r} at $0 \in J$ ($k \geq 1$).*
- (4) *The \mathcal{A} -equivalence classes of map-germs B_k^\pm, C_k^\pm, F_4 , and H_k in Table 1 can never be realized as singularities of the pedal unfolding $Un\text{-}ped_{\mathbf{r}, P}$.*
- (5) *The germ of pedal unfolding $Un\text{-}ped_{\mathbf{r}, P} : (I \times J, (0, 0)) \rightarrow S^2 \times J$ is \mathcal{A} -equivalent to the cuspidal edge in Table 2 if and only if $P(0) = \mathbf{r}(0)$ and $(P(J), P(0))$ coincides with $(\mathbf{r}(I), \mathbf{r}(0))$ as set-germs.*

If k is even, then it can be clearly seen that S_k^+ is \mathcal{A} -equivalent to S_k^- [15]. On the other hand, S_k^+ is not \mathcal{A} -equivalent to S_k^- if k is odd. Figure 2 shows that the curvature of \mathbf{r} at zero is greater than the curvature of P at zero if and only if the pedal unfolding $Un\text{-}ped_{\mathbf{r}, P}$ is \mathcal{A} -equivalent to S_k^- . Since S_1^\pm has been investigated independently in [6], it is reasonable to classify the \mathcal{A} -equivalence class of S_1^\pm as *Chen-Matsumoto-Mond singularity*.

Theorem 2. *Let I, J be open intervals containing $0 \in \mathbb{R}$, and let $\mathbf{r} : I \rightarrow S^2$ be a spherical unit speed curve such that $\mathbf{r}(0)$ is an ordinary inflection point. Furthermore, let $P : J \rightarrow S^2$ be a C^∞ immersion. Then, the following hold:*

- (1) *The germ of pedal unfolding $Un\text{-}ped_{\mathbf{r}, P} : (I \times J, (0, 0)) \rightarrow S^2 \times J$ is \mathcal{A} -equivalent to the cuspidal edge in Table 2 if and only if $P(0) \notin E_{\mathbf{r}(0)}$.*

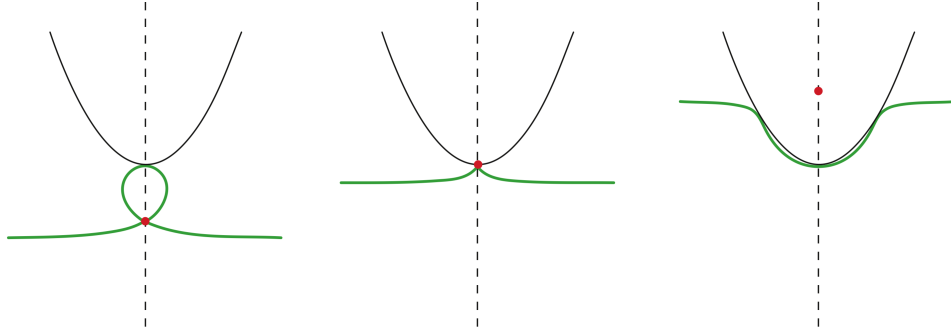


FIGURE 1. Cross-cap. Left: $\lambda = -\varepsilon$, Center: $\lambda = 0$, Right: $\lambda = \varepsilon$.

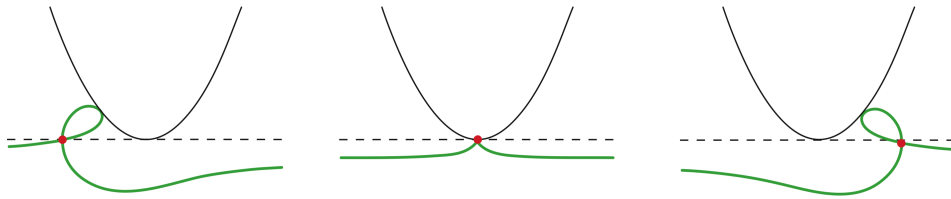


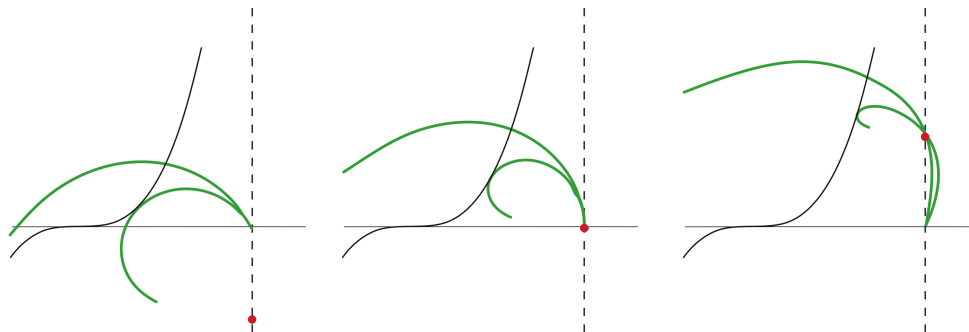
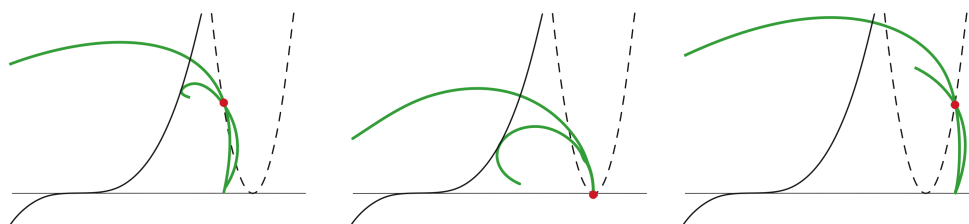
FIGURE 2. S_1^- . Left: $\lambda = -\varepsilon$, Center: $\lambda = 0$, Right: $\lambda = \varepsilon$.

