Interim Efficient Allocations under Uncertainty*

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Abstract

This paper considers an exchange economy under uncertainty with asymmetric information. Uncertainty is represented by multiple priors and posteriors of agents who have either Bewley's incomplete preferences or Gilboa-Schmeidler's maximin expected utility preferences. The main results characterize interim efficient allocations under uncertainty; that is, they provide conditions on the sets of posteriors, thus implicitly on the way how agents update the sets of priors, for non-existence of a trade which makes all agents better off at any realization of private information. For agents with the incomplete preferences, the condition is necessary and sufficient, but for agents with the maximin expected utility preferences, the condition is sufficient only. A couple of necessary conditions for the latter case are provided.
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1 Introduction

This paper considers an exchange economy under uncertainty with asymmetric information. There are a finite number of states and in each state there is a single good. There are a finite number of agents and each agent has private information about the states. We model uncertainty by so-called multiple priors; that is, for each agent, uncertainty is represented by sets of priors and sets of posteriors. The good is evaluated by concave utility index functions, from which agents derive either Bewley's [4] incomplete preferences (BI-preferences for short) or Gilboa-Schmeidler's [12] maximin expected utility preferences (MEU-preferences for short).

Prior sets induce preferences in the ex ante stage (before the receipt of private information) and posterior sets induce preferences in the interim stage (after the receipt of private information). An allocation is *ex ante efficient* if there is no feasible trade which makes all agents better off in the ex ante sage. Bewley [3] and Rigotti and Shannon [23] characterized ex ante efficient allocations by prior sets for agents with BI-preferences, and Billot et al. [2] characterized ex ante efficient allocations similarly for agents with MEU-preferences under no aggregate uncertainty.[1] An allocation is *interim efficient* if there is no feasible trade which makes all agents better off in the interim sage for any realization of private information. No attempt has been made to obtain a counterpart for interim efficient allocations as far as we are aware of.

The purpose of this paper is to provide characterizations of *interim efficient* allocations by posterior sets for agents with BI-preferences and MEU-preferences. The key concept in our characterizations is the *compatible prior set* of an agent, which is defined as the collection of all the probability distributions such that, for each piece of private information of the agent, the conditional probability distributions are in the corresponding posterior set of the agent. The compatible prior set of an agent coincides with the convex hull of all posteriors of the agent. The main results show the following: for agents with BI-preferences, an allocation is interim efficient if and only if it is ex ante efficient for agents possessing their compatible prior sets as their own prior sets; and for agents with MEU-preferences, an allocation is interim efficient if the same condition holds, but not vice versa. Thus, ex ante efficiency with respect to the compatible prior sets is necessary and sufficient for interim efficiency for the former case, but it is sufficient only for

[1] Rigotti et al. [24] generalized this result.

the latter case. To obtain a sharper result for the latter case, we restrict our attention to a limited class of allocations and provide a couple of necessary conditions for interim efficiency.

In the standard Bayesian models, Morris [21] and Feinberg [11] provided a characterization of interim efficient allocations,[2] which is closely related to the agreement theorem of Aumann [1]. The agreement theorem in this context asserts that if agents with linear utility index functions have a common prior, then an allocation is interim efficient. The result of Morris [21] and Feinberg [11] implies the converse: if an allocation is interim efficient for agents with linear utility index functions, then there is a prior which induces all the agents' posteriors; that is, it appears as if they share a fictitious common prior. Our results have the corresponding implication when utility index functions are linear; that is, for agents with BI-preferences, an allocation is interim efficient if and only if the compatible prior sets of all agents have a non-empty intersection, whose element is interpreted as a fictitious common prior.

Characterizations of interim efficient allocations are important in the context of the no trade theorem [20]: it asserts that any ex ante efficient allocation is interim efficient, as interpreted by Holmström and Myerson [17], and thus purely speculative trade is impossible. A simple and intuitive explanation for the no trade theorem is that agents in the standard Bayesian models are dynamically consistent. We show that an analogous but not identical explanation can be given for the multiple priors models, as follows. By combining the characterization of ex ante efficient allocations and that of interim efficient allocations, we can obtain a necessary and sufficient condition for any ex ante efficient allocation to be interim efficient for agents with BI-preferences.[3] Using this condition, we show that if agents with BI-preferences derive posterior sets from prior sets by prior-by-prior Bayesian updating, then the no trade theorem holds. We also argue that agents with BI-preferences are dynamically consistent indeed. On the other hand, for agents with MEU-preferences, the no trade theorem does not hold because agents are not dynamically consistent. Epstein and Schneider [9] and Wakai [27] identified a sufficient condition for agents to be dynamically consistent by introducing *rectangular prior sets*. The compatible prior sets in this paper turn out to be rectangular prior

[2]See also Samet [26] and Ng [22].

[3]On the no trade theorem in more general non-expected utility models, see Dow et al. [7], Ma [19], and Halevy [15].

sets, which explains why ex ante efficiency with respect to the compatible prior sets is sufficient for interim efficiency.

The organization of this paper is as follows. Section 2 sets up the model. Section 3 reports the characterization of interim efficient allocations for agents with BI-preferences, and Section 4 reports that for agents with MEU-preferences. Section 5 discusses issues of dynamic consistency, the agreement theorem, and conditional preferences in the multiple priors models.

2 Setup

In this section, we set up the model of an exchange economy under uncertainty with asymmetric information. There is a finite set of sates $\Omega = \{1, \ldots, n\}$.[4] Let $\Delta(\Omega)$ be the set of all probability distributions over Ω, and let $\mathcal{P} \subsetneq 2^{\Delta(\Omega)}$ be the collection of all non-empty, convex, and closed subsets of $\Delta(\Omega)$. For $p \in \Delta(\Omega)$ and $z \in \mathbb{R}^\Omega$, let $E_p[z] = \sum_{\omega \in \Omega} p(\omega)z(\omega)$ be the expected value of a random variable z with respect to p. We write $E_P[z] = \min_{p \in P} E_p[z]$ for $P \in \mathcal{P}$, which is the minimum expected value of z where the minimum is taken over P. Note that the minimum exists because each $P \in \mathcal{P}$ is compact. For a function $f : \mathbb{R} \to \mathbb{R}$ and $z \in \mathbb{R}^\Omega$, we write $E_p[f(z)] = \sum_{\omega \in \Omega} p(\omega)f(z(\omega))$ for $p \in \Delta(\Omega)$ and $E_P[f(z)] = \min_{p \in P} E_p[f(z)]$ for $P \in \mathcal{P}$ with some abuse of notation.

There is a finite set of agents $\mathcal{I} = \{1, \ldots, I\}$. Agent $i \in \mathcal{I}$ has an information partition $\Pi_i \subsetneq 2^\Omega$ of Ω with a generic element $\pi_i \in \Pi_i$. We write $\pi_i(\omega) \in \Pi_i$ for the partition element containing $\omega \in \Omega$; agent i observes $\pi_i(\omega)$ as private information when the true state is ω. Agent i has a set of priors $P_i \in \mathcal{P}$, which represents his prior beliefs, and a set of posteriors $\Phi_i(\pi_i) \in \mathcal{P}$ for each $\pi_i \in \Pi_i$, which represents his posterior beliefs after observing π_i. We write $\Phi_i = \{\Phi_i(\pi_i)\}_{\pi_i \in \Pi_i}$ for the collection of all posteriors of agent i.[5] For P_i, we assume that there is no $\pi_i \in \Pi_i$ such that $p(\pi_i) = 0$ for all $p \in P_i$, by which conditional probability distributions will be well defined. For Φ_i, we assume that $p(\pi_i) = 1$ for each $p \in \Phi_i(\pi_i)$ and $\pi_i \in \Pi_i$.[6]

There is a single good in the economy, and agent $i \in \mathcal{I}$ has a concave, strictly in-

[4]We use a finite set of states to avoid topological and measure theoretic complications.

[5]Examples of P_i and Φ_i are found in Section 3.

