Correlated Equilibrium and Concave Games*

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Abstract

This paper shows that if a game satisfies the sufficient condition for the existence and uniqueness of a pure-strategy Nash equilibrium provided by Rosen (1965) then the game has a unique correlated equilibrium, which places probability one on the unique pure-strategy Nash equilibrium. In addition, it shows that a weaker condition suffices for the uniqueness of a correlated equilibrium. The condition generalizes the sufficient condition for the uniqueness of a correlated equilibrium provided by Neyman (1997) for a potential game with a strictly concave potential function.

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1 Introduction

This paper explores conditions for uniqueness of a correlated equilibrium (Aumann, 1974, 1987) in a class of games where strategy sets are finite-dimensional convex sets and each player's payoff function is concave and continuously differentiable with respect to the player's own strategy. Liu (1996) showed that a Cournot oligopoly game with a linear demand function has a unique correlated equilibrium. Neyman (1997) studied a correlated equilibrium of a potential game (Monderer and Shapley, 1996) and showed that if a potential function is concave and payoff functions are bounded then any correlated equilibrium is a mixture of potential maximizers in Theorem 1 and that if a potential function is strictly concave and strategy sets are compact then the potential game has a unique correlated equilibrium, which places probability one on the unique potential maximizer, in Theorem 2. The latter, which is derived from the former, generalizes the result of Liu (1996) because a Cournot oligopoly game with a linear demand function is a potential game with a strictly concave potential function (Slade, 1994).

We study a correlated equilibrium of a class of games examined by Rosen (1965). For a given game, consider a vector each component of which is a partial derivative of each player's payoff function with respect to the player's own strategy and call it the payoff gradient of the game. The payoff gradient is said to be strictly monotone if the inner product of a difference of arbitrary two strategy profiles and the corresponding difference of the payoff gradient is strictly negative. Strict monotonicity of the payoff gradient implies strict concavity of each player's payoff function with respect to the player's own strategy. Theorem 2 of Rosen (1965) showed that if the payoff gradient is strictly monotone and strategy sets are compact then the game has a unique pure-strategy Nash equilibrium. The present paper shows that, under the same condition, the game has a unique correlated equilibrium, which places probability one on the unique pure-strategy Nash equilibrium. In addition, our main result (Proposition 5) shows that a weaker condition suffices for the uniqueness of a correlated equilibrium. This result generalizes Theorem 2 of Neyman (1997) because the payoff gradient of a potential game with a strictly concave potential function is strictly monotone.

To establish the main result, we first provide a sufficient condition for any correlated equilibrium to be a mixture of pure-strategy Nash equilibria, which differs from but overlaps with Theorem 1 of Neyman (1997). We then show that if the payoff gradient is strictly monotone and strategy sets are compact then the game satisfies the sufficient condition and thus any correlated equilibrium must place probability one on the unique pure-strategy Nash equilibrium.

The organization of this paper is as follows. Preliminary definitions and results are summarized in section 2. The concept of strict monotonicity for the payoff gradient is introduced in section 3. The results are reported in section 4.

2 Preliminaries

A game consists of a set of players $N = \{1, \ldots, n\}$, a measurable set of strategies $X_i \subseteq \mathbb{R}^{m_i}$ for $i \in N$ with a generic element $x_i = (x_{i1}, \ldots, x_{im_i})^\top$, and a measurable payoff function $u_i : X \to \mathbb{R}$ for $i \in N$ where $X = \prod_{i \in N} X_i$. It is assumed that X_i is a full-dimensional convex subset[1] of an Euclidean space $\mathbb{R}^{m_i}$. We write $X_{-i} = \prod_{j \neq i} X_j$ and $x_{-i} = (x_j)_{j \neq i} \in X_{-i}$. We will fix N and X throughout this paper and simply denote a game by $\mathbf{u} = (u_i)_{i \in N}$.

A pure-strategy Nash equilibrium of $\mathbf{u}$ is a strategy profile $x^* \in X$ such that, for all $x_i \in X_i$ and $i \in N$, $u_i(x^*) \geq u_i(x_i, x^*_{-i})$. A correlated equilibrium[2] of $\mathbf{u}$ is a probability distribution μ over X such that, for each $i \in N$ and any measurable function $\xi_i : X_i \to X_i$,

$$\int u_i(x) d\mu(x) \geq \int u_i(\xi_i(x_i), x_{-i}) d\mu(x).$$

A game $\mathbf{u}$ is a *smooth game* if, for each $i \in N$, $u_i(x)$ has continuous partial derivatives with respect to the components of x_i. In a smooth game $\mathbf{u}$, the first-order condition for a pure-strategy Nash equilibrium $x^* \in X$ is

$$\begin{aligned}\lim_{t \to +0} & \frac{u_i(x_i^* + t(x_i - x_i^*), x^*_{-i}) - u_i(x^*)}{t} \\ &= \nabla_i u_i(x^*)^\top (x_i - x_i^*) \leq 0 \text{ for all } x_i \in X_i \text{ and } i \in N \qquad (1)\end{aligned}$$

where $\nabla_i u_i = (\partial u_i / \partial x_{i1}, \ldots, \partial u_i / \partial x_{im_i})^\top$ denotes the gradient of $u_i(x)$ with respect to x_i. It is straightforward to check that (1) is equivalent to

$$\sum_{i \in N} \nabla_i u_i(x^*)^\top (x_i - x_i^*) \leq 0 \text{ for all } x \in X. \qquad (2)$$

[1] Even if X_i is not full-dimensional, we can use a reparametrization to get to the full-dimensional case.