TABLE 2.

<i>Germ</i>	<i>Name</i>
$g(s, \lambda) = (s^3, s^2, \lambda)$	<i>Cuspidal edge</i>
$g_0^+(s, \lambda) = (s^5 + \lambda s^3, s^2, \lambda)$	<i>Cuspidal cross-cap (Cuspidal S_0)</i>
$g_k^\pm(s, \lambda) = (s^5 \pm \lambda^{k+1} s^3, s^2, \lambda), (k \geq 1)$	<i>Cuspidal S_k^\pm</i>

- (2) The germ of pedal unfolding $Un\text{-}ped_{\mathbf{r},P} : (I \times J, (0,0)) \rightarrow S^2 \times J$ is \mathcal{A} -equivalent to the cuspidal cross-cap in Table 2 if and only if $P(0) \in E_{\mathbf{n}(0)} - \{\mathbf{r}(0)\}$ and P is transverse to $E_{\mathbf{n}(0)}$ at $P(0)$.
- (3) The germ of pedal unfolding $Un\text{-}ped_{\mathbf{r},P} : (I \times J, (0,0)) \rightarrow S^2 \times J$ is \mathcal{A} -equivalent to cuspidal S_k^\pm ($k \geq 1$) in Table 2 if and only if $P(0) \in E_{\mathbf{n}(0)} - \{\mathbf{r}(0)\}$ and P has $(k + 1)$ -point contact with $E_{\mathbf{n}(0)}$ ($k \geq 1$).

As in the case of S_k^\pm singularities, it can be clearly seen that cuspidal S_k^+ singularity is \mathcal{A} -equivalent to cuspidal S_k^- singularity if k is even. On the other hand, cuspidal S_k^+ singularity is not \mathcal{A} -equivalent to cuspidal S_k^- singularity if k is odd. Figure 4 shows that for a sufficiently small positive real number ε , there exists a positive real number δ such that the union of tangent lines $\cup_{s \in (-\varepsilon, \varepsilon)} E_{\mathbf{n}(s)}$ contains the images $P((-\delta, \delta))$ if and only if the map-germ $Un\text{-}ped_{\mathbf{r},P} : (I \times J, (0,0)) \rightarrow S^2 \times J$ is \mathcal{A} -equivalent to cuspidal S_k^- singularity. Since map-germ g_0^+ singularity

FIGURE 3. Cuspidal cross-cap. Left: $\lambda = -\varepsilon$, Center: $\lambda = 0$, Right: $\lambda = \varepsilon$.FIGURE 4. Cuspidal S_1^- . Left: $\lambda = -\varepsilon$, Center: $\lambda = 0$, Right: $\lambda = \varepsilon$.

is known as the normal form of the *cuspidal cross-cap* (see [11]), it is reasonable to classify the \mathcal{A} -equivalence class of the map-germ $g_{k,\pm}$ (resp., $g_{1,\pm}$) as *cuspidal S_k^\pm singularity* (resp., *cuspidal Chen-Matsumoto-Mond singularity*).

It can be clearly seen that the cuspidal edge, cuspidal cross-cap, and cuspidal S_k^\pm are not finitely \mathcal{A} -determined (but finitely \mathcal{K} -determined) by the Mather-Gaffney geometric characterization of finite determinacy, even though S_k^\pm singularity is $(k+2)$ - \mathcal{A} -determined [15] (for the definition of finite determinacy and Mather-Gaffney geometric characterization, see [23]). Thus, in order to prove Theorems 1 and 2 in a unified manner, it is difficult to directly use the standard techniques of the finite determinacy theory developed in [8, 9, 10, 13, 14, 15, 20, 23].

On the other hand, Saji succeeded in obtaining simple criteria for Chen-Matsumoto-Mond singularity and cuspidal S_k^\pm -singularities [22]. Although Saji's criteria are useful, the criteria for S_k^\pm singularities ($k \geq 2$) have not been provided by him; therefore, Saji's criteria are not suited to our purpose. In this study, we plan to develop a unified method for proving Theorems 1 and 2; hence, we adopt a recognition criterion for map-germs that appear as singularities of pedal unfoldings. It is important to note that this criterion has already been presented in a suitable form in [15].

The preliminary work required to prove Theorems 1 and 2 is presented in Section 2. Theorems 1 and 2 are proved in Sections 3 and 4, respectively.

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2. PRELIMINARIES

2.1. Spherical pedal curves. Let I , S^2 , and $\mathbf{r} : I \rightarrow S^2$ be an interval containing zero, the unit sphere in \mathbb{R}^3 , and a spherical unit speed curve respectively. Furthermore, let $\mathbf{t} : I \rightarrow S^2$, $\mathbf{n} : I \rightarrow S^2$ be map-germs, as described in Section 1. Then, we have the following Serret-Frenet type formula.