[6]It might be natural to assume some relationship between P_i and Φ_i by some updating rule, but we assume nothing a priori.

creasing, and continuously differentiable[7] utility index function $u_i : \mathbb{R}_+ \to \mathbb{R}$, which together with P_i and Φ_i induces Bewley's incomplete preferences [4] or Gilboa-Schmeidler's maximin expected utility preferences [12]. Let $x_i, x'_i \in \mathbb{R}^\Omega_+$ be contingent consumption bundles. Bewley's incomplete preferences (BI-preferences for short) are determined as follows: in the ex ante stage, agent i prefers x_i to x'_i if and only if $E_p[u_i(x_i)] > E_p[u_i(x'_i)]$ for each $p \in P_i$, or equivalently, $E_{P_i}[u_i(x_i) - u_i(x'_i)] > 0$; in the interim stage, agent i with private information $\pi_i \in \Pi_i$ prefers x_i to x'_i if and only if $E_p[u_i(x_i)] > E_p[u_i(x'_i)]$ for each $p \in \Phi_i(\pi_i)$, or equivalently, $E_{\Phi_i(\pi_i)}[u_i(x_i) - u_i(x'_i)] > 0$. Gilboa-Schmeidler's maximin expected utility preferences (MEU-preferences for short) are determined as follows: in the ex ante stage, agent i prefers x_i to x'_i if and only if $E_{P_i}[u_i(x_i)] > E_{P_i}[u_i(x'_i)]$; in the interim stage, agent i with private information $\pi_i \in \Pi_i$ prefers x_i to x'_i if and only if $E_{\Phi_i(\pi_i)}[u_i(x_i)] > E_{\Phi_i(\pi_i)}[u_i(x'_i)]$.

An allocation is $x = (x_1, \ldots, x_I) \in \mathbb{R}^{\Omega\times\mathcal{I}}_+$ where $x_i \in \mathbb{R}^\Omega_+$ is a contingent consumption bundle of agent $i \in \mathcal{I}$. To avoid cumbersome boundary arguments, we restrict our attention to an interior allocation $x \in \mathbb{R}^{\Omega\times\mathcal{I}}_{++}$ in the following analysis. We call $t = (t_1, \ldots, t_I) \in \mathbb{R}^{\Omega\times\mathcal{I}}$ a feasible trade at an allocation x if $\sum_{i\in\mathcal{I}} t_i = 0$ and $x + t$ is also an allocation.[8] We say that an allocation x is *ex ante efficient* if there is no feasible trade t at x such that, in the ex ante stage, agent i prefers $x_i + t_i$ to x_i for each $i \in \mathcal{I}$. We say that an allocation x is *interim efficient* if there is no feasible trade t at x such that, in the interim stage, agent i with any private information $\pi_i \in \Pi_i$ prefers $x_i + t_i$ to x_i for each $i \in \mathcal{I}$. Note that these concepts of efficiency are defined for both BI-preferences and MEU-preferences.

3 Efficiency with BI-preferences

In this section, we assume that all agents have BI-preferences. We first review the characterization of ex ante efficient allocations due to Bewley [3] and Rigotti and Shannon [23].

[7] A similar analysis can be done with continuity only; by concavity, u_i has the right derivative everywhere, and replace u'_i with the right derivative.

[8] Alternatively, one could start with fixed total endowments $\bar{e}$ and define a feasible allocation x to satisfy $\sum_{i\in\mathcal{I}} x_i = \bar{e}$.

For each $i \in \mathcal{I}$, $x_i \in \mathbb{R}^{\Omega}_{++}$, and $P \in \mathcal{P}$, let

$$\Xi_i(P, x_i) = \left\{ \left(\frac{p(\omega) u_i'(x_i(\omega))}{\sum_{\omega' \in \Omega} p(\omega') u_i'(x_i(\omega'))} \right)_{\omega \in \Omega} : p \in P \right\}, \tag{1}$$

which is the set of marginal-utility weighted priors at the bundle x_i. If $u_i(\cdot)$ is linear or $x_i(\cdot)$ is constant, then $u_i'(x_i(\cdot))$ is constant, and thus $\Xi_i(P, x_i) = P$.

Bewley [3] and Rigotti and Shannon [23] have established the following result.

Proposition 1 *Assume that all agents have BI-preferences. An interior allocation $x \in \mathbb{R}^{\Omega \times \mathcal{I}}_{++}$ is ex ante efficient if and only if*

$$\bigcap_{i \in \mathcal{I}} \Xi_i(P_i, x_i) \neq \emptyset. \tag{2}$$

To appreciate Proposition 1, recall that the fundamental theorems of welfare economics assert that efficiency is equivalent to the existence of a common supporting vector of agents' upper contour sets. It can be shown that $\Xi_i(P_i, x_i)$ is the set of all normalized supporting vectors of the upper contour set $\{t_i \in \mathbb{R}^{\Omega} : E_{P_i}[u_i(x_i + t_i) - u_i(x_i)] > 0\}$. Thus, (2) is equivalent to the existence of a common supporting vector. In this sense, Proposition 1 is essentially the fundamental theorems of welfare economics.[9]

We use Proposition 1 to characterize interim efficient allocations. The key concept in our characterization is a special set of probability distributions over Ω derived from Φ_i, which is defined as follows. For each $i \in \mathcal{I}$, $p \in \Delta(\Omega)$ is said to be a Φ_i-*compatible prior* if $p(\cdot|\pi_i) \in \Phi_i(\pi_i)$ for each $\pi_i \in \Pi_i$ with $p(\pi_i) > 0$ where $p(\cdot|\pi_i)$ is the conditional probability distribution of p given π_i, i.e., $p(E|\pi_i) = p(E \cap \pi_i)/p(\pi_i)$ for each $E \in 2^{\Omega}$. Let P_i^* be the collection of all Φ_i-compatible priors, which is our key concept. We refer to P_i^* as the Φ_i-*compatible prior set*.

Note that $p \in P_i^*$ if and only if there exists $q \in \Delta(\Omega)$ and $r(\cdot|\pi_i) \in \Phi_i(\pi_i)$ for each $\pi_i \in \Pi_i$ such that $p = \sum_{\pi_i \in \Pi_i} q(\pi_i) r(\cdot|\pi_i)$. Thus, we can write

$$P_i^* = \bigcup_{q \in \Delta(\Omega)} \sum_{\pi_i \in \Pi_i} q(\pi_i) \Phi_i(\pi_i), \tag{3}$$

where $\sum_{\pi_i \in \Pi_i} q(\pi_i)\Phi_i(\pi_i) = \left\{ \sum_{\pi_i \in \Pi_i} q(\pi_i) r(\cdot|\pi_i) : r(\cdot|\pi_i) \in \Phi_i(\pi_i) \right\}$. In addition, P_i^* is equal to the convex hull of $\bigcup_{\pi_i \in \Pi_i} \Phi_i(\pi_i)$, i.e.,

$$P_i^* = \mathrm{co}\Big(\bigcup_{\pi_i \in \Pi_i} \Phi_i(\pi_i) \Big), \tag{4}$$

[9]This point has also been discussed by Dana [5, 6] and Rigotti et al. [24] among others.

by (3) and the following result in convex analysis.[10]

Lemma 1 *Let $\{C_k\}_{k=1}^m$ be a collection of non-empty convex sets in $\mathbb{R}^n$. Then,*

$$\bigcup_{\sum_k \lambda_k=1,\, \lambda_k \geq 0} \Big(\sum_{k=1}^m \lambda_k C_k\Big) = \mathrm{co}\Big(\bigcup_{k=1}^m C_k\Big).$$

Therefore, P_i^* is non-empty, convex, and closed, i.e., $P_i^* \in \mathcal{P}$, since $\Phi_i(\pi_i)$ is non-empty, convex, and closed for each $\pi_i \in \Pi_i$. The expression (4) results in the following lemma.