[2] A generalized definition of a correlated equilibrium for an infinite game is proposed by Hart and Schmeidler (1989).

The problem of solving this type of an inequality is called the variational inequality problem[3] and the following sufficient condition for the existence of a solution is well-known.[4]

Lemma 1 *Let* $\mathbf{u}$ *be a smooth game. If* X_i *is compact for all* $i \in N$ *then there exists* $x^* \in X$ *satisfying* (2).

A game $\mathbf{u}$ is a *concave game* (Rosen, 1965) if, for each $i \in N$, $u_i(x)$ is concave in x_i for every fixed $x_{-i} \in X_{-i}$. It can be readily shown that if $\mathbf{u}$ is a smooth concave game then the first-order condition (1) is necessary and sufficient for a pure-strategy Nash equilibrium and thus the set of solutions to the inequality problem (2) coincides with the set of pure-strategy Nash equilibria.[5]

A game $\mathbf{u}$ is a *potential game* (Monderer and Shapley, 1996) if there exists a *potential function* $f : X \to \mathbb{R}$ such that $u_i(x_i, x_{-i}) - u_i(x'_i, x_{-i}) = f(x_i, x_{-i}) - f(x'_i, x_{-i})$ for all $x_i, x'_i \in X_i$, $x_{-i} \in X_{-i}$, and $i \in N$. As shown by Monderer and Shapley (1996), a smooth game $\mathbf{u}$ is a potential game with a potential function f if and only if $\nabla_i u_i = \nabla_i f$ for all $i \in N$. This implies that the first-order condition for a pure-strategy Nash equilibrium and that for a potential maximizer $x^* \in \arg\max_{x \in X} f(x)$ coincide. From the equivalence of the two first-order conditions, we can derive the following lemma, which Neyman (1997) established, by noting that a smooth potential game with a concave potential function is a smooth concave game.[6]

Lemma 2 *In a smooth potential game with a concave potential function, a strategy profile is a pure-strategy Nash equilibrium if and only if it is a potential maximizer.*

Neyman (1997) studied a correlated equilibrium of a smooth potential game with a concave or strictly concave potential function and obtained the following two results,

[3]Let $S \subseteq \mathbb{R}^m$ be a convex set and let $F : S \to \mathbb{R}^m$ be a mapping. The variational inequality problem is to find $x^* \in S$ such that $F(x^*)^\top (x - x^*) \geq 0$ for all $x \in S$. It has been shown that a pure-strategy Nash equilibrium is a solution to the variational inequality problem with $F = (-\nabla_i u_i)_{i \in N}$ (cf. Hartman and Stampacchia, 1966; Gabay and Moulin, 1980).

[4]See Nagurney (1993), for example.

[5]Accordingly, a smooth concave game with compact strategy sets has a pure-strategy Nash equilibrium by Lemma 1, whereas Kakutani fixed point theorem directly shows that a concave game with compact strategy sets, which is not necessarily a smooth game, has a pure-strategy Nash equilibrium if $u_i(x)$ is continuous in x for all $i \in N$.

[6]If a potential function f is concave then $f(tx_i + (1-t)x'_i, x_{-i}) - f(x'_i, x_{-i}) \geq t(f(x_i, x_{-i}) - f(x'_i, x_{-i}))$. Hence $u_i(tx_i + (1-t)x'_i, x_{-i}) - u_i(x'_i, x_{-i}) \geq t(u_i(x_i, x_{-i}) - u_i(x'_i, x_{-i}))$, which implies that $u_i(x)$ is concave in x_i.

the latter of which is derived from the former. Neyman (1997) obtained Lemma 2 as a corollary of Proposition 1.

Proposition 1 *Let* $\mathbf{u}$ *be a smooth potential game with bounded payoff functions. If a potential function of* $\mathbf{u}$ *is concave then any correlated equilibrium of* $\mathbf{u}$ *is a mixture of potential maximizers.*

Proposition 2 *Let* $\mathbf{u}$ *be a smooth potential game with compact strategy sets. If a potential function of* $\mathbf{u}$ *is strictly concave then* $\mathbf{u}$ *has a unique correlated equilibrium, which places probability one on the unique potential maximizer.*

3 Strict monotonicity of the payoff gradient

Let $S \subseteq \mathbb{R}^m$ be a convex set and let $F : S \to \mathbb{R}^m$ be a mapping. A mapping F is said to be *strictly monotone* if $(F(x) - F(y))^\top (x - y) > 0$ for all $x, y \in S$ with $x \neq y$. The following sufficient condition for strict monotonicity is well-known.[7]

Lemma 3 *If a mapping* $F : S \to \mathbb{R}^m$ *is continuously differentiable and the Jacobian matrix of* F *is positive definite for all* $x \in S$ *then* F *is strictly monotone.*

Let us call $(\nabla_i u_i)_{i \in N}$ the payoff gradient of a smooth game $\mathbf{u}$. We say that, with some abuse of language, the payoff gradient of $\mathbf{u}$ is strictly monotone if a mapping $x \mapsto (-\nabla_i u_i(x))_{i \in N}$ is strictly monotone, i.e.,

$$\sum_{i \in N} (\nabla_i u_i(x) - \nabla_i u_i(y))^\top (x_i - y_i) < 0 \text{ for all } x, y \in X \text{ with } x \neq y. \tag{3}$$

