Discrete Concavity for Potential Games*

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Abstract

This paper proposes a discrete analogue of concavity appropriate for potential games with discrete strategy sets. It guarantees that every Nash equilibrium maximizes a potential function.
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1 Introduction

Consider a potential game (Monderer and Shapley, 1996) with a concave potential function where strategy sets are closed intervals of the real line. The first-order condition for a Nash equilibrium coincides with that for a potential maximizer, and it guarantees global optimality of the potential function. Thus, a strategy profile is a Nash equilibrium if and only if it maximizes the potential function (cf. Radner, 1962; Neyman, 1997). Now restrict strategies to the integers and consider a restricted potential game. In a naive sense, the restricted potential function can be seen as concave, but some Nash equilibrium may not maximize it.

For example, consider a two-player potential game with $u_1(x_1, x_2) = u_2(x_1, x_2) = v(x_1, x_2) = -x_1^2 - x_2^2 + x_1x_2 + x_1/3 + x_2/2$ where u_1 and u_2 are payoff functions, v is a potential function, and the set of strategies is a closed interval $[0, 1]$ for each player. It can be readily shown that v is strictly concave, and that this game has a unique Nash equilibrium $(x_1, x_2) = (7/18, 4/9)$, which maximizes v over $[0, 1] \times [0, 1]$. By restricting strategies to $\{0, 1\}$ for each player, we have the following game.

	0	1
0	$0, 0$	$-2/3, -2/3$
1	$-1/2, -1/2$	$-1/6, -1/6$

The restricted game has two pure-strategy Nash equilibria $(x_1, x_2) = (0, 0)$ and $(x_1, x_2) = (1, 1)$; the former maximizes v over $\{0, 1\} \times \{0, 1\}$ and the latter does not.

The above observation leads us to the following question: what is a discrete analogue of concavity, or "discrete concavity" for short, appropriate for a potential game, which guarantees that every Nash equilibrium maximizes a potential function? The purpose of this paper is to give an answer to this question. In the area of discrete optimization, several different types of discrete concavity have been proposed (Miller, 1971; Favati and Tardella, 1990; Murota, 1996, 1998; Murota and Shioura, 1999; Fujishige and Murota, 2000). But they do not conform to our purpose.[1]

We introduce the notion of larger midpoint property (LMP) for a function defined on a discrete space, which is analogous to midpoint concavity. We show that if a function satisfies the LMP condition, then local optimality in a sense similar to that of a concave function implies global optimality. If a potential function satisfies the LMP condition, then a Nash equilibrium maximizes the potential function because a Nash equilibrium is a local maximum. If a potential maximizer is unique, so is the Nash equilibrium.

[1]This does not imply that they are irrelevant in economics. Some discrete concavity notions are useful. See Tamura (2004) for a survey.

2 Discrete concavity

Let $N = \{1, \ldots, n\}$ be a finite set. A function $f : \mathbb{R}^N \to \mathbb{R}$ satisfies midpoint concavity if $f((x+y)/2) \geq (f(x) + f(y))/2$ for $x, y \in \mathbb{R}^N$. We consider a similar property for a function defined on a discrete space $X \subseteq \mathbb{Z}^N$ where $X = \prod_{i \in N} X_i$, $X_i = \{x_i \in \mathbb{Z} : \underline{x}_i \leq x_i \leq \overline{x}_i\} \subseteq \mathbb{Z}$, and $\underline{x}_i, \overline{x}_i \in \mathbb{Z} \cup \{-\infty, +\infty\}$. Let $\|x\| = \sum_{i \in N} |x_i|$ be the ℓ_1-norm of a vector $x \in \mathbb{Z}^N$.

We say that a function $f : X \to \mathbb{R}$ satisfies the *larger midpoint property* (LMP) if, for any $x, y \in X$ with $\|x - y\| = 2$,

$$\max_{z \in X : \|x-z\| = \|y-z\| = 1} f(z) \geq t f(x) + (1-t) f(y) \ (\exists t \in (0,1)). \tag{1}$$

Note that, in defining LMP, we postulate that the midpoint of $x, y \in X$ with $\|x - y\| = 2$ is $z \in X$ satisfying $\|x - z\| = \|y - z\| = 1$.