Lemma 2 *For any $z \in \mathbb{R}^\Omega$, it holds that $E_{P_i^*}[z] = \min_{\pi_i \in \Pi_i} E_{\Phi_i(\pi_i)}[z]$.*

Proof. By (4), $E_{P_i^*}[z] = \min_{p \in P_i^*} E_p[z] = \min_{p \in \mathrm{co}(\bigcup_{\pi_i \in \Pi_i} \Phi_i(\pi_i))} E_p[z]$. Since the minimum $\min_{p \in \mathrm{co}(\bigcup_{\pi_i \in \Pi_i} \Phi_i(\pi_i))} E_p[z]$ is attained at an extreme point of $\mathrm{co}(\bigcup_{\pi_i \in \Pi_i} \Phi_i(\pi_i))$,

$$\min_{p \in \mathrm{co}(\bigcup_{\pi_i \in \Pi_i} \Phi_i(\pi_i))} E_p[z] = \min_{p \in \bigcup_{\pi_i \in \Pi_i} \Phi_i(\pi_i)} E_p[z] = \min_{\pi_i \in \Pi_i} \min_{p \in \Phi_i(\pi_i)} E_p[z] = \min_{\pi_i \in \Pi_i} E_{\Phi_i(\pi_i)}[z]. \blacksquare$$

By Lemma 2, a characterization of interim efficient allocations can be reduced to that of ex ante efficient allocations with respect to a set of priors P_i^*, which is "fictitious" in the sense that P_i^* may be different from the "true" set of priors P_i. This leads us to the following main result of this paper.

Proposition 2 *Assume that all agents have BI-preferences. An interior allocation $x \in \mathbb{R}_{++}^{\Omega \times \mathcal{I}}$ is interim efficient if and only if*

$$\bigcap_{i \in \mathcal{I}} \Xi_i(P_i^*, x_i) \neq \emptyset. \tag{5}$$

Proof. By Proposition 1, (5) holds if and only if there is no feasible trade t at x such that $E_{P_i^*}[u_i(x_i+t_i)-u_i(x_i)] > 0$ for each $i \in \mathcal{I}$. Setting $z = (u_i(x_i(\omega)+t_i(\omega))-u_i(x_i(\omega)))_{\omega \in \Omega} \in \mathbb{R}^\Omega$ in Lemma 2, we see that this is true if and only if there is no feasible trade t at x such that $E_{\Phi_i(\pi_i)}[u_i(x_i+t_i) - u_i(x_i)] > 0$ for each $\pi_i \in \Pi_i$ and $i \in \mathcal{I}$. Therefore, (5) holds if and only if x is interim efficient. $\blacksquare$

[10]See Theorem 3.3 of Rockafellar [25], for example.

For example, suppose that u_i is linear for each $i \in \mathcal{I}$. Then, $\Xi_i(P_i^*, x_i) = P_i^*$ for each $x_i \in \mathbb{R}^{\Omega}_{++}$ and $i \in \mathcal{I}$, and thus the condition (5) is reduced to $\bigcap_{i\in\mathcal{I}} P_i^* \neq \emptyset$. Especially, when $\Phi_i(\pi_i)$ is a singleton for each $\pi_i \in \Pi_i$, $\bigcap_{i\in\mathcal{I}} P_i^* \neq \emptyset$ is equivalent to the existence of a fictitious common prior $p \in \Delta(\Omega)$ such that $\Phi_i(\pi_i) = \{p(\cdot|\pi_i)\}$ for each $\pi_i \in \Pi_i$ with $p(\pi_i) > 0$ and $i \in \mathcal{I}$. In this case, Proposition 2 says that the existence of a fictitious common prior is necessary and sufficient for interim efficiency, which is the result obtained by Morris [21] and Feinberg [11].

We use Proposition 1 and Proposition 2 to study the possibility of purely speculative trade. Recall that interim efficiency of an allocation implies non-existence of a trade to which all agents agree in the interim stage under any realization of private information. Thus, if any ex ante efficient allocation is interim efficient, speculative trade is impossible. In the standard Bayesian models, any ex ante efficient allocation is automatically interim efficient as shown by Milgrom and Stokey [20], which we refer to as the no trade theorem in this paper. The following result provides a necessary and sufficient condition for the no trade theorem to hold in our model. Since we do not assume any updating rule yet, the condition is stated in terms of the relationship between $P_1, \ldots, P_I$ and $\Phi_1, \ldots, \Phi_I$. We omit a proof because it is a direct consequence of Proposition 1 and Proposition 2.

Corollary 3 *Assume that all agents have BI-preferences. The following two conditions are equivalent:* (i) *any ex ante efficient allocation* $x \in \mathbb{R}^{\Omega\times\mathcal{I}}_{++}$ *is interim efficient;* (ii) *for* $x \in \mathbb{R}^{\Omega\times\mathcal{I}}_{++}$, $\bigcap_{i\in\mathcal{I}} \Xi_i(P_i, x_i) \neq \emptyset$ *implies* $\bigcap_{i\in\mathcal{I}} \Xi_i(P_i^*, x_i) \neq \emptyset$.

Since P_i^* is derived from Φ_i, the condition (ii) above is a requirement for the relationship between $P_1, \ldots, P_I$ and $\Phi_1, \ldots, \Phi_I$. Thus, it is interesting to ask if the condition (ii) is satisfied for a given updating rule of multiple priors, which induces Φ_i from P_i. Note that, in the standard Bayesian models, the condition (ii) is always true. We study this question for two popular updating rules of multiple priors in the remainder of this section.[11]

One updating rule of multiple priors is the full Bayesian updating rule. We say that Φ_i is the *full Bayesian updating* (FB-updating for short) of P_i if

$$\Phi_i(\pi_i) = \mathrm{cl}\{p(\cdot|\pi_i) : p \in P_i \text{ with } p(\pi_i) > 0\} \text{ for each } \pi_i \in \Pi_i,$$

[11]See Gilboa and Schmeidler [13] for the study of updating rules, for example.

where $\mathrm{cl}\, P$ denotes the closure of $P \subseteq \Delta(\Omega)$. The FB-updating is the collection of all conditional probability distributions of the priors in P_i. It can be checked that $\Phi_i(\pi_i)$ is a non-empty,[12] convex, and closed subset of $\Delta(\Omega)$ and thus $\Phi_i(\pi_i) \in \mathcal{P}$. Observe that Φ_i is the FB-updating of the Φ_i-compatible prior set P_i^*. Furthermore, P_i^* is maximal in the following sense.

Lemma 3 *If Φ_i is the FB-updating of P_i, then $P_i \subseteq P_i^*$.*

Proof. Recall that P_i^* consists of all probability distributions over Ω of the form $p = \sum_{\pi_i \in \Pi_i} q(\pi_i) r(\cdot|\pi_i)$ where $q \in \Delta(\Omega)$ and $r(\cdot|\pi_i) \in \Phi_i(\pi_i)$. If Φ_i is the FB-updating of P_i, then, for any $p \in P_i$, $p = \sum_{\pi_i \in \Pi_i} p(\pi_i) r(\cdot|\pi_i)$ with $r(\cdot|\pi_i) \in \Phi_i(\pi_i)$ and $r(\cdot|\pi_i) = p(\cdot|\pi_i)$ if $p(\pi_i) > 0$. This implies that $p \in P_i^*$ and thus $P_i \subseteq P_i^*$. ∎

Corollary 3 and Lemma 3 implies that any ex ante efficient allocation is interim efficient if all agents adopt the FB-updating.

Proposition 4 *Assume that all agents have BI-preferences and that Φ_i is the FB-updating of P_i for each $i \in \mathcal{I}$. Then, any ex ante efficient allocation $x \in \mathbb{R}_{++}^{\Omega \times \mathcal{I}}$ is interim efficient.*

Proof. By Lemma 3, $P_i \subseteq P_i^*$ for each $i \in \mathcal{I}$. This implies that $\Xi_i(P_i, x_i) \subseteq \Xi_i(P_i^*, x_i)$ for each $x_i \in \mathbb{R}_{++}^{\Omega}$. Thus, if $\bigcap_{i \in \mathcal{I}} \Xi_i(P_i, x_i) \neq \emptyset$, then $\bigcap_{i \in \mathcal{I}} \Xi_i(P_i^*, x_i) \neq \emptyset$. Therefore, by Corollary 3, any ex ante efficient allocation is interim efficient. ∎

Especially, suppose that u_i is linear and that Φ_i is the FB-updating of P_i for each $i \in \mathcal{I}$. If agents' prior sets are common, i.e., $P_i = P_j$ for all $i, j \in \mathcal{I}$, then any interior allocation is interim efficient by Proposition 1 and Proposition 4. The converse is not necessarily true. In the next example, though there is no common prior set P such that Φ_i is the FB-updating of P for each $i \in \mathcal{I}$, any interior allocation is interim efficient. This is in a sharp contrast with the standard Bayesian models where $\Phi_i(\pi_i)$ is a singleton for each π_i and i, where $\bigcap_{i \in \mathcal{I}} P_i^* \neq \emptyset$ implies the existence of $p \in \Delta(\Omega)$ such that Φ_i is the FB-updating of $\{p\}$ for each $i \in \mathcal{I}$.