Let $c_i \in \mathbb{R}_{++}$ be a constant for each $i \in N$ and call $(c_i \nabla_i u_i)_{i \in N}$ the c-weighted payoff gradient of $\mathbf{u}$ where $c = (c_i)_{i \in N}$. We say that the c-weighted payoff gradient of $\mathbf{u}$ is strictly monotone if a mapping $x \mapsto (-c_i \nabla_i u_i(x))_{i \in N}$ is strictly monotone,[8] i.e.,

$$\sum_{i \in N} c_i (\nabla_i u_i(x) - \nabla_i u_i(y))^\top (x_i - y_i) < 0 \text{ for all } x, y \in X \text{ with } x \neq y. \tag{4}$$

Note that if $c_i = c_j$ for all $i, j \in N$ then (4) implies (3).

[7]See Nagurney (1993), for example.

[8]Rosen (1965) called this property "diagonal strict concavity."

Let $\gamma_i : X_i \to \mathbb{R}_{++}$ be a function for each $i \in N$ and call $(\gamma_i \nabla_i u_i)_{i\in N}$ the γ-weighted payoff gradient of $\mathbf{u}$ where $\gamma = (\gamma_i)_{i\in N}$. We say that the γ-weighted payoff gradient of $\mathbf{u}$ is strictly monotone if a mapping $x \mapsto (-\gamma_i(x_i)\nabla_i u_i(x))_{i\in N}$ is strictly monotone, i.e.,

$$\sum_{i\in N}(\gamma_i(x_i)\nabla_i u_i(x) - \gamma_i(y_i)\nabla_i u_i(y))^\top (x_i - y_i) < 0 \text{ for all } x, y \in X \text{ with } x \neq y. \quad (5)$$

Note that if $\gamma_i(x_i) = c_i \in \mathbb{R}_{++}$ for all $x_i \in X_i$ and $i \in N$ then (5) implies (4).

Rosen (1965) showed that strict monotonicity of the c-weighted payoff gradient leads to the uniqueness of a pure-strategy Nash equilibrium.

Proposition 3 *Let* $\mathbf{u}$ *be a smooth game with compact strategy sets. If there exists a constant* $c_i \in \mathbb{R}_{++}$ *for each* $i \in N$ *such that the* c*-weighted payoff gradient of* $\mathbf{u}$ *is strictly monotone then* $\mathbf{u}$ *has a unique pure-strategy Nash equilibrium. Especially, if the payoff gradient of* $\mathbf{u}$ *is strictly monotone then* $\mathbf{u}$ *has a unique pure-strategy Nash equilibrium.*

In the next section, we shall show that strict monotonicity of the γ-weighted payoff gradient leads to the uniqueness of a correlated equilibrium.

Before closing this section, we discuss two implications of strict monotonicity.[9] In a smooth potential game, strict monotonicity of the payoff gradient is equivalent to strict concavity of a potential function.

Lemma 4 *Let* $\mathbf{u}$ *be a smooth potential game. A potential function of* $\mathbf{u}$ *is strictly concave if and only if the payoff gradient of* $\mathbf{u}$ *is strictly monotone.*

Proof. Let f be a potential function and suppose that f is strictly concave. For $x \neq y$, $\sum_{i\in N} \nabla_i f(x)^\top (y_i - x_i) > f(y) - f(x)$ and $\sum_{i\in N} \nabla_i f(y)^\top (x_i - y_i) > f(x) - f(y)$. Adding these two inequalities, we have

$$\sum_{i\in N}(\nabla_i f(x) - \nabla_i f(y))^\top (x_i - y_i) = \sum_{i\in N}(\nabla_i u_i(x) - \nabla_i u_i(y))^\top (x_i - y_i) < 0$$

since $\nabla_i f = \nabla_i u_i$, which implies that the payoff gradient of $\mathbf{u}$ is strictly monotone.

Conversely, suppose that the payoff gradient of $\mathbf{u}$ is strictly monotone. Fix $x, y \in X$ with $x \neq y$. Let $\phi(t) = f(x + t(y - x))$ for $t \in [0, 1]$. Then, ϕ is differentiable and, by

[9] I thank a referee for pointing out the next two lemmas with proofs.

the mean-value theorem, there exist $0 < \theta_1 < 1/2 < \theta_2 < 1$ such that $\phi(1/2) - \phi(0) = \phi'(\theta_1)/2$ and $\phi(1) - \phi(1/2) = \phi'(\theta_2)/2$, which are rewritten as

$$f((x+y)/2) - f(x) = \sum_{i \in N} \nabla_i f(x + \theta_1(y-x))^\top (y_i - x_i)/2, \tag{6}$$

$$f(y) - f((x+y)/2) = \sum_{i \in N} \nabla_i f(x + \theta_2(y-x))^\top (y_i - x_i)/2. \tag{7}$$

On the other hand, since the payoff gradient is strictly monotone,

$$\sum_{i \in N} (\nabla_i u_i(x + \theta_2(y-x)) - \nabla_i u_i(x + \theta_1(y-x)))^\top (\theta_2 - \theta_1)(y_i - x_i) < 0$$

and thus

$$\sum_{i \in N} \nabla_i f(x + \theta_2(y-x))^\top (y_i - x_i) < \sum_{i \in N} \nabla_i f(x + \theta_1(y-x))^\top (y_i - x_i)$$

because $\theta_2 - \theta_1 > 0$ and $\nabla_i u_i = \nabla_i f$. This inequality, (6), and (7) imply that $f((x+y)/2) > (f(x) + f(y))/2$. Therefore, f must be strictly concave by the continuity of f. ∎

In a smooth game, strict monotonicity of the c-weighted payoff gradient implies strict concavity of each player's payoff function with respect to the player's own strategy.