For example, a separable concave function satisfies LMP. A separable concave function is defined as $f : X \to \mathbb{R}$ of the form $f(x) = \sum_{i \in N} f_i(x_i)$ where $f_i(x_i) \geq (f_i(x_i - 1) + f_i(x_i + 1))/2$ for all $x_i \neq \underline{x}_i, \overline{x}_i$. Let $x, y \in X$ be such that $\|x - y\| = 2$. If $|x_i - y_i| = 2$ for some $i \in N$, then the definition immediately implies (1). If $|x_i - y_i| = |x_j - y_j| = 1$ for some $i \neq j$, then

$$\begin{aligned}\max_{z \in X : \|x-z\| = \|y-z\| = 1} f(z) &= \max\{f_i(x_i) + f_j(y_j), f_i(y_i) + f_j(x_j)\} + \sum_{k \neq i,j} f_k(x_k) \\ &\geq \frac{f_i(x_i) + f_j(x_j) + f_i(y_i) + f_j(y_j)}{2} + \sum_{k \neq i,j} f_k(x_k) = \frac{f(x) + f(y)}{2}.\end{aligned}$$

Thus, f satisfies LMP.

The following lemma restates the LMP condition.

Lemma 1 *A function $f : X \to \mathbb{R}$ satisfies LMP if and only if, for any $x, y \in X$ with $\|x - y\| = 2$,*

$$\max_{z \in X : \|x-z\| = \|y-z\| = 1} f(z) \begin{cases} > \min\{f(x), f(y)\} & \text{if } f(x) \neq f(y), \\ \geq f(x) = f(y) & \text{otherwise.} \end{cases} \tag{2}$$

Proof. If $f(x) = f(y)$ then $f(x) = f(y) = t f(x) + (1-t) f(y)$ and thus (1) and (2) are equivalent. Suppose that $f(x) > f(y)$. If (1) is true, then $t f(x) + (1-t) f(y) > f(y)$ for all $t \in (0,1)$ and thus (2) is true. If (2) is true, then we can choose sufficiently small $t > 0$ such that $t f(x) + (1-t) f(y)$ is sufficiently close to $f(y)$ and thus (1) is true. Therefore, (1) and (2) are equivalent. ■

If $f : X \to \mathbb{R}$ satisfies LMP, then local optimality implies global optimality.[2]

Proposition 1 *Suppose that $f : X \to \mathbb{R}$ satisfies LMP. Then, $f(x) \geq f(y)$ for all $y \in X$ with $\|x - y\| \leq 1$ if and only if $f(x) \geq f(y)$ for all $y \in X$.*

Proof. The "if" part is obvious and we show the "only if" part. Let $x \in X$ satisfy $f(x) \geq f(y)$ for all $y \in X$ with $\|x - y\| \leq 1$. For $y \in X$ with $d \equiv \|x - y\| \geq 2$, construct a sequence $\{x^k \in X\}_{k=0}^d$ such that $x^0 = x$ and $x^d = y$ by the following steps: for $0 \leq k \leq d-1$, choose

$$x^{k+1} \in \arg \max_{\|x^k - z\| = 1,\, \|y - z\| = d-k-1} f(z).$$

Note that $\|x^k - x^{k+l}\| = l$ for all $0 \leq k \leq d$ and $0 \leq l \leq d-k$.

Let $z \in X$ be such that $\|x^k - z\| = \|x^{k+2} - z\| = 1$. Then $d - k = \|x^k - y\| = \|x^k - z + z - y\| \leq \|x^k - z\| + \|z - y\| = 1 + \|z - y\|$ and $\|z - y\| = \|z - x^{k+2} + x^{k+2} - y\| \leq \|z - x^{k+2}\| + \|x^{k+2} - y\| = d - k - 1$. Thus, $\|z - y\| = d - k - 1$. This implies that

$$f(x^{k+1}) = \max_{\|x^k - z\| = 1,\, \|y - z\| = d-k-1} f(z) \geq \max_{\|x^k - z\| = \|x^{k+2} - z\| = 1} f(z).$$

Since f satisfies LMP, it must be true that, for $0 \leq k \leq d-2$,

$$f(x^{k+1}) \begin{cases} > \min\{f(x^k), f(x^{k+2})\} \text{ if } f(x^k) \neq f(x^{k+2}), \\ \geq f(x^k) = f(x^{k+2}) \text{ otherwise} \end{cases}$$

by Lemma 1. Using the above property, we show that $f(x^k) \geq f(x^{k+1})$ for all $0 \leq k \leq d-1$. Since $\|x - x^1\| = 1$, $f(x) = f(x^0) \geq f(x^1)$. Suppose that $f(x^k) \geq f(x^{k+1})$ for some $0 \leq k \leq d-2$. If $f(x^k) \neq f(x^{k+2})$, then $f(x^{k+1}) > \min\{f(x^k), f(x^{k+2})\} = f(x^{k+2})$ since $f(x^k) \geq f(x^{k+1})$. If $f(x^k) = f(x^{k+2})$, then $f(x^{k+1}) \geq f(x^k) = f(x^{k+2})$. By induction, $f(x) = f(x^0) \geq f(x^1) \geq \cdots \geq f(x^{d-1}) \geq f(x^d) = f(y)$. ■

