A Note on the Validity of Mean-Variance Criterion

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IT IS FREQUENTLY ASSERTED that mean-variance criterion is a valid efficiency criterion if and only if distributions are normal or utility functions are quadratic. For example, the proposition of the sufficiency appears in Fama and Miller [1], Chap. 6; the proposition of the necessity appears in Tobin [18], Samuelson [7] and Kroll et al. [5]; furthermore, Hakansson [3], Levy and Markowitz [6] claim the equivalence. However, these authers did not give the rigorous proof and, as far as I know, it does not exist.

The purpose of this paper is to show that the sufficiency is correct, but the necessity is not correct.

It is assumed throughout that any utility function $u(x)$ is of the von Neumann-Morgenstern type, nondecreasing and concave. The risks considered are random variables X with given probability distributions $F(x)$.

Mean-variance criterion is called *valid* if it can achieve consistency with the von Neumann-Morgenstern postulates; increasing variance σ^2 (or μ) decreases (or increases) the expected utility $Eu(X)$ for a given expected return μ (or σ^2).

Of course, we must assume that the distribution functions are completely determined by the two first moments μ, σ^2; otherwise, mean-variance criterion would not necessarily be valid.

THEOREM 1. If the utility functions are quadratic, then mean-variance criterion is valid.

Proof: Consider the quadratic utility function

$$u(x) = a+bx-cx^2$$

where $b>0$, $c>0$, and x is constrainted to amounts less than $b/2c$. Admittedly, the domain of $f(x)$ is also an interval $(-\infty, b/2c)$.
Then we get

$$\begin{aligned} Eu(X) &= \int_{-\infty}^{b/2c} (a+bx-cx^2)f(x)dx \\ &= a+b\mu-c(\sigma^2+\mu^2)\,. \end{aligned}$$

If μ and σ^2 show a functional relationship, the result is really trivial by the quadratic form of $u(x)$. Therefore, we assume that μ and σ^2 are independent variables.
Taking derivatives we find

$$\frac{\partial Eu(X)}{\partial \sigma} = -2c\sigma < 0, \quad \frac{\partial Eu(X)}{\partial \mu} = b-2c\mu > 0\,.$$

Q.E.D.

Before proceeding to the next theorem, we need the following preliminaries (Hammond [4]):

LEMMA (Hammond). Suppose that

(a) the probability distribution functions of random variables X and Y cross at most once

(b) $E(X) \leqq E(Y)$

(c) as w increases from $-\infty$, $F_Y(w)-F_X(w)$ is first negative

(d) a utility function is nondecreasing and concave

Then $Eu(X) < Eu(Y)$.

THEOREM 2. If the distributions are normal, then mean-variance criterion is valid.

Proof: Suppose that X and Y are independent, normal $N(0, \sigma_1^2)$ and $N(0, \sigma_2^2)$, $\sigma_1^2 > \sigma_2^2$ respectively. Then, we get

$$\frac{d}{dw}[F_Y(w)-F_X(w)]$$

$$=\frac{1}{\sqrt{2\pi}\sigma_2} \exp\left\{-\frac{w^2}{2\sigma_2^2}\right\}-\frac{1}{\sqrt{2\pi}\sigma_1} \exp\left\{-\frac{w^2}{2\sigma_1^2}\right\}.$$

Therefore, the sign of values of $(d/dw)[F_Y(w)-F_X(w)]$ changes from $-$, through $+$, to $-$. Then we can show that, by $F_Y(-\infty)-F_X(-\infty) = 0$ and $F_Y(\infty)-F_X(\infty) = 0$, X and Y satisfy the conditions (a) and (c) of the Hammond's lemma.

Therefore, $(\partial Eu(X))/(\partial\sigma) < 0$.

Next, suppose that X and Y are independent, normal $N(\mu_1, \sigma^2)$ and $N(\mu_2, \sigma^2)$, $\mu_1 > \mu_2$. Clearly, $F_X(w) < F_Y(w)$ for all w. Therefore,

$$E_X[u(w)]-E_Y[u(w)]=-\int_{-\infty}^{\infty} [F_X(w)-F_Y(w)]u'(w)dw>0. \qquad \text{Q.E.D.}$$

THEOREM 3. A valid mean-variance criterion does not necessarily entail the normal distributions or the quadratic utility functions.

Proof: We shall give the following counter-example. The lognormal distribution defines a two-parameter family

$$f(x)=\frac{1}{\sqrt{2\pi}sx} \exp\left\{-\frac{(\log x-m)^2}{2s^2}\right\}$$

where m is the expected value of $\log X$ and s^2 is the variance of $\log X$. There exists a one-to-one mapping between a $m-s^2$ pair and the corresponding moments of X, $\mu-\sigma^2$ pair:

$$\mu=e^{m+s^2/2}, \quad \sigma^2=(e^{m+s^2/2})^2(e^{s^2}-1).$$

Now we assume that the utility function is $u(x)=\log x$. After some computations, we have (Feldstein [2])

$$Eu(X)=\log\mu-\frac{1}{2}\log\left(\frac{\sigma^2}{\mu^2}+1\right).$$

Since $\partial Eu(X)/\partial\sigma<0$ and $\partial Eu(X)/\partial\mu>0$, mean-variance criterion is valid while the distribution is not normal and the utility function is not quadratic. Q.E.D.

References

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