Lemma 5 *Let* $\mathbf{u}$ *be a smooth game. If there exists a constant* $c_i \in \mathbb{R}_{++}$ *for each* $i \in N$ *such that the* c*-weighted payoff gradient of* $\mathbf{u}$ *is strictly monotone then, for each* $i \in N$, $u_i(x)$ *is strictly concave in* x_i *for every fixed* $x_{-i} \in X_{-i}$.

Proof. Fix arbitrary $i \in N$ and $x_{-i} \in X_{-i}$ and consider a game with a singleton player set $\{i\}$, a strategy set X_i, and a payoff function $u_i(\cdot, x_{-i}) : X_i \to \mathbb{R}$. This game is trivially a potential game with a potential function $u_i(\cdot, x_{-i})$. Note that the payoff gradient of this game is strictly monotone. Thus, the potential function is strictly concave by Lemma 4. This implies that $u_i(x)$ is strictly concave in x_i. ∎

4 Results

We provide a sufficient condition for any correlated equilibrium to be a mixture of pure-strategy Nash equilibria.

Proposition 4 *Let* $\mathbf{u}$ *be a smooth game with bounded payoff functions. Assume that there exists a pure-strategy Nash equilibrium* $x^* \in X$ *and a bounded measurable function* $\gamma_i : X_i \to \mathbb{R}_{++}$ *for each* $i \in N$ *such that:*

(i) $$\sum_{i \in N} \gamma_i(x_i) \nabla_i u_i(x)^\top (x_i^* - x_i) \begin{cases} \geq 0 & \text{for all } x \in X, \\ > 0 & \text{if } x \text{ is not a pure-strategy Nash equilibrium,} \end{cases}$$

(ii) $$\inf_{(x,t) \in X \times (0,1]} \frac{u_i(x_i + t(x_i^* - x_i), x_{-i}) - u_i(x)}{t} > -\infty \text{ for all } i \in N.$$

Then, any correlated equilibrium of $\mathbf{u}$ *is a mixture of pure-strategy Nash equilibria.*

Proof. Let μ be a probability distribution over X such that $\mu(Y) > 0$ for some measurable set $Y \subseteq X$ containing no pure-strategy Nash equilibria. It is enough to show that μ is not a correlated equilibrium. By (i),

$$\int \sum_{i \in N} \gamma_i(x_i) \nabla_i u_i(x)^\top (x_i^* - x_i) d\mu(x) > 0.$$

Thus, there exists $i \in N$ such that

$$\int \gamma_i(x_i) \nabla_i u_i(x)^\top (x_i^* - x_i) d\mu(x) > 0.$$

By (ii), $\inf_{(x,t) \in X \times (0,1]} \gamma_i(x_i)(u_i(x_i + t(x_i^* - x_i), x_{-i}) - u_i(x))/t > -\infty$ since γ_i is bounded. Thus, by the Lebesgue-Fatou Lemma,

$$\begin{aligned} \liminf_{t \to +0} \int \gamma_i(x_i) & \frac{u_i(x_i + t(x_i^* - x_i), x_{-i}) - u_i(x)}{t} d\mu(x) \\ & \geq \int \liminf_{t \to +0} \gamma_i(x_i) \frac{u_i(x_i + t(x_i^* - x_i), x_{-i}) - u_i(x)}{t} d\mu(x) \\ & = \int \gamma_i(x_i) \nabla_i u_i(x)^\top (x_i^* - x_i) d\mu(x) > 0. \end{aligned}$$

Therefore, there exists $t > 0$ such that

$$\int \gamma_i(x_i) \big(u_i(x_i + t(x_i^* - x_i), x_{-i}) - u_i(x) \big) d\mu(x) > 0. \tag{8}$$

Set $\xi_i(x_i) = x_i + t(x_i^* - x_i)$ for all $x_i \in X_i$.

For a measurable function $f : X \to \mathbb{R}$, let $E_{\mu(x)}[f(x)|x_i]$ denote the conditional expected value of $f(x)$ given $x_i \in X_i$ with respect to μ. Define a measurable set

$$S_i = \{x_i \in X_i \mid E_{\mu(x)}[u_i(\xi_i(x_i), x_{-i}) - u_i(x)|x_i] \geq 0\}$$

and write $1_{S_i} : X_i \to \{0, 1\}$ for its indicator function. Let $\bar{\gamma}_i = \sup_{x_i \in S_i} \gamma_i(x_i) < \infty$. Then,

$$\begin{aligned} E_{\mu(x)}[1_{S_i}(x_i)(u_i(\xi_i(x_i), x_{-i}) - u_i(x))|x_i] &\geq \frac{\gamma_i(x_i)}{\bar{\gamma}_i} E_{\mu(x)}[1_{S_i}(x_i)(u_i(\xi_i(x_i), x_{-i}) - u_i(x))|x_i] \\ &\geq \frac{\gamma_i(x_i)}{\bar{\gamma}_i} E_{\mu(x)}[u_i(\xi_i(x_i), x_{-i}) - u_i(x)|x_i] \\ &= \frac{1}{\bar{\gamma}_i} E_{\mu(x)}[\gamma_i(x_i)(u_i(\xi_i(x_i), x_{-i}) - u_i(x))|x_i]. \end{aligned}$$