In the area of discrete optimization, several types of discrete concavity have been proposed, which do not imply LMP and are not implied by LMP. Functions satisfying discrete concavity have the property that local optimality implies global optimality. But

[2] Note that a global optimum in this proposition is not necessarily unique. Even if we replace a weak inequality in (1) with a strict one, a global optimum may not be unique. For example, let $n = 1$ and $X_1 = \{0, 1, 2\}$. Then, $f : X \to \mathbb{R}$ with $f(0) = 0$, $f(1) = 1$, and $f(2) = 1$ satisfies (1) with a strict inequality but a global optimum is not unique.

the meaning of local optimality is different from that in Proposition 1 and thus $x \in X$ may not maximize f even if $f(x) \geq f(y)$ for all $y \in X$ with $\|x - y\| \leq 1$.[3]

Miller (1971) was a forerunner to study discrete concavity. For $x \in \mathbb{R}^N$, let $N(x) = \{z \in \mathbb{Z}^N : \lfloor x \rfloor \leq z \leq \lceil x \rceil\}$ where $\lfloor x \rfloor$ denotes the vector obtained by rounding down and $\lceil x \rceil$ by rounding up the components of x to the nearest integers. Miller (1971) called $f : X \to \mathbb{R}$ a discretely-concave function if, for any $x, y \in X$, it holds that

$$\max_{z \in N(tx+(1-t)y)} f(z) \geq tf(x) + (1-t)f(y) \ (\forall t \in [0,1]). \tag{3}$$

Miller (1971) showed that if $f(x) \geq f(x + d)$ for all $d \in \{-1, 0, 1\}^N$, then a discretely-concave function f achieves its maximum at x. Favati and Tardella (1990) introduced integrally-concave functions and showed that the same local optimality condition guarantees global optimality of integrally-concave functions because these functions form a special class of discretely-concave functions.

A discretely-concave function does not necessarily satisfy LMP. For example, it can be readily checked that the restricted potential function $v : \{0,1\} \times \{0,1\} \to \mathbb{R}$ discussed in the introduction is a discretely-concave function, while $\max\{v(0,1), v(1,0)\} = -1/2$ and $\min\{v(0,0), v(1,1)\} = -1/6$, which violates (2) for $x = (0,0)$ and $y = (1,1)$. Also, a function satisfying LMP is not necessarily a discretely-concave function. For example, let $N = \{1,2\}$, $X_1 = \{0,1,2\}$, and $X_2 = \{0,1\}$. Define $f : X \to \mathbb{R}$ such that $f(0,0) = 10$, $f(0,1) = 2$, $f(1,0) = 2$, $f(1,1) = 1$, $f(2,0) = 1$, and $f(2,1) = 0$. It can be readily checked that f satisfies LMP, while $\max_{z \in N(0.5(0,0)+0.5(2,1))} f(z) = 2$ and $0.5f(0,0) + 0.5f(2,1) = 5$, which violates (3) for $x = (0,0)$ and $y = (2,1)$.

Recently, Murota (1998) advocates "discrete convex analysis," where L-convex functions (Murota, 1998) and M-convex functions (Murota, 1996) play central roles. L$^\natural$-concave functions and M$^\natural$-concave functions,[4] introduced respectively by Fujishige and Murota (2000) and Murota and Shioura (1999), are variants of L-concave functions and M-concave functions. The following results are shown in these papers. If $f(x) \geq f(x \pm d)$ for all $d \in \{0,1\}^N$, then an L-concave or L$^\natural$-concave function f achieves its maximum at x. If $f(x) \geq f(x + d)$ for all $d \in \{e^k - e^l : k, l \in N\} \cup \{\pm e^k : k \in N\}$ where $e^k = (e^k_i)_{i \in N} \in X$ is such that $e^i_i = 1$ and $e^k_i = 0$ for $k \neq i$, then an M-concave or M$^\natural$-concave function f achieves its maximum at x.[5]

[3] Corollary 2.2 of Altman *et al.* (2000) states that if $f(x) \geq f(y)$ for all $y \in X$ with $\|x - y\| \leq 1$, then a multimodular function f achieves its maximum at x. However, Murota (2005) finds a counterexample against it and provides a correct local optimality condition.

[4] M$^\natural$-concavity is found to be very useful in studying economies with indivisible goods. See Danilov *et al.* (2001).

[5] The local optimality conditions for L-concave and M-concave functions can be relaxed.

Note that these local optimality conditions are different from that in Proposition 1. It can also be readily shown that any of these types of discrete concavity do not imply LMP and are not implied by LMP. For the relationship among various types of discrete concavity, see Murota and Shioura (2001) and Murota (2003). See also Ui (2006) who provides a unified framework to understand the relationship between various types of discrete concavity and local optimality.