Lemma 2.1 ([17]).

$$\begin{pmatrix} \mathbf{r}'(s) \\ \mathbf{t}'(s) \\ \mathbf{n}'(s) \end{pmatrix} = \begin{pmatrix} 0 & 1 & 0 \\ -1 & 0 & \kappa(s) \\ 0 & -\kappa(s) & 0 \end{pmatrix} \begin{pmatrix} \mathbf{r}(s) \\ \mathbf{t}(s) \\ \mathbf{n}(s) \end{pmatrix}.$$

By Lemma 2.1, the dual curve germ $\mathbf{n} : (I, 0) \rightarrow S^2$ is non-singular at 0 if and only if $\kappa(0) \neq 0$. By using Lemma 2.1 recursively, we obtain the following:

Lemma 2.2. (1) *Suppose that $\kappa(0) \neq 0$. Then, the properties $\mathbf{r}(0) \cdot \mathbf{n}'(0) = 0$, $\mathbf{r}(0) \cdot \mathbf{n}''(0) \neq 0$, and $\mathbf{t}(0) \cdot \mathbf{n}'(0) \neq 0$ hold.*
 (2) *Suppose that $\kappa(0) = 0$ and $\kappa'(0) \neq 0$. Then, the properties $\mathbf{r}(0) \cdot \mathbf{n}'(0) = \mathbf{r}(0) \cdot \mathbf{n}''(0) = 0$, $\mathbf{r}(0) \cdot \mathbf{n}^{(3)}(0) \neq 0$, $\mathbf{t}(0) \cdot \mathbf{n}'(0) = 0$, and $\mathbf{t}(0) \cdot \mathbf{n}''(0) \neq 0$ hold.*

Let P be a point of $S^2 - \{\pm \mathbf{n}(s) \mid s \in I\}$.

Lemma 2.3 ([17]). *The pedal curve of \mathbf{r} relative to the pedal point P is given by the following expression:*

$$ped_{\mathbf{r},P}(s) = \frac{1}{\sqrt{1 - (P \cdot \mathbf{n}(s))^2}} (P - (P \cdot \mathbf{n}(s))\mathbf{n}(s)).$$

Let Ψ_P be the C^∞ map from $S^2 - \{\pm P\}$ to S^2 , given by

$$\Psi_P(X) = \frac{1}{\sqrt{1 - (P \cdot X)^2}} (P - (P \cdot X)X).$$

The map Ψ_P , which has been introduced and used in [17, 18, 19] (the hyperbolic version of Ψ_P has been introduced and investigated independently in [12]), has the following distinctive properties :

- (1) $X \cdot \Psi_P(X) = 0$ for any $X \in S^2 - \{\pm P\}$.
- (2) $\Psi_P(X) \in \mathbb{R}P + \mathbb{R}X$ for any $X \in S^2 - \{\pm P\}$.
- (3) $P \cdot \Psi_P(X) > 0$ for any $X \in S^2 - \{\pm P\}$.

By property 3, $\Psi_P(S^2 - \{\pm P\})$ lies inside the open hemisphere centered at P . By properties 1 and 2, $\Psi_P(E_P) = P$. Let the open hemisphere centered at P be denoted by H_P , and put $B_P = \pi(S^2 - \{\pm P\})$, where $\pi : S^2 \rightarrow P^2(\mathbb{R})$ is the canonical projection. Since $\Psi_P(X) = \Psi_P(-X)$, the map Ψ_P canonically induces the map $\tilde{\Psi}_P : B_P \rightarrow H_P$. Then, by Lemma 2.3, $ped_{\mathbf{r},P}$ is factored into three maps as follows:

$$ped_{\mathbf{r},P}(s) = \tilde{\Psi}_P \circ \pi \circ \mathbf{n}(s).$$

Let $p : B \rightarrow \mathbb{R}^2$ be the blow up centered at the origin in \mathbb{R}^2 .

Lemma 2.4 ([17]). *Let P be a point of S^2 . Then, there exist C^∞ diffeomorphisms $h_1 : B_P \rightarrow B$ and $h_2 : H_P \rightarrow \mathbb{R}^2$ such that the equality $h_2 \circ \tilde{\Psi}_P = p \circ h_1$ holds, and the set $\pi(E_P)$ is mapped to the exceptional set of p by h_1 .*

2.2. Criterion for recognition problem due to Mond. Let $T : \mathbb{R}^2 \rightarrow \mathbb{R}^2$ be the linear transformation of the form $T(s, \lambda) = (-s, \lambda)$. Two C^∞ function germs $p_1, p_2 : (\mathbb{R}^2, 0) \rightarrow (\mathbb{R}, 0)$ are said to be \mathcal{K}^T -equivalent if there exist a germ of C^∞ diffeomorphism $h : (\mathbb{R}^2, 0) \rightarrow (\mathbb{R}^2, 0)$ of the form $h \circ T = T \circ h$ and a C^∞ function-germ $M : (\mathbb{R}^2, (0, 0)) \rightarrow \mathbb{R}$ of the form $M \circ T = M$, $M(0, 0) \neq 0$ such that $p_1 \circ h(s, \lambda) = M(s, \lambda)p_2(s, \lambda)$ ([15]).