This and (8) imply that

$$\int 1_{S_i}(x_i)\big(u_i(\xi_i(x_i), x_{-i}) - u_i(x)\big) d\mu(x) \geq \frac{1}{\bar{\gamma}_i} \int \gamma_i(x_i)\big(u_i(\xi_i(x_i), x_{-i}) - u_i(x)\big) d\mu(x) > 0.$$

Let $\xi'_i : X_i \to X_i$ be such that $\xi'_i(x_i) = \xi_i(x_i)$ if $x_i \in S_i$ and $\xi'_i(x_i) = x_i$ otherwise. Then,

$$\int \big(u_i(\xi'_i(x_i), x_{-i}) - u_i(x)\big) d\mu(x) = \int 1_{S_i}(x_i)\big(u_i(\xi_i(x_i), x_{-i}) - u_i(x)\big) d\mu(x) > 0$$

and thus μ is not a correlated equilibrium, which completes the proof. ∎

As the next lemma shows, a smooth potential game with bounded payoff functions satisfies the sufficient condition for any correlated equilibrium to be a mixture of pure-strategy Nash equilibria given by Proposition 4 if its potential function is concave and a potential maximizer exists. On the other hand, Proposition 1 does not assume the existence of a potential maximizer a priori: it asserts that if a correlated equilibrium exists then a potential maximizer also exists and any correlated equilibrium is a mixture of potential maximizers, i.e., pure-strategy Nash equilibria. In this sense, Proposition 4 with the following lemma partially explains Proposition 1.

Lemma 6 *Let* $\mathbf{u}$ *be a smooth potential game with bounded payoff functions. If a potential function of* $\mathbf{u}$ *is concave and a potential maximizer exists then, for a potential maximizer* $x^* \in X$ *and* $\gamma_i : X_i \to \mathbb{R}_{++}$ *with* $\gamma_i(x_i) = 1$ *for all* $x_i \in X_i$ *and* $i \in N$*, the conditions* (i) *and* (ii) *in Proposition 4 are true.*

Proof. Let f be a potential function and write $X^* = \arg\max_{x \in X} f(x)$, which is non-empty by the assumption and coincides with the set of pure-strategy Nash equilibria by Lemma 2. Let $x^* \in X^*$. Then,

$$\sum_{i \in N} \nabla_i u_i(x)^\top (x_i^* - x_i) = \sum_{i \in N} \nabla_i f(x)^\top (x_i^* - x_i) \geq f(x^*) - f(x) \geq 0$$

for all $x \in X$ by the concavity of f. If $x \notin X^*$ then $\sum_{i \in N} \nabla_i u_i(x)^\top (x_i^* - x_i) \geq f(x^*) - f(x) > 0$, which establishes (i). Next, note that $(u_i(x_i + t(x_i^* - x_i), x_{-i}) - u_i(x))/t = (f(x_i + t(x_i^* - x_i), x_{-i}) - f(x))/t$ is decreasing in $t \in (0, 1]$ since f is concave. Thus,

$$\inf_{(x,t) \in X \times (0,1]} \frac{u_i(x_i + t(x_i^* - x_i), x_{-i}) - u_i(x)}{t} \geq \inf_{x \in X} \big(u_i(x_i^*, x_{-i}) - u_i(x)\big) > -\infty$$

since u_i is bounded, which establishes (ii). ∎

Using Proposition 4, we show that strict monotonicity of the γ-weighted payoff gradient leads to the uniqueness of a correlated equilibrium.

Proposition 5 *Let* $\mathbf{u}$ *be a smooth game with compact strategy sets. If there exists a bounded measurable function* $\gamma_i : X_i \to \mathbb{R}_{++}$ *for each* $i \in N$ *such that the* γ*-weighed payoff gradient of* $\mathbf{u}$ *is strictly monotone then* $\mathbf{u}$ *has a unique correlated equilibrium, which places probability one on a unique pure-strategy Nash equilibrium. Especially, if the payoff gradient of* $\mathbf{u}$ *is strictly monotone then* $\mathbf{u}$ *has a unique correlated equilibrium.*

Proposition 5 generalizes Proposition 2 because, by Lemma 4, the payoff gradient of a smooth potential game with a strictly concave potential function is strictly monotone. Proposition 5 also generalizes Proposition 3 because the c-weighted payoff gradient is a special case of the γ-weighted payoff gradient.

To prove Proposition 5, we first show the existence[10] and uniqueness of a pure-strategy Nash equilibrium.