[12] Recall that, by the assumption on P_i, $\max_{p \in P_i} p(\pi_i) > 0$ for each $\pi_i \in \Pi_i$.

Example 1 Let $\Omega = \{1, 2, 3, 4\}$ and $\mathcal{I} = \{1, 2\}$. For agent 1, let $\Pi_1 = \{\{1, 2\}, \{3, 4\}\}$, $P_1 = \{p \in \Delta(\Omega) : p(1) = p(3) = 1/6,\ p(2) \geq 1/6,\ p(4) \geq 1/6\}$, $\Phi_1(\{1, 2\}) = \{p \in \Delta(\Omega) : p(1)+p(2) = 1,\ 1/4 \leq p(1) \leq 1/2\}$, and $\Phi_1(\{3, 4\}) = \{p \in \Delta(\Omega) : p(3)+p(4) = 1,\ 1/4 \leq p(3) \leq 1/2\}$. For agent 2, let $\Pi_2 = \{\Omega\}$ and $P_2 = \Phi_2(\Omega) = \{(1/8, 3/8, 1/8, 3/8)\}$. It can be checked that Φ_i is the FB-updating of P_i for each i. Note that $P_1^* = \{(\alpha q, \alpha(1 - q), (1 - \alpha)r, (1 - \alpha)(1 - r)) \in \Delta(\Omega) : q, r \in [1/4, 1/2],\ \alpha \in [0, 1]\}$ and $P_2^* = \Phi_2(\Omega) = \{(1/8, 3/8, 1/8, 3/8)\}$. Let u_i be linear for each i. Then, by Proposition 2, any interior allocation is interim efficient because $P_1^* \cap P_2^* = P_2^* \neq \emptyset$. On the other hand, by Proposition 1, no interior allocation is ex ante efficient because $P_1 \cap P_2 = \emptyset$. Moreover, in this example, there is no common prior set $P \in \mathcal{P}$ such that Φ_i is the FB-updating of P for each i. In fact, if Φ_1 is the FB-updating of P, then P cannot be a singleton, and if Φ_2 is the FB-updating of P, then $P = \Phi_2(\Omega)$, which is a singleton. Thus, P cannot be a common prior set which induces both Φ_1 and Φ_2.

Another updating rule of multiple priors is the maximum likelihood updating rule. We say that Φ_i is the *maximum likelihood updating* (ML-updating for short) of P_i if

$$\Phi_i(\pi_i) = \mathrm{cl}\{p(\cdot|\pi_i) : p \in \arg\max_{p' \in P_i} p'(\pi_i)\} \text{ for each } \pi_i \in \Pi_i.$$

The ML-updating is the collection of all conditional probability distributions of the priors in P_i that maximize the likelihood of the observed private information. As the next example of the ML-updating shows, an ex ante efficient allocation is not necessarily interim efficient even if agents have a common prior set, which is different from the result for the FB-updating.

Example 2 Let Ω, $\mathcal{I}$, and Π_i be those given in Example 1 for each $i \in \mathcal{I}$. Let $P_1 = P_2 = \{p \in \Delta(\Omega) : p(1) = p(3) = 1/6,\ p(2) \geq 1/6,\ p(4) \geq 1/6\}$, $\Phi_1(\{1, 2\}) = \{(1/4, 3/4, 0, 0)\}$, $\Phi_1(\{3, 4\}) = \{(0, 0, 1/4, 3/4)\}$, and $\Phi_2(\Omega) = P_2$. It can be checked that Φ_i is the ML-updating of P_i for each i. Note that $P_1^* = \{(\alpha/4, 3\alpha/4, (1-\alpha)/4, 3(1-\alpha)/4) : \alpha \in [0, 1]\}$ and $P_2^* = P_2$. Let u_i be linear for each i. By Proposition 1, any interior allocation is ex ante efficient since $P_1 = P_2$, and by Proposition 2, no interior allocation is interim efficient since $P_1^* \cap P_2^* = \emptyset$.

4 Efficiency with MEU-preferences

In this section, we assume that all agents have MEU-preferences. We first review and generalize the characterization of ex ante efficient allocations due to Billot et al. [2].

For a set of priors $P_i \in \mathcal{P}$ and a contingent consumption bundle $x_i \in \mathbb{R}^{\Omega}_{+}$ of agent $i \in \mathcal{I}$, we call $p \in P_i$ an active prior of agent i in P_i at x_i if p minimizes $E_p[u_i(x_i)]$ over P_i, i.e., $E_p[u_i(x_i)] = E_{P_i}[u_i(x_i)]$. Let $P_i(x_i)$ be the collection of all active priors of agent i in P_i at x_i; that is,

$$P_i(x_i) = \arg\min_{p \in P_i} E_p[u_i(x_i)].$$

Since $E_p[u_i(x_i)]$ is linear in $p \in P_i$ and $P_i \in \mathcal{P}$ is non-empty, convex, and closed, $P_i(x_i)$ is also non-empty, convex, and closed, i.e., $P_i(x_i) \in \mathcal{P}$. Note that if $x_i(\cdot)$ is constant, then $E_p[u_i(x_i)]$ is also constant over $p \in \Delta(\Omega)$ and thus $P_i(x_i) = P_i$.

The following proposition characterizes ex ante efficient allocations.

Proposition 5 *Assume that all agents have MEU-preferences. An interior allocation $x \in \mathbb{R}^{\Omega\times\mathcal{I}}_{++}$ is ex ante efficient if and only if*

$$\bigcap_{i\in\mathcal{I}} \Xi_i(P_i(x_i), x_i) \neq \emptyset. \tag{6}$$

As Proposition 1 is essentially the fundamental theorems of welfare economics, so is Proposition 5 where $\Xi_i(P_i(x_i), x_i)$ is the set of all normalized supporting vectors of the upper contour set $\{t_i \in \mathbb{R}^{\Omega} : E_{P_i}[u_i(x_i + t_i)] > E_{P_i}[u_i(x_i)]\}$. Billot et al. [2] showed a special case[13] of Proposition 5 in which an interior allocation $x \in \mathbb{R}^{\Omega\times\mathcal{I}}_{++}$ has the *full insurance property*, i.e., $x_i(\cdot)$ is constant for each $i \in \mathcal{I}$. In this case, every prior is active, by which the condition (6) is reduced to $\bigcap_{i\in\mathcal{I}} \Xi_i(P_i(x_i), x_i) = \bigcap_{i\in\mathcal{I}} \Xi_i(P_i, x_i) = \bigcap_{i\in\mathcal{I}} P_i \neq \emptyset$, which is the condition Billot et al. [2] found. In the context of asset pricing models with a representative agent (with MEU-preferences), it is well known that active priors determine the supporting vectors [8, 10]. Proposition 5 is a natural consequence of this, and can be readily proved. We give a sketch of a proof in the appendix.

Let $P_i^*(x_i) = \arg\min_{p\in P_i^*} E_p[u_i(x_i)]$ denote the sets of all active Φ_i-compatible priors. By replacing $P_i(x_i)$ with $P_i^*(x_i)$ in (6), we obtain the following characterization of interim efficient allocations.

[13] The state space of Billot et al. [2] is a general measurable space.

Proposition 6 *Assume that all agents have MEU-preferences. An interior allocation* $x \in \mathbb{R}^{\Omega \times \mathcal{I}}_{++}$ *is interim efficient if*

$$\bigcap_{i \in \mathcal{I}} \Xi_i(P_i^*(x_i), x_i) \neq \emptyset. \tag{7}$$

If an interior allocation $x \in \mathbb{R}^{\Omega \times \mathcal{I}}_{++}$ *is interim efficient and the interim expected utility* $E_{\Phi_i(\pi_i)}[u_i(x_i)]$ *is constant over* $\pi_i \in \Pi_i$ *for each* $i \in \mathcal{I}$*, then* (7) *holds.*

We give a proof of this proposition later. The reader may wonder if the condition (7) is also necessary for interim efficiency in general. But it is not the case. In fact, in some cases, an allocation is interim efficient even if (7) is not true, as shown in the next proposition. We say that an allocation $x \in \mathbb{R}^{\Omega \times \mathcal{I}}_{+}$ has the *full insurance property in the interim stage* if, for each $\pi_i \in \Pi_i$ and each $i \in \mathcal{I}$, the restriction of $x_i(\cdot)$ to π_i is constant. Clearly, if x has the full insurance property, then it has it in the interim stage, but not vice versa.