Theorem 3 ([15]). *Two C^∞ map-germs of the following form*

$$f_i(s, \lambda) = (sp_i(s^2, \lambda), s^2, \lambda) \quad \text{where } p_i(s^2, \lambda) \notin m_2^\infty, \quad (i = 1, 2)$$

are \mathcal{A} -equivalent if and only if the function-germs $p_i(s^2, \lambda)$ are \mathcal{K}^T -equivalent.

Note that Theorem 3 provides a criterion for the \mathcal{A} -equivalence of C^∞ map-germs of the forms $(s, \lambda) \mapsto (\varphi(s, \lambda), s^2, \lambda)$ ($\varphi : (\mathbb{R}^2, 0) \rightarrow (\mathbb{R}, 0)$ is a C^∞ function-germ) on the basis of the Malgrange preparation theorem (for the Malgrange preparation theorem, see [4, 23]).

3. PROOF OF THEOREM 1

Since $\mathbf{r}(0)$ is not an inflection point, the dual germ $\mathbf{n} : (I, 0) \rightarrow S^2$ is a C^∞ immersive germ.

Proof of assertion 1 of Theorem 1.

Suppose that $P(0)$ does not belong to $E_{\mathbf{n}(0)}$. Then, by Lemma 2.4, the restriction $\Psi_{P(0)}|_{S^2 - \{\pm P(0)\} - E_{P(0)}}$ is C^∞ immersive. Thus, by Lemma 2.3, the map-germ $ped_{\mathbf{r}, P(0)} : (I, 0) \rightarrow S^2$ is also C^∞ immersive. Therefore, the map-germ $Un-ped_{\mathbf{r}, P} : (I \times J, (0, 0)) \rightarrow S^2 \times J$ is also C^∞ immersive. Next, suppose that $P(0) \in E_{\mathbf{n}(0)} - \mathbf{r}(0)$. Then, the image of the dual \mathbf{n} and $E_{P(0)}$ intersect transversely at $\mathbf{n}(0)$. Thus, by Lemmata 2.3 and 2.4, the map-germ $ped_{\mathbf{r}, P(0)} : (I, 0) \rightarrow S^2$ is C^∞ immersive. Therefore, the map-germ $Un-ped_{\mathbf{r}, P} : (I \times J, (0, 0)) \rightarrow S^2 \times J$ is also C^∞ immersive.

Conversely, suppose that the map-germ $Un-ped_{\mathbf{r}, P} : (I \times J, (0, 0)) \rightarrow S^2 \times J$ is C^∞ immersive. Then, in particular, the map-germ $ped_{\mathbf{r}, P(0)} : (I, 0) \rightarrow S^2$ is also C^∞ immersive. In order to conclude the proof of assertion 1 of Theorem 1, it is sufficient to show that the assumption $P(0) = \mathbf{r}(0)$ implies a contradiction. The assumption $P(0) = \mathbf{r}(0)$ implies that the image of \mathbf{n} is tangent to $E_{P(0)}$ at $\mathbf{n}(0)$. By Lemma 2.4, the map-germ $ped_{\mathbf{r}, P(0)} : (I, 0) \rightarrow S^2$ must be singular; this is a contradiction. \square

Proof of assertion 5 of Theorem 1.

Suppose that both $P(0) = \mathbf{r}(0)$ and $(P(J), P(0)) = (\mathbf{r}(I), \mathbf{r}(0))$ as set-germs hold. Then, for any $\lambda \in J$, $ped_{\mathbf{r}, P(\lambda)} : (I, 0) \rightarrow S^2$ is \mathcal{A} -equivalent to the ordinary cusp $s \mapsto (s^3, s^2)$ by [17] (also, see [19]). Thus, by using the Malgrange preparation theorem and Theorem 3, the map-germ $Un-ped_{\mathbf{r}, P} : (I \times J, (0, 0)) \rightarrow S^2 \times J$ is \mathcal{A} -equivalent to the cuspidal edge $(s, \lambda) \mapsto (s^3, s^2, \lambda)$.

Conversely, suppose that the map-germ $Un-ped_{\mathbf{r}, P} : (I \times J, (0, 0)) \rightarrow S^2 \times J$ is \mathcal{A} -equivalent to the cuspidal edge. Then, in particular, for any sufficiently small $\lambda_0 \in J$, there exists a sufficiently small $s_0 \in I$ such that the map-germ $ped_{\mathbf{r}, P(\lambda_0)} : (I, s_0) \rightarrow S^2$ is singular. Since $\mathbf{r}(0)$ is not an inflection point, by Lemma 2.4, $E_{P(\lambda_0)} = S^2 \cap (\mathbb{R}\mathbf{t}(s_0) + \mathbb{R}\mathbf{n}(s_0))$. Therefore, $P(\lambda_0) = \mathbf{r}(s_0)$. \square

Proof of assertions 2 and 3 of Theorem 1.