3 An application to potential games

A game consists of a set of players $N = \{1, \dots, n\}$, a set of strategies X_i for $i \in N$, and a payoff function $u_i : X \to \mathbb{R}$ for $i \in N$. Simply denote a game by $\mathbf{u} = (u_i)_{i \in N}$. We write $X_{-i} = \prod_{j \neq i} X_j$ and $x_{-i} = (x_j)_{j \neq i} \in X_{-i}$. A strategy profile $x \in X$ is a Nash equilibrium of $\mathbf{u}$ if $u_i(x_i, x_{-i}) \geq u_i(x'_i, x_{-i})$ for all $x'_i \in X_i$ and $i \in N$.

Monderer and Shapley (1996) introduced potential games and ordinal potential games. A game $\mathbf{u}$ is a *potential game* with a *potential function* $v : X \to \mathbb{R}$ provided $u_i(x_i, x_{-i}) - u_i(x'_i, x_{-i}) = v(x_i, x_{-i}) - v(x'_i, x_{-i})$ for all $x_i, x'_i \in X_i$, $x_{-i} \in X_{-i}$, and $i \in N$. A game $\mathbf{u}$ is an *ordinal potential game* with an *ordinal potential function* $v : X \to \mathbb{R}$ provided $u_i(x_i, x_{-i}) - u_i(x'_i, x_{-i}) > 0$ if and only if $v(x_i, x_{-i}) - v(x'_i, x_{-i}) > 0$ for all $x_i, x'_i \in X_i$, $x_{-i} \in X_{-i}$, and $i \in N$. Clearly, a potential game is an ordinal potential game. It is straightforward to check that an ordinal potential game $\mathbf{u}$ with an ordinal potential function v satisfies

$$\arg\max_{x_i \in X_i} u_i(x_i, x_{-i}) = \arg\max_{x_i \in X_i} v(x_i, x_{-i}) \text{ for all } x_{-i} \in X_{-i} \text{ and } i \in N. \tag{4}$$

Voorneveld (2000) calls a game $\mathbf{u}$ a *best-response potential game* with a *best-response potential function* $v : X \to \mathbb{R}$ if $\mathbf{u}$ and v satisfies (4). The condition (4) implies the following lemma (Monderer and Shapley, 1996; Voorneveld, 2000).

Lemma 2 *Let $\mathbf{u}$ be a best-response potential game with a best-response potential function v. If $x \in X$ maximizes v, then it is a Nash equilibrium.*

The converse of the above lemma is not necessarily true. The LMP condition provides a sufficient condition for any Nash equilibrium to be a best-response potential maximizer, which is an immediate consequence of Proposition 1.

Proposition 2 *Let $\mathbf{u}$ be a best-response potential game with a best-response potential function v. Suppose that v satisfies LMP. Then, $x \in X$ maximizes v if and only if it is a Nash equilibrium.*

Proof. The "only if" part is true by Lemma 2. We show the "if" part. Let $x \in X$ be a Nash equilibrium. Then $x_i \in \arg\max_{x_i' \in X_i} u_i(x_i', x_{-i})$ for all $i \in N$, which together with (4) implies that $x_i \in \arg\max_{x_i' \in X_i} v(x_i', x_{-i})$ for all $i \in N$. For $y \in X$ with $\|x - y\| = 1$, there exists $i \in N$ such that $y = (y_i, x_{-i})$ and thus $v(x) \geq v(y_i, x_{-i}) = v(y)$. By Proposition 1, $v(x) \geq v(y)$ for all $y \in X$. ■

Remark 1 A similar claim does not hold if $\mathbf{u}$ is a pseudo-potential game introduced by Dubey *et al.* (2006).[6] A game $\mathbf{u}$ is a pseudo-potential game with a pseudo-potential function $v : X \to \mathbb{R}$ provided $\arg\max_{x_i \in X_i} u_i(x_i, x_{-i}) \supseteq \arg\max_{x_i \in X_i} v(x_i, x_{-i})$ for all $x_{-i} \in X_{-i}$ and $i \in N$. Different from a best-response potential game, a pseudo-potential game allows the existence of a Nash equilibrium $x \in X$ such that $x_i \notin \arg\max_{x_i' \in X_i} v(x_i', x_{-i})$ for some $i \in N$. Such a Nash equilibrium does not maximize v even if v satisfies LMP. Thus, the LMP condition does not provide a sufficient condition for any Nash equilibrium to be a pseudo-potential maximizer.

Remark 2 Consider a potential game with a concave and continuously differentiable potential function where strategy sets are closed intervals of the real line. Neyman (1997) showed that a correlated equilibrium of this game is a mixture of potential maximizers. This result is a consequence of both concavity and differentiability of a potential function. It is an open question whether a similar claim is true in the discrete case.

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[6]For a characterization of pseudo-potential games, see Schipper (2004).

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