Proposition 7 *Assume that all agents have MEU-preferences. Let an interior allocation* $x \in \mathbb{R}^{\Omega \times \mathcal{I}}_{++}$ *have the full insurance property in the interim stage. Then,* x *is interim efficient if and only if*

$$\bigcap_{i \in \mathcal{I}} \Xi_i(P_i^*, x_i) \neq \emptyset. \tag{8}$$

A proof is supplied later in this section. Note that, since $P_i^*(x_i) \subsetneq P_i^*$ generically, (8) is strictly weaker than (7). This shows that the condition (7) is not necessary for interim efficiency.

To understand the role of the assumptions on allocations in Proposition 6 and Proposition 7, consider the following example.

Example 3 Let Ω, $\mathcal{I}$, and Π_i be those given in Example 1 for each $i \in \mathcal{I}$. Let P_1 and Φ_1 be those given in Example 1 and set $P_2 = \Phi_2(\Omega) = P_1$. Note that Φ_i is the FB-updating of P_i for each $i \in \mathcal{I}$. Let $u_i(c) = c$ for each $c \in \mathbb{R}_+$ and $i \in \mathcal{I}$.

Suppose that $x_1 = (1, 3, 1, 3)$ and $x_2 = (3, 1, 3, 1)$. Note that $E_{\Phi_1(\{1,2\})}[u_1(x_1)] = E_{\Phi_1(\{3,4\})}[u_1(x_1)] = 2$ and thus $E_{\Phi_i(\pi_i)}[u_i(x_i)]$ is constant over $\pi_i \in \Pi_i$ for each $i \in \mathcal{I}$. We have $P_1(x_1) = P_1$, $P_2(x_2) = P_2$, $P_1^*(x_1) = \{(\alpha/2, \alpha/2, (1-\alpha)/2, (1-\alpha)/2) \in \Delta(\Omega) : \alpha \in [0, 1]\}$, and $P_2^*(x_2) = P_2(x_2) = P_2$. Thus, $P_1(x_1) \cap P_2(x_2) \neq \emptyset$ and $P_1^*(x_1) \cap P_2^*(x_2) = \emptyset$.

By Proposition 5, x is ex ante efficient, and by Proposition 6, x is not interim efficient. Thus, the no trade theorem fails under the FB-updating.

Suppose that $x_1 = (1, 1, 3, 3)$ and $x_2 = (1, 1, 1, 1)$. Note that x has the full insurance property in the interim stage. We have $P_1(x_1) = \{(1/6, 1/2, 1/6, 1/6)\}$, $P_2(x_2) = P_2$, $P_1^*(x_1) = \{(q, 1-q, 0, 0) : q \in [1/4, 1/2]\}$, and $P_2^*(x_2) = P_2(x_2) = P_2$. Thus, $P_1(x_1) \cap P_2(x_2) \neq \emptyset$, $P_1^*(x_1) \cap P_2^*(x_2) = \emptyset$, and $P_1^* \cap P_2^* \neq \emptyset$. By Proposition 5, x is ex ante efficient, and by Proposition 7, x is interim efficient. So this is an instance where (8) is strictly weaker than (7).

We can use Proposition 6 and Proposition 7 to study the possibility of speculative trade for agents with MEU-preferences analogously to the case of BI-preferences. The following corollaries are the counter parts of Corollary 3 in the previous section. We omit proofs because they are direct consequences of Proposition 5, Proposition 6, and Proposition 7.

Corollary 8 *Assume that all agents have MEU-preferences. Let an interior allocation $x \in \mathbb{R}_{++}^{\Omega \times \mathcal{I}}$ be such that the interim expected utility $E_{\Phi_i(\pi_i)}[u_i(x_i)]$ is constant over $\pi_i \in \Pi_i$ for each $i \in \mathcal{I}$. The following two conditions are equivalent:* (i) *if x is ex ante efficient, then it is interim efficient;* (ii) *if $\bigcap_{i \in \mathcal{I}} \Xi_i(P_i(x_i), x_i) \neq \emptyset$, then $\bigcap_{i \in \mathcal{I}} \Xi_i(P_i^*(x_i), x_i) \neq \emptyset$.*

Corollary 9 *Assume that all agents have MEU-preferences. Let an interior allocation $x \in \mathbb{R}_{++}^{\Omega \times \mathcal{I}}$ have the full insurance property in the interim stage. The following two conditions are equivalent:* (i) *if x is ex ante efficient, then it is interim efficient;* (ii) *if $\bigcap_{i \in \mathcal{I}} \Xi_i(P_i(x_i), x_i) \neq \emptyset$, then $\bigcap_{i \in \mathcal{I}} \Xi_i(P_i^*, x_i) \neq \emptyset$.*

Interestingly enough, it turns out that the FB-updating satisfies the condition (ii) in Corollary 9. Consequently, we have the following result, which is a counterpart of Proposition 4.

Proposition 10 *Assume that all agents have MEU-preferences and that Φ_i is the FB-updating of P_i for each $i \in \mathcal{I}$. Let an interior allocation $x \in \mathbb{R}_{++}^{\Omega \times \mathcal{I}}$ have the full insurance property in the interim stage. If x is ex ante efficient, then it is interim efficient.*

Proof. Since P_i is the FB-updating of Φ_i, $P_i(x_i) \subseteq P_i \subseteq P_i^*$ by Lemma 3. Thus, $\bigcap_{i \in \mathcal{I}} \Xi_i(P_i(x_i), x_i) \subseteq \bigcap_{i \in \mathcal{I}} \Xi_i(P_i^*, x_i)$, which implies (ii) in Corollary 9. ∎

In the remainder of this section, we prove Proposition 6 and Proposition 7. We first prove Proposition 6 using the following lemma.

Lemma 4 *For any $z, z' \in \mathbb{R}^\Omega$, if $E_{\Phi_i(\pi_i)}[z] > E_{\Phi_i(\pi_i)}[z']$ for each $\pi_i \in \Pi_i$, then $E_{P_i^*}[z] > E_{P_i^*}[z']$. Suppose that $E_{\Phi_i(\pi_i)}[z']$ is constant over $\pi_i \in \Pi_i$. Then, $E_{\Phi_i(\pi_i)}[z] > E_{\Phi_i(\pi_i)}[z']$ for each $\pi_i \in \Pi_i$ if and only if $E_{P_i^*}[z] > E_{P_i^*}[z']$.*

Proof. By Lemma 2, $E_{P_i^*}[z] = \min_{\pi_i \in \Pi_i} E_{\Phi_i(\pi_i)}[z]$ and $E_{P_i^*}[z'] = \min_{\pi_i \in \Pi_i} E_{\Phi_i(\pi_i)}[z']$. This implies that if $E_{\Phi_i(\pi_i)}[z] > E_{\Phi_i(\pi_i)}[z']$ for each $\pi_i \in \Pi_i$, then $E_{P_i^*}[z] > E_{P_i^*}[z']$.