By composing an appropriate rotation without the loss of generality, it can be assumed that $\mathbf{r}(0) = (0, 1, 0)$, $\mathbf{t}(0) = (0, 0, 1)$, $\mathbf{n}(0) = (-1, 0, 0)$. For a point Q of S^2 , put $H(Q) = \{X \in S^2 \mid Q \cdot X \geq 0\}$, and let $\alpha_{\mathbf{n}(0)} : H(\mathbf{n}(0)) - E_{\mathbf{n}(0)} \rightarrow \{-1\} \times \mathbb{R}^2$ be the central projection relative to $\mathbf{n}(0)$. Then, by Lemma 2.2, the germ of composition $\alpha_{\mathbf{n}(0)} \circ \mathbf{n}$ is of the form

$$\alpha_{\mathbf{n}(0)} \circ \mathbf{n}(s) = (s + \varphi_1(s), s^2 + \varphi_2(s)) \quad (\varphi_j(s) = o(s^j)).$$

Since $\varphi_2(s) = o(s^2)$, the map-germ given by $h\left(s\sqrt{1 + \frac{\varphi_2(s)}{s^2}}\right) = s$ is a well-defined germ of local C^∞ diffeomorphism. Thus, there exists a C^∞ map-germ $\tilde{\varphi}_1 : (I, 0) \rightarrow \mathbb{R}$ such that

$$\alpha_{\mathbf{n}(0)} \circ \mathbf{n} \circ h(s) = (s + \tilde{\varphi}_1(s), s^2) \quad (\tilde{\varphi}_1(s) = o(s)).$$

Let $\alpha_{P(0)} : H(P(0)) - E_{P(0)} \rightarrow \mathbb{R} \times \{1\} \times \mathbb{R}$ be the central projection relative to $P(0)$. By the form mentioned above and Lemma 2.4, the germ of composition $\alpha_{P(0)} \circ \text{ped}_{\mathbf{r}, P(0)}$ is \mathcal{A} -equivalent to a map-germ of the following form:

$$s \mapsto ((s + \tilde{\varphi}_1(s))s^2, s^2).$$

Next, we investigate the influence of moving the pedal points $P(\lambda)$. Suppose that $P(0) = \mathbf{r}(0)$ and P has $(k+1)$ -point contact with \mathbf{r} at $0 \in J$ ($k \geq 0$). In other words, suppose that there exist a sufficiently small neighborhood U of $\mathbf{r}(0)$ in S^2 and a C^∞ function $F : U \rightarrow \mathbb{R}$ such that $F \circ \mathbf{r}(s) \equiv 0$ ($\forall s \in I \cap \mathbf{r}^{-1}(U)$), $F \circ P(0) = (F \circ P)'(0) = \cdots = (F \circ P)^{(k)}(0) = 0$, and $(F \circ P)^{(k+1)}(0) \neq 0$. Since $\mathbf{r} : I \rightarrow S^2$ is a unit speed curve, it can be assumed that F is non-singular provided that I (resp., U) is a sufficiently small neighborhood of 0 (resp., $\mathbf{r}(0)$). Then, there exists a sufficiently small neighborhood $\tilde{U} \subset U$ of $\mathbf{r}(0)$ such that for any $X \in \tilde{U}$, the integral curve of $-\text{grad}(F)$ starting from X lies within \tilde{U} until it reaches the image of the unit speed curve $\mathbf{r}(I)$. Let this reaching point be denoted by $\gamma(X)$ and define the map $\Gamma : \tilde{U} \rightarrow I$ as $\Gamma(X) = \mathbf{r}^{-1} \circ \gamma(X)$. Then, $(\tilde{U}, (\Gamma, F))$ can be used as a chart at $\mathbf{r}(0)$ since the map $(\Gamma, F) : \tilde{U} \rightarrow I \times \mathbb{R}$ is non-singular. By using the chart $(\tilde{U}, (\Gamma, F))$ and by the proof of assertion 5 of Theorem 1, the germ of composition

$$(s, \lambda) \mapsto (\alpha_{P(0)} \circ \text{ped}_{\mathbf{r}, P(\lambda)} \circ h(s), \lambda)$$

is \mathcal{A} -equivalent to a map-germ of the following form:

$$(a) \quad (s, \lambda) \mapsto ((s + \tilde{\varphi}_1(s)) (s^2 \pm F \circ P(\lambda)), s^2 \pm F \circ P(\lambda), \lambda).$$

Furthermore, by the Malgrange preparation theorem and Theorem 3, a map-germ of the form (a) must be \mathcal{A} -equivalent to the map-germ $f_k^\pm(s, \lambda) = (s(s^2 \pm \lambda^{k+1}), s^2, \lambda)$.

Conversely, we suppose that the germ of pedal unfolding $Un\text{-ped}_{\mathbf{r}, P} : (I \times J, (0, 0)) \rightarrow S^2 \times J$ is \mathcal{A} -equivalent to S_k^\pm ($k \geq 0$), $P(0) = \mathbf{r}(0)$ and that P does not have $(k+1)$ -point contact with \mathbf{r} at $0 \in J$. Then, by the proof presented above, for any positive integer ℓ , P does not have $(\ell+1)$ -point contact with \mathbf{r} at $0 \in J$. In particular, there exists a C^∞ immersion $\tilde{P} : J \rightarrow S^2$ such that \tilde{P} is sufficiently near P under the Whitney C^∞ topology, and \tilde{P} has $(k+2)$ -point contact with \mathbf{r} at $0 \in J$. By the proof of the implication described above, it can be concluded that S_k^\pm singularity is adjacent to S_{k+1}^\pm singularity; however, this contradicts the adjacency diagram given in [15]. \square

Proof of assertion 4 of Theorem 1.