Lemma 7 *Let* $\mathbf{u}$ *be a smooth game with compact strategy sets. If there exists a function* $\gamma_i : X_i \to \mathbb{R}_{++}$ *for each* $i \in N$ *such that the* γ*-weighed payoff gradient of* $\mathbf{u}$ *is strictly monotone then* $\mathbf{u}$ *has a unique pure-strategy Nash equilibrium.*

[10]The game in Proposition 5 is not necessarily a concave game and thus we cannot directly use the existence result by Rosen (1965).

Proof. First, we show that $\mathbf{u}$ has a pure-strategy Nash equilibrium. By Lemma 1, there exists $x^* \in X$ satisfying (2), which is equivalent to (1). Thus, it is enough to show that x^* is a pure-strategy Nash equilibrium. Fix $i \in N$ and $x_i \neq x_i^*$. By strict monotonicity, we have

$$(\gamma_i(x_i^*)\nabla_i u_i(x^*) - \gamma_i(x_i + t(x_i^* - x_i))\nabla_i u_i(x_i + t(x_i^* - x_i), x_{-i}^*))^\top (1-t)(x_i^* - x_i) < 0$$

for $t \in [0,1)$ by letting $x = x^*$ and $y = (x_i + t(x_i^* - x_i), x_{-i}^*)$ in (5). Hence, by (1),

$$\nabla_i u_i(x_i + t(x_i^* - x_i), x_{-i}^*)^\top (x_i^* - x_i) > \frac{\gamma_i(x_i^*)}{\gamma_i(x_i + t(x_i^* - x_i))} \nabla_i u_i(x^*)^\top (x_i^* - x_i) \geq 0$$

and thus

$$\frac{d}{dt} u_i(x_i + t(x_i^* - x_i), x_{-i}^*) = \nabla_i u_i(x_i + t(x_i^* - x_i), x_{-i}^*)^\top (x_i^* - x_i) > 0$$

for all $t \in [0,1)$. Therefore, $u_i(x^*) \geq u_i(x_i, x_{-i}^*)$, which implies that x^* is a pure-strategy Nash equilibrium because $x_i \in X_i$ and $i \in N$ are chosen arbitrarily.

Next, we show that a pure-strategy Nash equilibrium is unique. Let $x^*, y^* \in X$ be two pure-strategy Nash equilibria. By (1), it holds that $\gamma_i(x_i^*)\nabla_i u_i(x^*)^\top (y_i^* - x_i^*) \leq 0$ and $\gamma_i(y_i^*)\nabla_i u_i(y^*)^\top (x_i^* - y_i^*) \leq 0$ for all $i \in N$. By adding them and taking a sum over $i \in N$, we have $\sum_{i \in N}(\gamma_i(x_i^*)\nabla_i u_i(x^*) - \gamma_i(y_i^*)\nabla_i u_i(y^*))^\top (x_i^* - y_i^*) \geq 0$. On the other hand, if $x^* \neq y^*$ then $\sum_{i \in N}(\gamma_i(x_i^*)\nabla_i u_i(x^*) - \gamma_i(y_i^*)\nabla_i u_i(y^*))^\top (x_i^* - y_i^*) < 0$ by strict monotonicity. Thus, x^* and y^* must coincide, which completes the proof. ∎

We are now ready to prove Proposition 5.

Proof of Proposition 5. We show that $\mathbf{u}$ satisfies the sufficient condition for any correlated equilibrium to be a mixture of pure-strategy Nash equilibria given by Proposition 4. By Lemma 7, $\mathbf{u}$ has a unique pure-strategy Nash equilibrium $x^* \in X$. For any $x \neq x^*$,

$$\sum_{i \in N}(\gamma_i(x_i^*)\nabla_i u_i(x^*) - \gamma_i(x_i)\nabla_i u_i(x))^\top (x_i^* - x_i) < 0$$

by strict monotonicity and thus

$$\sum_{i \in N} \gamma_i(x_i)\nabla_i u_i(x)^\top (x_i^* - x_i) > \sum_{i \in N} \gamma_i(x_i^*)\nabla_i u_i(x^*)^\top (x_i^* - x_i) \geq 0$$

by (1), which establishes (i).

Fix $i \in N$. By the mean-value theorem, for any $x \in X$ and $t \in (0,1]$, there exists $\theta \in (0,t)$ such that $(u_i(x_i+t(x_i^*-x_i),x_{-i})-u_i(x))/t = \nabla_i u_i(x_i+\theta(x_i^*-x_i),x_{-i})^\top(x_i^*-x_i)$. Thus,

$$\inf_{(x,t)\in X\times(0,1]} \frac{u_i(x_i+t(x_i^*-x_i),x_{-i})-u_i(x)}{t}$$
$$\geq \min_{(x,\theta)\in X\times[0,1]} \nabla_i u_i(x_i+\theta(x_i^*-x_i),x_{-i})^\top(x_i^*-x_i) > -\infty$$

since X is compact and $\nabla_i u_i$ is continuous, which establishes (ii).

Therefore, by Proposition 4, any correlated equilibrium of $\mathbf{u}$ must place probability one on the unique pure-strategy Nash equilibrium x^*, which completes the proof. ∎

Using Lemma 3, we can obtain a sufficient condition for strict monotonicity of the γ-weighed payoff gradient which is in some cases easier to verify than (5) if payoff functions are twice continuously differentiable. By considering a special case with $X_i \subseteq \mathbb{R}$ for all $i \in N$, we have the following corollary of Proposition 5.