Let $z' \in \mathbb{R}^\Omega$ satisfy $E_{\Phi_i(\pi_i)}[z'] = c \in \mathbb{R}$ for each $\pi_i \in \Pi_i$. Note that $E_{P_i^*}[z'] = \min_{\pi_i \in \Pi_i} E_{\Phi_i(\pi_i)}[z'] = c$. Suppose that $E_{\Phi_i(\pi_i')}[z] \leq c$ for some $\pi_i' \in \Pi_i$. Then, $E_{P_i^*}[z] = \min_{\pi_i \in \Pi_i} E_{\Phi_i(\pi_i)}[z] \leq E_{\Phi_i(\pi_i')}[z] \leq c = E_{P_i^*}[z']$. To summarize, if $E_{\Phi_i(\pi_i')}[z] \leq E_{\Phi_i(\pi_i')}[z']$ for some $\pi_i' \in \Pi_i$ then $E_{P_i^*}[z] \leq E_{P_i^*}[z']$. This implies that if $E_{P_i^*}[z] > E_{P_i^*}[z']$ then $E_{\Phi_i(\pi_i)}[z] > E_{\Phi_i(\pi_i)}[z']$ for each $\pi_i \in \Pi_i$. ∎

Proof of Proposition 6. Assume that (7) holds. Seeking a contradiction, suppose that x is not interim efficient. Then, there exists a feasible trade t at x such that $E_{\Phi_i(\pi_i)}[u_i(x_i + t_i)] > E_{\Phi_i(\pi_i)}[u_i(x_i)]$ for each $\pi_i \in \Pi_i$ and $i \in \mathcal{I}$. By Lemma 4, this implies that $E_{P_i^*}[u_i(x_i + t_i)] > E_{P_i^*}[u_i(x_i)]$ for each $i \in \mathcal{I}$. On the other hand, Proposition 5 implies that if (7) is true, then there is no feasible trade t at x such that $E_{P_i^*}[u_i(x_i + t_i)] > E_{P_i^*}[u_i(x_i)]$ for each $i \in \mathcal{I}$, a contradiction. Thus, the first half of the proposition is established.

To establish the second half, assume that x is interim efficient and $E_{\Phi_i(\pi_i)}[u_i(x_i)]$ is constant over $\pi_i \in \Pi_i$ for each $i \in \mathcal{I}$. It is enough to show that there is no feasible trade t at x such that $E_{P_i^*}[u_i(x_i + t_i)] > E_{P_i^*}[u_i(x_i)]$ for each $i \in \mathcal{I}$ because if this is true then (7) holds by Proposition 5. Seeking a contradiction, suppose otherwise and let t be a feasible trade at x such that $E_{P_i^*}[u_i(x_i + t_i)] > E_{P_i^*}[u_i(x_i)]$ for each $i \in \mathcal{I}$. Then, by Lemma 4, $E_{\Phi_i(\pi_i)}[u_i(x_i + t_i)] > E_{\Phi_i(\pi_i)}[u_i(x_i)]$ for each $\pi_i \in \Pi_i$, which contradicts to the interim efficiency of x. Thus, the second half of the proposition is established. ∎

In the proof of Proposition 7, the following lemma is essential.

Lemma 5 *Let $x_i, x_i' \in \mathbb{R}_+^\Omega$ be contingent consumption bundles of agent $i \in \mathcal{I}$. Fix $\pi_i \in \Pi_i$ and suppose that the restriction of $x_i(\cdot)$ to π_i is constant. Then, in the interim stage when agent i's private information is π_i, agent i with MEU-preferences prefers x_i'*

to x_i if and only if agent i with BI-preferences prefers x'_i to x_i; that is, $E_{\Phi_i(\pi_i)}[u_i(x'_i)] - E_{\Phi_i(\pi_i)}[u_i(x_i)] > 0$ if and only if $E_{\Phi_i(\pi_i)}[u_i(x'_i) - u_i(x_i)] > 0$.

Proof. Let $x_i(\omega) = c$ for each $\omega \in \pi_i$. Then,

$$\begin{aligned} &E_{\Phi_i(\pi_i)}[u_i(x'_i)] - E_{\Phi_i(\pi_i)}[u_i(x_i)] \\ &\qquad = E_{\Phi_i(\pi_i)}[u_i(x'_i)] - u_i(c) = E_{\Phi_i(\pi_i)}[u_i(x'_i) - u_i(c)] = E_{\Phi_i(\pi_i)}[u_i(x'_i) - u_i(x_i)]. \end{aligned}$$

Therefore, $E_{\Phi_i(\pi_i)}[u_i(x'_i)] - E_{\Phi_i(\pi_i)}[u_i(x_i)] > 0$ if and only if $E_{\Phi_i(\pi_i)}[u_i(x'_i) - u_i(x_i)] > 0$. ∎

Proof of Proposition 7. Let an interior allocation $x \in \mathbb{R}_{++}^{\Omega \times \mathcal{I}}$ have the full insurance property in the interim stage. By Lemma 5, for a feasible trade t at x, $E_{\Phi_i(\pi_i)}[u_i(x_i + t_i)] - E_{\Phi_i(\pi_i)}[u_i(x_i)] > 0$ if and only if $E_{\Phi_i(\pi_i)}[u_i(x_i + t_i) - u_i(x_i)] > 0$ for each $\pi_i \in \Pi_i$ and $i \in \mathcal{I}$. This implies that x is interim efficient with MEU-preferences if and only if it is interim efficient with BI-preferences. Therefore, by Proposition 2, x is interim efficient with MEU-preferences if and only if (8) holds. ∎

5 Discussions

5.1 Dynamic consistency

Proposition 4 have established the no trade theorem in the multiple priors models with BI-preferences and FB-updating. As is well known, the no trade theorem in the standard Bayesian models is a consequence of dynamic consistency of agents' behavior. Thus, it is natural to ask whether Proposition 4 can be understood as a consequence of some kind of dynamic consistency. In this subsection, we provide an affirmative answer to this question.

Let $x_i, x'_i \in \mathbb{R}_+^\Omega$ be contingent consumption bundles. We define dynamic consistency as follows. Agent $i \in \mathcal{I}$ is said to be *dynamically consistent* if agent i prefers x_i to x'_i in the ex ante stage whenever agent i with every private information $\pi_i \in \Pi_i$ prefers x_i to x'_i in the interim stage. If every agent is dynamically consistent, then any ex ante efficient allocation is interim efficient.[14] The following lemma shows that agents with BI-preferences and FB-updating is dynamically consistent.

[14] If an allocation x is not interim efficient, there exists another allocation x' such that every player with every private information prefers x'_i to x_i in the interim stage. If agents are dynamically consistent, then every player prefers x'_i to x_i in the ex ante stage, implying that x is not ex ante efficient.

Lemma 6 *Assume that agent $i \in \mathcal{I}$ has BI-preferences and that Φ_i is the FB-updating of P_i. Then, agent i is dynamically consistent.*

Proof. Suppose that agent i with each $\pi_i \in \Pi_i$ prefers x_i to x_i' in the interim stage. Then, $E_{\Phi_i(\pi_i)}[u_i(x_i) - u_i(x_i')] > 0$ for each $\pi_i \in \Pi_i$. Since Φ_i is the FB-updating of P_i, for each $p \in P_i$ and $\pi_i \in \Pi_i$ with $p(\pi_i) > 0$,

$$\sum_{\omega \in \pi_i} \Big(u_i(x_i(\omega)) - u_i(x_i'(\omega)) \Big) p(\omega|\pi_i) \geq E_{\Phi_i(\pi_i)}[u_i(x_i) - u_i(x_i')] > 0.$$

This implies that

$$\begin{aligned} E_{P_i}[u_i(x_i) - u_i(x_i')] &= \min_{p \in P_i} \sum_{\omega \in \Omega} \Big(u_i(x_i(\omega)) - u_i(x_i'(\omega)) \Big) p(\omega) \\ &= \min_{p \in P_i} \sum_{\pi_i \in \Pi_i : p(\pi_i) > 0} \left(\sum_{\omega \in \pi_i} \Big(u_i(x_i(\omega)) - u_i(x_i'(\omega)) \Big) p(\omega|\pi_i) \right) p(\pi_i) > 0. \end{aligned}$$

Therefore, agent i prefers x_i to x_i' in the ex ante stage. ∎

On the other hand, agents with MEU-preferences are not necessarily dynamically consistent. Epstein and Schneider [9] and Wakai [27] identified a class of priors and posteriors with which agents are dynamically consistent. A set of priors $P_i \in \mathcal{P}$ is said to be a Φ_i-*rectangular prior set*[15] if

$$P_i = \bigcup_{p \in P_i} \sum_{\pi_i \in \Pi_i} p(\pi_i) \Phi_i(\pi_i). \tag{9}$$

Note that if P_i is a Φ_i-rectangular prior set, then Φ_i must be the FB-updating of P_i, but not vice versa; that is, if Φ_i is the FB-updating of P_i then

$$P_i \subseteq \bigcup_{p \in P_i} \sum_{\pi_i \in \Pi_i} p(\pi_i) \Phi_i(\pi_i)$$

holds where this set inclusion may be strict in general. As shown by Epstein and Schneider [9] and Wakai [27], agents with MEU-preferences and rectangular prior sets are dynamically consistent. Based upon this, Wakai [27] showed the following result.