Suppose that the map-germ $Un\text{-ped}_{\mathbf{r}, P} : (I \times J, (0, 0)) \rightarrow S^2 \times J$ is \mathcal{A} -equivalent to one of B_k^\pm, C_k^\pm, F_4 , and H_k . Then, by assertions 1, 2, and 3 of Theorem 1, the given immersion $P : J \rightarrow S^2$ must satisfy not only $P(0) = \mathbf{r}(0)$ but also the condition that for any positive integer ℓ , P does not have $(\ell+1)$ -point contact with \mathbf{r} at $0 \in J$. Thus, for any positive integer ℓ , there exists a C^∞ immersion $\tilde{P} : J \rightarrow S^2$ such that \tilde{P} is sufficiently near P under the Whitney C^∞ topology, and \tilde{P} has the $(\ell+1)$ -contact with \mathbf{r} at $0 \in J$. Hence, it can be concluded that one of B_k^\pm, C_k^\pm, F_4 , and H_k singularity is adjacent to S_ℓ^\pm singularity for any positive integer ℓ ; however, this contradicts the adjacency diagram given in [15]. \square

4. PROOF OF THEOREM 2

Since $\mathbf{r}(0)$ is an ordinary inflection point, by Lemma 2.2 and the Malgrange preparation theorem, the dual germ $\mathbf{n} : (I, 0) \rightarrow S^2$ is \mathcal{A} -equivalent to the ordinary cusp $s \mapsto (s^3, s^2)$.

Proof of assertion 1 of Theorem 2.

Suppose that $P(0)$ does not belong to $E_{\mathbf{n}(0)}$. Then, for any sufficiently small $\lambda_0 \in J$, $P(\lambda_0)$ lies outside $E_{\mathbf{n}(0)}$. This implies that by Lemma 2.4, the map-germ $\Psi_{P(\lambda_0)}$ at $\mathbf{n}(0)$ is non-singular. Thus, by Lemma 2.3, the map-germ $ped_{\mathbf{r}, P(\lambda_0)} : (I, 0) \rightarrow S^2$ is also \mathcal{A} -equivalent to the ordinary cusp. Therefore, by Theorem 3, the map-germ $Un-ped_{\mathbf{r}, P} : (I \times J, (0, 0)) \rightarrow S^2 \times J$ is \mathcal{A} -equivalent to the cuspidal edge.

Conversely, suppose that the map-germ $Un-ped_{\mathbf{r}, P} : (I \times J, (0, 0)) \rightarrow S^2 \times J$ is \mathcal{A} -equivalent to the map-germ $g(s, \lambda) = (s^3, s^2, \lambda)$; we show that $P(0) \in E_{\mathbf{n}(0)}$ implies a contradiction under this assumption. The property $P(0) \in E_{\mathbf{n}(0)}$ implies that $\mathbf{n}(0) \in E_{P(0)}$. Since the dual germ $\mathbf{n} : (I, 0) \rightarrow S^2$ is \mathcal{A} -equivalent to the ordinary cusp $s \mapsto (s^3, s^2)$, by Lemma 2.4, $\mathbf{n}(0) \in E_{P(0)}$ implies that $j^3(Un-ped_{\mathbf{r}, P})(0)$ is not \mathcal{A}^3 -equivalent to $j^3g(0)$. This contradicts the assumption that $Un-ped_{\mathbf{r}, P} : (I \times J, (0, 0)) \rightarrow S^2 \times J$ is \mathcal{A} -equivalent to the map-germ $g(s, \lambda) = (s^3, s^2, \lambda)$. \square

Proof of "if" parts of assertions 2 and 3 of Theorem 1.

Since $P(0)$ belongs to $E_{\mathbf{n}(0)} - \{\mathbf{r}(0)\}$, by composing an appropriate rotation without the loss of generality, it can be assumed that $\mathbf{n}(0) = (-1, 0, 0)$ and $P(0) = (0, 0, 1)$. Let $\alpha_{\mathbf{n}(0)} : H(\mathbf{n}(0)) - E_{\mathbf{n}(0)} \rightarrow \{-1\} \times \mathbb{R}^2$ be the central projection relative to $\mathbf{n}(0)$. Then, by Lemma 2.2, the germ of composition $\alpha_{\mathbf{n}(0)} \circ \mathbf{n}$ is of the form

$$\alpha_{\mathbf{n}(0)} \circ \mathbf{n}(s) = (as^2 + bs^3 + \varphi_1(s), cs^2 + ds^3 + \varphi_2(s)),$$

where $bc \neq 0$ and $\varphi_j(s) = o(s^3)$. Since $c \neq 0$, there exists a germ of C^∞ diffeomorphism $h : (I, 0) \rightarrow (I, 0)$ such that

$$\alpha_{\mathbf{n}(0)} \circ \mathbf{n} \circ h(s) = (\tilde{a}s^2 + \tilde{b}s^3 + \tilde{\varphi}_1(s), s^2),$$

where $\tilde{b} \neq 0$ and $\tilde{\varphi}_1(s) = o(s^3)$. Let $\alpha_{P(0)} : H(P(0)) - E_{P(0)} \rightarrow \mathbb{R}^2 \times \{1\}$ be the central projection relative to $P(0)$. By the form mentioned above and Lemma 2.4, the germ of composition $\alpha_{P(0)} \circ ped_{\mathbf{r}, P(0)}$ is \mathcal{A} -equivalent to a map-germ of the following form:

$$s \mapsto \left((\tilde{a}s^2 + \tilde{b}s^3 + \tilde{\varphi}_1(s))s^2, s^2 \right).$$