Corollary 6 *Let* $\mathbf{u}$ *be a smooth game. Suppose that* $X_i \subseteq \mathbb{R}$ *is a closed bounded interval for all* $i \in N$ *and that payoff functions are twice continuously differentiable. If there exists a continuously differentiable function* $\gamma_i : X_i \to \mathbb{R}_{++}$ *for each* $i \in N$ *such that a matrix*

$$\left[\delta_{ij}\frac{d\gamma_i(x_i)}{dx_i}\frac{\partial u_i(x)}{\partial x_i}\right] + \left[\gamma_i(x_i)\frac{\partial^2 u_i(x)}{\partial x_i \partial x_j}\right] \tag{9}$$

is negative definite for all $x \in X$ *where* δ_{ij} *is the Kronecker delta then* $\mathbf{u}$ *has a unique correlated equilibrium, which places probability one on a unique pure-strategy Nash equilibrium. Especially, if a matrix* $[\partial^2 u_i(x)/\partial x_i \partial x_j]$ *is negative definite for all* $x \in X$ *then* $\mathbf{u}$ *has a unique correlated equilibrium.*

Proof. Note that $\gamma_i : X_i \to \mathbb{R}_{++}$ is a bounded measurable function for each $i \in N$. Consider a mapping $x \mapsto (-\gamma_i(x_i)\nabla_i u_i(x))_{i\in N}$. Then, (9) is the Jacobian matrix multiplied by -1. Thus, if (9) is negative definite for all $x \in X$ then the mapping is strictly monotone by Lemma 3. This implies that the γ-weighted payoff gradient of $\mathbf{u}$ is strictly monotone, which completes the proof by Proposition 5. ∎

As shown by Monderer and Shapley (1996), if the matrix $[\partial^2 u_i(x)/\partial x_i \partial x_j]$ is symmetric for all $x \in X$ then $\mathbf{u}$ is a potential game and $[\partial^2 u_i(x)/\partial x_i \partial x_j]$ coincides with the Hessian matrix of a potential function. Thus, if $[\partial^2 u_i(x)/\partial x_i \partial x_j]$ is symmetric and negative definite for all $x \in X$ then $\mathbf{u}$ is a smooth potential game with a strictly concave potential function and thus a correlated equilibrium of $\mathbf{u}$ is unique by Proposition 2. Corollary 6 says that $[\partial^2 u_i(x)/\partial x_i \partial x_j]$ needs not be symmetric for the uniqueness of a correlated equilibrium.

Finally, we discuss two examples.

Example 1 Consider a Cournot oligopoly game with differentiated products in which a strategy of firm $i \in N$ is a quantity of differentiated product $i \in N$ to produce. So let $X_i \subseteq \mathbb{R}_+$ be a closed bounded interval for all $i \in N$. An inverse demand function for product i is denoted by $p_i : X \to \mathbb{R}_+$ and a cost function of firm i is denoted by $c_i : X_i \to \mathbb{R}_+$. It is assumed that both of them are twice continuously differentiable and that $d^2 c_i(x_i)/dx_i^2 \geq 0$ for all $x_i \in X_i$. The payoff function $u_i : X \to \mathbb{R}$ of firm i is given by $u_i(x) = p_i(x)x_i - c_i(x_i)$.

The matrix (9) is calculated as

$$\left[\delta_{ij} \frac{d\gamma_i(x_i)}{dx_i} \frac{\partial p_i(x)x_i}{\partial x_i}\right] + \left[\gamma_i(x_i) \frac{\partial^2 p_i(x)x_i}{\partial x_i \partial x_j}\right] - \left[\delta_{ij} \frac{d}{dx_i}\left(\gamma_i(x_i) \frac{dc_i(x_i)}{dx_i}\right)\right].$$

If $\gamma(x_i) = 1$ for each $i \in N$ then the above is reduced to

$$\left[\frac{\partial^2 p_i(x)x_i}{\partial x_i \partial x_j}\right] - \left[\delta_{ij} \frac{d^2 c_i(x_i)}{dx_i^2}\right].$$

Thus, if a matrix $\left[\partial^2 p_i(x)x_i/\partial x_i \partial x_j\right]$ is negative definite for all $x \in X$ then $\mathbf{u}$ has a unique correlated equilibrium by Corollary 6 since a matrix $\left[\delta_{ij} d^2 c_i(x_i)/dx_i^2\right]$ is positive semidefinite for all $x \in X$. As a special case, consider a linear inverse demand function $p_i(x) = \sum_{j \in N} a_{ij} x_j + b_i$ for all $i \in N$. In this case, $\partial^2 p_i(x)x_i/\partial x_i^2 = 2a_{ii}$ and $\partial^2 p_i(x)x_i/\partial x_i \partial x_j = a_{ij}$ for $i \neq j$. Thus, if a matrix $[(1 + \delta_{ij})a_{ij}]$ is negative definite then $\mathbf{u}$ has a unique correlated equilibrium. Note that if $[(1 + \delta_{ij})a_{ij}]$ is symmetric, i.e., $a_{ij} = a_{ji}$ for all $i, j \in N$, then $[\partial^2 u_i(x)/\partial x_i \partial x_j]$ is symmetric and thus $\mathbf{u}$ is a potential game.