[15]We adopt this term from Epstein and Schneider [9].

Proposition 11 *Assume that all agents have MEU-preferences and that P_i is a Φ_i-rectangular prior set for each $i \in \mathcal{I}$. Then, any ex ante efficient allocation $x \in \mathbb{R}^{\Omega\times\mathcal{I}}_{++}$ is interim efficient.*

Proposition 11 explains the first part of Proposition 6 because P_i^* is a Φ_i-rectangular prior set. The following result is immediate from (3) and (9).

Lemma 7 *The Φ_i-compatible prior set P_i^* is a Φ_i-rectangular prior set such that $P_i \subseteq P_i^*$ for any Φ_i-rectangular prior set P_i. That is, P_i^* is the largest Φ_i-rectangular prior set.*

Suppose that $P_i = P_i^*$ for each $i \in \mathcal{I}$. If an interior allocation $x \in \mathbb{R}^{\Omega\times\mathcal{I}}_{++}$ satisfies (7), then x is ex ante efficient by Proposition 5. Thus, dynamic consistency implies that it is also interim efficient, which corresponds to the first part of Proposition 6.

5.2 Conditional preferences

Up to this point, it has been assumed that posterior beliefs induce interim preferences. But there is a direct way to derive interim preferences from ex ante preferences without using posterior beliefs, which we refer to as conditional preferences. When general preferences are considered (that is, beliefs are not necessarily specified separately), conditional preferences are regarded as a natural candidate for interim preferences. In this subsection, we briefly discuss an implication of our results for the "conditional preferences" approach.

Let ex ante preferences of agent $i \in \mathcal{I}$ be given, which may be any preferences. For two contingent consumption bundles $x_i, x_i' \in \mathbb{R}^\Omega_+$ and an event $E \in 2^\Omega$, let $x_{iE}x_i' \in \mathbb{R}^\Omega_+$ be the contingent consumption bundle defined by $x_{iE}x_i'(\omega) = x_i(\omega)$ if $\omega \in E$ and $x_{iE}x_i'(\omega) = x_i'(\omega)$ otherwise. That is, agent i having $x_{iE}x_i'$ consumes x_i on E and x_i' on $\Omega\backslash E$. We say that agent i conditionally prefers x_i to x_i' on π_i if agent i prefers $x_{i\pi_i}x_i'$ to x_i' in the ex ante stage. Define interim preferences by the following rule: in the interim stage, agent i with private information $\pi_i \in \Pi_i$ prefers x_i to x_i' if and only if agent i conditionally prefers x_i to x_i' on π_i. We call this type of induced interim preferences *conditional preferences.* We say that an allocation is *conditionally efficient* if it is interim efficient with the understanding that interim preferences of all agents are conditional preferences.

By construction, conditional preferences are uniquely determined from ex ante preferences. On the other hand, given a set of priors, there is a variety of ways of specifying a set of posteriors in our setup. Thus in general, interim preferences induced by posterior beliefs do not necessarily coincide with conditional preferences. But in some cases, they do. An important example is an agent with BI-preferences and the FB-updating.

Lemma 8 *Assume that agent $i \in \mathcal{I}$ has BI-preferences and that Φ_i is the FB-updating of P_i. For $x_i, x'_i \in \mathbb{R}^\Omega_+$ and $\pi_i \in \Pi_i$, if agent i conditionally prefers x_i to x'_i on π_i, then agent i with private information π_i prefers x_i to x'_i in the interim stage. If $p(\pi_i) > 0$ for each $p \in P_i$ and agent i with private information π_i prefers x_i to x'_i in the interim stage, then agent i conditionally prefers x_i to x'_i on π_i.*

Proof. By definition, agent i conditionally prefers x_i to x'_i on π_i if and only if

$$\begin{aligned} E_{P_i}[u_i(x_i \pi_i x'_i) - u_i(x'_i)] &= \min_{p \in P_i} \sum_{\omega \in \Omega} p(\omega) \Big(u_i(x_i \pi_i x'_i(\omega)) - u_i(x'_i(\omega)) \Big) \\ &= \min_{p \in P_i} \sum_{\omega \in \pi_i} p(\omega) \Big(u_i(x_i(\omega)) - u_i(x'_i(\omega)) \Big) \\ &= \min_{p \in P_i} p(\pi_i) \sum_{\omega \in \pi_i} p(\omega|\pi_i) \Big(u_i(x_i(\omega)) - u_i(x'_i(\omega)) \Big) > 0. \end{aligned}$$

Since Φ_i is the FB-updating of P_i, the above inequality implies $E_{\Phi_i(\pi_i)}[u_i(x_i) - u_i(x'_i)] > 0$. Conversely, if $p(\pi_i) > 0$ for each $p \in P_i$, then $E_{\Phi_i(\pi_i)}[u_i(x_i) - u_i(x'_i)] > 0$ implies the above inequality. ∎

Using Lemma 8, we can translate our results for agents with BI-preferences and the FB-updating into those for agents with conditional BI-preferences. To see this, assume that all agents have BI-preferences and adopt the FB-updating. By Lemma 8, any interim efficient allocation is conditionally efficient. Thus, by Proposition 4, any ex ante efficient allocation is conditionally efficient.

Ma [19] and Halevy [15] considered general *complete* ex ante preferences and studied under what condition any ex ante efficient allocation is conditionally efficient. A sufficient condition given by Ma [19] and Halevy [15] is essentially the same as the weak decomposability axiom introduced by Grant et al. [14]. As shown by Grant et al. [14], weak decomposability is equivalent to dynamic consistency in the sense defined in the previous subsection with the understanding that interim preferences are replaced with conditional preferences. This implies that, if all agents have weakly decomposable ex ante preferences, then any ex ante efficient allocation is conditionally efficient, which is essentially the "no trade theorem" of Ma [19] and Halevy [15]. Although these works assume complete preferences, careful reading reveals that the completeness assumption is not essential in their arguments. In fact, one can show that ex ante BI-preferences are weakly decomposable, which is consistent with the above discussion on conditional efficiency with ex ante BI-preferences.

5.3 The agreement theorem

Proposition 2 is related to the agreement theorem of Aumann [1] because the agreement theorem suggests that the existence of a common prior is sufficient for interim efficiency in the standard Bayesian models. In fact, as a corollary of Proposition 2, we can obtain a multiple priors version of the agreement theorem.

Corollary 12 *Suppose that* $\bigcap_{i\in\mathcal{I}} P_i^* \neq \emptyset$. *Fix an event* $E \in 2^\Omega$. *If, for each* $i \in \mathcal{I}$, $\underline{p}_i \equiv \min_{p\in\Phi_i(\pi_i)} p(E)$ *and* $\overline{p}_i \equiv \max_{p\in\Phi_i(\pi_i)} p(E)$ *are constant over all* $\pi_i \in \Pi_i$, *then* $\underline{p}_i \leq \overline{p}_j$ *for all* $i, j \in \mathcal{I}$.