Next, we investigate the influence of moving the pedal points $P(\lambda)$. Suppose that $P(0) = \mathbf{r}(0)$ and P has $(k+1)$ -point contact with $E_{\mathbf{n}(0)}$ at $0 \in J$ ($k \geq 0$). Since $E_{\mathbf{n}(0)}$ is defined by the equation $\mathbf{n}(0) \cdot X = 0$, the assumption of $(k+1)$ -point contact implies that $\mathbf{n}(0) \cdot P(0) = \mathbf{n}(0) \cdot P'(0) = \dots = \mathbf{n}(0) \cdot P^{(k)}(0) = 0$ and $\mathbf{n}(0) \cdot P^{(k+1)}(0) \neq 0$. Then, as in the proof of assertions 2 and 3 of Theorem 1, the germ of composition

$$(s, \lambda) \mapsto (\alpha_{P(0)} \circ ped_{\mathbf{r}, P(\lambda)}(s), \lambda)$$

is \mathcal{A} -equivalent to the germ of the following form:

$$(b) \quad (s, \lambda) \mapsto \left((\tilde{a}s^2 + \tilde{b}s^3 + \varphi_1(s)) (s^2 \pm \mathbf{n}(0) \cdot P(\lambda)), s^2 \pm \mathbf{n}(0) \cdot P(\lambda), \lambda \right).$$

Furthermore, by the Malgrange preparation theorem and Theorem 3, a map-germ of the form mentioned in (b) must be \mathcal{A} -equivalent to the map-germ $g_k^\pm(s, \lambda) = (s^3(s^2 \pm \lambda^{k+1}), s^2, \lambda)$. \square

The “only if” parts of assertions 2 and 3 of Theorem 2 can be proved as follows. Put $\tilde{g}(s, \lambda) = s^2$, $\tilde{g}_0(s, \lambda) = s^4 + \lambda s^2$, $\tilde{g}_{2i+1}^\pm(s, \lambda) = s^4 \pm \lambda^{2i+2} s^2$, and $\tilde{g}_{2i}^+(s, \lambda) = s^4 + \lambda^{2i+1} s^2$. Then, it can be clearly seen that any two distinct elements of the following set are not \mathcal{K}^T -equivalent.

$$\{\tilde{g}, \tilde{g}_0, \tilde{g}_1^+, \tilde{g}_1^-, \tilde{g}_2^+, \tilde{g}_3^+, \tilde{g}_3^-, \dots\}.$$

Hence, by Theorem 3, any two distinct elements of the set of the cuspidal edge, cuspidal crosscap, cuspidal S_1^+ , cuspidal S_1^- , cuspidal S_2^+ , cuspidal S_3^+ , cuspidal $S_3^- \dots$ are not \mathcal{A} -equivalent. Furthermore, by Theorem 3 and the form of $g_0, g_1^\pm, g_2^\pm, \dots$ in Table 2, the following adjacency diagram is obtained.

$$(c) \quad \dots \longrightarrow \text{cuspidal } S_k \longrightarrow \dots \longrightarrow \text{cuspidal } S_1 \longrightarrow \text{cuspidal } S_0.$$

Proof of “only if” parts of the assertions 2, 3 of Theorem 2.

As in the proof of the “only if” parts of assertions 2 and 3 of Theorem 1, we suppose that $Un\text{-}ped_{\mathbf{r}, P} : (I \times J, (0, 0)) \rightarrow S^2 \times J$ is \mathcal{A} -equivalent to cuspidal S_k^\pm ($k \geq 0$), $P(0) \in E_{\mathbf{n}(0)} - \{\mathbf{r}(0)\}$, and P does not have $(k+1)$ -point contact with $E_{\mathbf{n}(0)}$ at $0 \in J$. Then, by the “if” parts of assertions 2, 3 of Theorem 2, for any non-negative integer ℓ , P does not have $(\ell+1)$ -point contact with $E_{\mathbf{n}(0)}$ at $0 \in J$. In particular, for any non-negative integer ℓ , there exists a C^∞ immersion $\tilde{P} : J \rightarrow S^2$ such that \tilde{P} is sufficiently near P under the Whitney C^∞ topology, and \tilde{P} has $(\ell+1)$ -point contact with $E_{\mathbf{n}(0)}$ at $0 \in J$. Hence, it can be concluded that cuspidal S_k^\pm singularity is adjacent to cuspidal S_ℓ^\pm singularity for any positive integer ℓ ; however, this contradicts diagram (c).

Next, suppose that $Un\text{-}ped_{\mathbf{r}, P} : (I \times J, (0, 0)) \rightarrow S^2 \times J$ is \mathcal{A} -equivalent to cuspidal S_k^\pm ($k \geq 0$) and $P(0) = \{\mathbf{r}(0)\}$. In this case, the tangent cone of $\mathbf{n}(I)$ at $\mathbf{n}(0)$ coincides with E_P . Thus, by Lemma 2.4, $j^2(Un\text{-}ped_{\mathbf{r}, P})(0)$ is not \mathcal{A}^2 -equivalent to $j^2 g_k^\pm(0)$; this contradicts the assumption that $Un\text{-}ped_{\mathbf{r}, P} : (I \times J, (0, 0)) \rightarrow S^2 \times J$ is \mathcal{A} -equivalent to the map-germ $g_k^\pm(s, \lambda) = (s^5 \pm \lambda^{k+1} s^3, s^2, \lambda)$. \square

Remarks

It is possible to adopt the criteria given in [16] or an argument similar to that given in [7] to prove Theorems 1 and 2. However, the criteria in [16] are too general to be directly applied to our study, and the argument in [7] seems to be somewhat ad hoc. Thus, in order to apply them to our study, considerable preliminary work is required, the proofs of which are time-consuming and complicated. On the other hand, Theorem 3 is the most suitable criterion for our study. Moreover, the calculations with respect to \mathcal{K}^T -equivalence are relatively straightforward; hence, by using Theorem 3, we can prove both Theorem 1 and Theorem 2 in a coherent and unified manner.