Example 2 Let $X_i \subseteq \mathbb{R}$ be a closed bounded interval for all $i \in N$ and let $\mathbf{u}$ be a smooth game such that the payoff gradient of $\mathbf{u}$ is strictly monotone. Consider another game $\mathbf{v} = (v_i)_{i \in N}$ such that, for all $x \in X$ and $i \in N$,

$$v_i(x) = w_i(x_i)u_i(x) - \int_{c_i}^{x_i} \frac{dw_i(t)}{dt} u_i(t, x_{-i})dt + z_i(x_{-i}) \tag{10}$$

where $w_i : X_i \to \mathbb{R}_{++}$ is a continuously differentiable function, $z_i : X_{-i} \to \mathbb{R}$ is a bounded measurable function, and $c_i \in X_i$. Then it holds that $\nabla_i v_i(x) = w_i(x_i)\nabla_i u_i(x)$ for all $x \in X$ and $i \in N$. Since a mapping $x \mapsto (-\nabla_i u_i(x))_{i\in N}$ is strictly monotone, so is a mapping $x \mapsto (-\nabla_i v_i(x)/w_i(x_i))_{i\in N}$. This implies that the γ-weighted payoff gradient of $\mathbf{v}$ is strictly monotone with $\gamma_i(x_i) = 1/w_i(x_i)$ for all $x_i \in N$ and $i \in N$. Therefore, not only $\mathbf{u}$ but also $\mathbf{v}$ have a unique correlated equilibrium by Proposition 5.

For example, assume that $\min X_i > 0$ and let $w_i(x_i) = x_i$ for all $x_i \in X_i$ and $i \in N$. Then (10) is rewritten as

$$v_i(x) = x_i u_i(x) - \int_{c_i}^{x_i} u_i(t, x_{-i})dt + z_i(x_{-i}). \tag{11}$$

Furthermore, let $u_i(x) = -\partial f_i(x)/\partial x_i$ and $z_i(x_{-i}) = f_i(c_i, x_{-i})$ for all $x \in X$ and $i \in N$ where $f_i : X \to \mathbb{R}$ is a twice continuously differentiable function. Then, (11) is rewritten as

$$v_i(x) = f_i(x) - x_i \frac{\partial f_i(x)}{\partial x_i}.$$

One possible interpretation is that $x_i \in X_i$ is a quantity of a good consumed by player i, $f_i(x)$ is player i's benefit of consumption where there exists a consumption externality, and $x_i(\partial f_i(x)/\partial x_i)$ is player i's consumption expenditure where the price of the good is set at the marginal benefit of consumption and player i knows that the price depends upon the choice of x_i. Note that, in the game $\mathbf{v}$, each player chooses his consumption to maximize the benefit minus the cost, whereas, in the game $\mathbf{u} = (-\partial f_i/\partial x_i)_{i\in N}$, each player chooses his consumption to minimize the marginal benefit. By Proposition 5, if the payoff gradient of $\mathbf{u}$ is strictly monotone then not only $\mathbf{u}$ but also $\mathbf{v}$ have a unique correlated equilibrium.

In the construction of this example, for a game $\mathbf{v}$ of which γ-weighted payoff gradient is strictly monotone, there exists a game $\mathbf{u}$ of which payoff gradient is strictly monotone such that $\nabla_i u_i(x) = \gamma_i(x_i)\nabla_i v_i(x)$.[11] It should be noted that this is not always true if $X_i \subseteq \mathbb{R}^{m_i}$ with $m_i \geq 2$: in this case, for given $\mathbf{v}$ and γ, there may not exist $\mathbf{u}$ such that $\nabla_i u_i(x) = \gamma_i(x_i)\nabla_i v_i(x)$.

[11] It can be readily shown that $\mathbf{u}$ and $\mathbf{v}$ have the same best-response correspondence. See Morris and Ui (2004).

References

Aumann, R. J., 1974. Subjectivity and correlation in randomized strategies. J. Math. Econ. 1, 67–96.

Aumann, R. J., 1987. Correlated equilibrium as an expression of Bayesian rationality. Econometrica 55, 1–18.

Gabay, D., Moulin, H., 1980. On the uniqueness and stability of Nash equilibria in noncooperative games. In: Bensoussan, A., Kleindorfer, P., Tapiero, C. S. (Eds.), Applied Stochastic Control in Econometrics and Management Science. North-Holland, Amsterdam, pp. 271–294.

Hart, S., Schmeidler, D., 1989. Existence of correlated equilibria. Math. Oper. Res. 14, 18–25.

Hartman, P., Stampacchia, G., 1966. On some nonlinear elliptic differential functional equations. Acta Math. 115, 271–310.

Liu, L., 1996. Correlated equilibrium of Cournot oligopoly competition. J. Econ. Theory 68, 544–548.

Monderer, D., Shapley, L. S., 1996. Potential games. Games Econ. Behav. 14, 124–143.

Morris, S., Ui, T., 2004. Best response equivalence. Games Econ. Behav. 49, 260–287.

Nagurney, A., 1993. Network Economics: A Variational Inequality Approach. Kluwer Academic Publishers, Dordrecht.

Neyman, A., 1997. Correlated equilibrium and potential games. Int. J. Game Theory 26, 223–227.

Rosen, J. B., 1965. Existence and uniqueness of equilibrium points for concave N-person games. Econometrica 33, 520–534.

Slade, M. E., 1994. What does an oligopoly maximize?, J. Ind. Econ. 42, 45–61.