Proof. Suppose on the contrary that $\overline{p}_j < \underline{p}_i$ for some $i, j \in \mathcal{I}$ with $i \neq j$. We can choose $c_i, c_j \in \mathbb{R}_{++}$ such that $\overline{p}_j < c_j < c_i < \underline{p}_i$. Let $\delta_E \in \mathbb{R}^\Omega$ be the indicator function of $E \in 2^\Omega$, i.e., $\delta_E(\omega) = 1$ if $\omega \in E$ and $\delta_E(\omega) = 0$ otherwise. Note that $\underline{p}_i = E_{\Phi_i(\pi_i)}[\delta_E]$ and $\overline{p}_j = -E_{\Phi_j(\pi_j)}[-\delta_E]$ for each $\pi_i \in \Pi_i$ and $\pi_j \in \Pi_j$. Let a trade $t = (t_1, \ldots, t_I) \in \mathbb{R}^{\Omega\times\mathcal{I}}$ be such that $t_i = \delta_E - c_i + (c_i - c_j)/I$, $t_j = c_j - \delta_E + (c_i - c_j)/I$, and $t_k = (c_i - c_j)/I$ for $k \neq i, j$. Note that $\sum_{k\in\mathcal{I}} t_k = 0$ and $E_{\Phi_k(\pi_k)}[t_k] > 0$ for each $\pi_k \in \Pi_k$ and $k \in \mathcal{I}$. Now consider agents with linear utility index functions $u_k(c) = c$ for each $c \in \mathbb{R}_+$ and $k \in \mathcal{I}$. Then, for any interior allocation $x \in \mathcal{R}_{++}^{\Omega\times\mathcal{I}}$ at which t is a feasible trade, $E_{\Phi_k(\pi_k)}[u_k(x_k + t_k)] \geq E_{\Phi_k(\pi_k)}[x_k] + E_{\Phi_k(\pi_k)}[t_k] > E_{\Phi_k(\pi_k)}[x_k] = E_{\Phi_k(\pi_k)}[u_k(x_k)]$ for each $\pi_k \in \Pi_k$ and $k \in \mathcal{I}$, which implies that x is not interim efficient. This contradicts to $\bigcap_{k\in\mathcal{I}} P_k^* \neq \emptyset$ by Proposition 2. ∎

Note that if $\Phi_i(\pi_i)$ is a singleton for each $\pi_i \in \Pi_i$ and $i \in \mathcal{I}$, then $\underline{p}_i = \overline{p}_i$. In this case, Corollary 12 says that if $\bigcap_{i\in\mathcal{I}} P_i^* \neq \emptyset$ and all agents' posterior probabilities of $E \in 2^\Omega$ are constant over all $\omega \in \Omega$, then they must coincide, which is the agreement theorem of

Aumann [1].[16]

Appendix: proof of Proposition 5

We use the fundamental theorems of welfare economics of the following form.

Lemma A *Let $U_i : \mathbb{R}^{\Omega}_{+} \to \mathbb{R}$ be a continuous, strictly increasing, concave utility function. Let $x \in \mathbb{R}^{\Omega\times\mathcal{I}}_{++}$ be an interior allocation. There is no feasible trade $t \in \mathbb{R}^{\Omega\times\mathcal{I}}$ at x such that $U_i(x_i + t_i) > U_i(x_i)$ for each $i \in \mathcal{I}$ if and only if there exists a price vector $q \in \mathbb{R}^{\Omega}$ with $q \neq 0$ such that, for each $i \in \mathcal{I}$, $q \cdot (x'_i - x_i) \geq 0$ for all $x'_i \in \mathbb{R}^{\Omega}_{+}$ with $U_i(x'_i) \geq U_i(x_i)$.*

We start with checking that a price vector q is parallel to a supergradient of U_i. For a concave function $f : \mathbb{R}^n \to \mathbb{R}$, a vector $s \in \mathbb{R}^n$ is a *supergradient* of f at $x \in \mathbb{R}^n$ if

$$f(y) \leq f(x) + s \cdot (y - x) \text{ for all } y \in \mathbb{R}^n.$$

If f is differentiable at x, then a supergradient is nothing but the gradient. The *superdifferential* of f at x, denoted by $\partial f(x)$, is the collection of all supergradients at x:

$$\partial f(x) = \{s \in \mathbb{R}^n : f(y) \leq f(x) + s \cdot (y - x) \text{ for all } y \in \mathbb{R}^n\}.$$

By the next lemma, a price vector is parallel to a supergradient of a utility function (see Theorem 1.3.5 of Lesson D in Hiriart-Urruty and Lemaréchal [16]).

Lemma B *Let $f : \mathbb{R}^n \to \mathbb{R}$ be a concave function and suppose $0 \notin \partial f(x)$. Then, $q \cdot (y - x) \geq 0$ for all $y \in \mathbb{R}^n$ with $f(y) \geq f(x)$ if and only if $q = \lambda s$ for some $\lambda \geq 0$ and $s \in \partial f(x)$.*

Let $U_i : \mathbb{R}^{\Omega}_{+} \to \mathbb{R}$ be such that $U_i(x_i) = E_{P_i}(u_i(x_i))$ for each $x_i \in \mathbb{R}^{\Omega}_{+}$ and $i \in \mathcal{I}$. It is straightforward to check that U_i is continuous, strictly increasing, and concave. Thus, by Lemma A and Lemma B, an interior allocation $x \in \mathbb{R}^{\Omega\times\mathcal{I}}_{++}$ is ex ante efficient if and

[16] Kajii and Ui [18] considered a special case of Corollary 12 assuming that posteriors are the FB-updating of priors.

only if

$$\bigcap_{i\in\mathcal{I}}\{q\in\mathbb{R}^{\Omega}: q\cdot(x_i'-x_i)\geq 0 \text{ for all } x_i'\in\mathbb{R}^{\Omega}_{+} \text{ with } U_i(x_i')\geq U_i(x_i)\}$$
$$=\bigcap_{i\in\mathcal{I}}\{q\in\mathbb{R}^{\Omega}: q=\lambda s \text{ for } s\in\partial U_i(x_i),\ \lambda\geq 0\}\neq\{0\}.$$

The above turns out to be equivalent to (6). To see this, we evaluate $\partial U_i(x_i)$ using the following lemma (see Theorem 4.4.2 of Lesson D in Hiriart-Urruty and Lemaréchal [16]).

Lemma C *Let J be a compact set in some metric space, and $\{f_j\}_{j\in J}$ be a collection of concave functions from $\mathbb{R}^n$ to $\mathbb{R}$ such that functions $j\mapsto f_j(x)$ are lower semi-continuous (in the metric space topology) for each $x\in\mathbb{R}^n$. Define a function $f:\mathbb{R}^n\to\mathbb{R}$ such that*

$$f(x)=\inf_{j\in J} f_j(x),$$

and let $J(x)=\{j\in J: f_j(x)=f(x)\}$. Assume that $f(x)>-\infty$ for all $x\in\mathbb{R}^n$. Then, $\partial f(x)$ is a convex hull of $\bigcup_{j\in J(x)}\partial f_j(x)$, i.e.,

$$\partial f(x)=\mathrm{co}\Big(\bigcup_{j\in J(x)}\partial f_j(x)\Big).$$

By Lemma C,

$$\begin{aligned}\partial U_i(x_i)&=\partial E_{P_i}(x_i)\\&=\partial\Big(\min_{p\in P_i}\sum_{\omega\in\Omega}p(\omega)u_i(x_i(\omega))\Big)\\&=\mathrm{co}\Big(\bigcup_{p\in P_i(x_i)}\partial\Big(\sum_{\omega\in\Omega}p(\omega)u_i(x_i(\omega))\Big)\Big)\\&=\mathrm{co}\Big(\bigcup_{p\in P_i(x_i)}\{(p(\omega)u_i'(x_i(\omega)))_{\omega\in\Omega}\}\Big)\\&=\{(p(\omega)u_i'(x_i(\omega)))_{\omega\in\Omega}: p\in P_i(x_i)\},\end{aligned}$$

where the last equality holds since $P_i(x_i)$ is convex. Therefore, an interior allocation $x\in\mathbb{R}^{\Omega\times\mathcal{I}}_{++}$ is ex ante efficient if and only if

$$\bigcap_{i\in\mathcal{I}}\{q\in\mathbb{R}^{\Omega}: q=\lambda s \text{ for } s\in\partial U_i(x_i),\ \lambda\geq 0\}$$
$$=\bigcap_{i\in\mathcal{I}}\{\lambda(p(\omega)u_i'(x_i(\omega)))_{\omega\in\Omega}\in\mathbb{R}^{\Omega}: p\in P_i(x_i),\ \lambda\geq 0\}\neq\{0\}.$$

This is true if and only if

$$\bigcap_{i\in\mathcal{I}} \left\{ \left(\frac{p(\omega)u_i'(x_i(\omega))}{\sum_{\omega'\in\Omega} p(\omega')u_i'(x_i(\omega'))} \right)_{\omega\in\Omega} : p \in P_i(x_i) \right\} \neq \emptyset,$$

which is (6).

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