REFERENCES

- [1] V. I. Arnol'd, *The geometry of spherical curves and the algebra of quaternions*, Russian Math. Surveys, **50** (1995), 1–68. DOI: 10.1070/RM1995v050n01ABEH001662
- [2] V. I. Arnol'd, *Singularities of Caustics and Wave fronts*, Springer, Berlin-New York, 2001.
- [3] V. I. Arnol'd, V. V. Goryunov, O. V. Lyashko, V. A. Vasil'ev, *Dynamical Systems VIII*, Encyclopaedia of Mathematical Sciences, **39**, Springer-Verlag, Berlin Heidelberg New York, 1989.
- [4] V. I. Arnol'd, S. M. Gusein-Zade, A. N. Varchenko, *Singularities of Differentiable Maps I*, Monographs in Mathematics, **82**, Birkhäuser, Boston Basel Stuttgart, 1985.
- [5] J. W. Bruce, P. J. Giblin, *Curves and Singularities (second edition)*, Cambridge University Press, Cambridge, 1992.
- [6] X. Y. Chen and T. Matumoto, *On generic 1-parameter families of C^∞ -maps of an n -manifold into a $(2n-1)$ -manifold*, Hiroshima Math. J., **14** (1984), 547–550.

- [7] S. Fujimori, K. Saji, M. Umehara and K. Yamada, *Cuspidal cross caps and singularities of maximal surfaces*, Math. Z., **259** (2008), 827–848. DOI: 10.1007/s00209-007-0250-0
- [8] T. Gaffney, *On the order of determination of a finitely determined germs*, Invent. Math., **37** (1976), 83–92. DOI: 10.1007/BF01418963
- [9] T. Gaffney, *A note on the order of determination of a finitely determined germs*, Invent. Math., **59** (1979), 127–130. DOI: 10.1007/BF01403059
- [10] T. Gaffney and A. A. du Plessis, *More on the determinacy of smooth map-germs*, Invent. Math., **66** (1982), 137–163. DOI: 10.1007/BF01404761
- [11] S. Izumiya, K. Saji and M. Takahashi, *Horospherical flat surfaces in hyperbolic 3-space*, J. Math. Soc. Japan, **62** (2010), 789–849. DOI: 10.2969/jmsj/06230789
- [12] S. Izumiya and F. Tari, *Projections of hypersurfaces in the hyperbolic space to hyperhorospheres and hyperplanes*, Rev. Mat. Iberoam. **24** (2008) 895–920.
- [13] J. Mather, *Stability of C^∞ mappings, III, Finitely determined map-germs*, Publ. Math. I. H. E. S., **35** (1969), 127–156. DOI: 10.1007/BF02698926
- [14] J. Mather, *Stability of C^∞ mappings, IV, Classification of stable map-germs by \mathbf{R} -algebras*, Publ. Math. I. H. E. S., **37** (1970), 223–248. DOI: 10.1007/BF02684889
- [15] D. Mond, *On the classification of germs of maps from \mathbb{R}^2 to \mathbb{R}^3* , Proc. London Math. Soc., **50** (1985), 333–369. DOI: 10.1112/plms/s3-50.2.333
- [16] T. Nishimura, *Criteria for right-left equivalence of smooth map-germs*, Topology, **40** (2001), 433–462. DOI: 10.1016/S0040-9383%2899%2900068-3
- [17] T. Nishimura, *Normal forms for singularities of pedal curves produced by non-singular dual curve germs in S^n* , Geom Dedicata, **133** (2008), 59–66. DOI: 10.1007/s10711-008-9233-5
- [18] T. Nishimura, *Singularities of pedal curves produced by singular dual curve germs in S^n* , Demonstratio Math., **43** (2010), 447–459.
- [19] T. Nishimura and K. Kitagawa, *Classification of singularities of pedal curves in S^2* , The Natural Sciences, Journal of the Faculty of Education and Human Sciences, Yokohama National University, **10** (2008), 39–55. (available from <http://hdl.handle.net/10131/4067>).
- [20] A. A. du Plessis, *On the determinacy of smooth map-germs*, Invent. Math., **58** (1980), 107–160. DOI: 10.1007/BF01403166
- [21] I. R. Porteous, *Geometric Differentiation (second edition)*, Cambridge University Press, Cambridge, 2001.
- [22] K. Saji, *Criteria for cuspidal S_k singularities and its applications*, preprint.
- [23] C. T. C. Wall, *Finite determinacy of smooth map-germs*, Bull. London Math. Soc., **13** (1981), 481–539. DOI: 10.1112/blms/13.6.481

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