Equivalence of the Dempster-Shafer Rule and the Maximum Likelihood Rule Implies Convexity*

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Abstract

It is known that if a capacity is convex then the Dempster-Shafer update rule for the capacity is equivalent to the maximum likelihood update rule for the core of the capacity. This paper shows that the converse is also true; that is, a capacity must be convex if these two rules are equivalent.
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1 Introduction

The Dempster-Shafer update rule (Dempster, 1967; Shafer, 1976) is one of the plausible update rules for *non-additive probability measures* or *capacities*.[1] The maximum likelihood update rule, on the other hand, is one of the plausible update rules for *collections of probability measures*. Gibloa and Schmeidler (1993) have shown that if a capacity is convex then the Dempster-Shafer update rule for the capacity is equivalent to the maximum likelihood update rule for the core of the capacity in the following sense:[2] the Dempster-Shafer updated capacity is the lower envelope of probabilities obtained by the maximum likelihood update rule.

This paper considers the converse and offers a characterization of convex capacities through update rules. The main result establishes that the equivalence of the two update rules implies convexity of the underlying capacity. Thus, a capacity is convex if and only if the Dempster-Shafer update rule for the capacity is equivalent to the maximum likelihood update rule for the core of the capacity. Technically, we exploit Shapley's characterization of cores of convex games, which is reviewed in section 2. The main result is given in section 3.

2 Core of convex games

Let $(\Omega, \mathcal{F})$ be a measurable space where $\mathcal{F}$ is an algebra on Ω. A function $v : \mathcal{F} \to \mathbb{R}$ with $v(\emptyset) = 0$ is called a *game*. A game $v : \mathcal{F} \to \mathbb{R}$ is *convex* if

$$v(S) + v(T) \leq v(S \cup T) + v(S \cap T) \text{ for all } S, T \in \mathcal{F}.$$

A game v is *additive* if the above inequality is always satisfied as an equality, and we refer to an additive game as a *measure*. Let $v' : 2^\Omega \to \mathbb{R}$ be the conjugate of v, i.e., $v'(E) = v(\Omega) - v(\Omega \backslash E)$ for all $E \in \mathcal{F}$ where $S^c = \Omega \backslash S$.

We denote by $C(v)$ the *core* of v:

$$C(v) = \{p : p \text{ is a measure with } p(\Omega) = v(\Omega) \text{ and } p(S) \geq v(S) \text{ for all } S \in \mathcal{F}\}.$$

For each $E \in \mathcal{F}$, define

$$C_E(v) = \{p \in C(v) : p(E) = v(E)\}.$$

[1]Further elaborations have been done by various authors. See for instance Wakker (2000) and its references. Preference based characterizations have also been studied: see Gibloa and Schmeidler (1993), Chateauneuf *et al.* (2001), and Wang (2003).

[2]Denneberg (1994) gave an alternative proof.

Note that $C_\Omega(v) = C_\emptyset(v) = C(v)$.

In general, $C(v)$ may be empty, and even if $C(v)$ is non-empty, $C_E(v)$ may be empty. It is well known that if a game is convex then $C_E(v) \neq \emptyset$ for all $E \in \mathcal{F}$, or equivalently, $v(E) = \min_{p \in C(v)} p(E)$ for all $E \in \mathcal{F}$, but not vice versa. So a stronger condition is equivalent to convexity, as shown in the following characterization result due to Shapley (1971) for a finite space[3] and Schmeidler (1984) for an infinite space.

Proposition 1 *A game v is convex if and only if, for any increasing sequence $S_1 \subseteq S_2 \subseteq \cdots \subseteq S_n \subseteq \Omega$ with $S_1, S_2, \ldots, S_n \in \mathcal{F}$, we have*

$$C_{S_1} \cap C_{S_2} \cap \cdots \cap C_{S_n} \neq \emptyset.$$

A simple modification yields the following lemma.

Lemma 2 *A game v is convex if and only if, for any $S_1 \subseteq S_2 \subseteq \Omega$ with $S_1, S_2 \in \mathcal{F}$, we have*

$$C_{S_1} \cap C_{S_2} \neq \emptyset.$$

Proof. It is enough to show the "if" part. For $S, T \in \mathcal{F}$, let $S_1 = S \cap T$ and $S_2 = S \cup T$. Then, $S_1 \subseteq S_2$ and thus there exists $p \in C_{S\cap T}(v) \cap C_{S\cup T}(v)$. Therefore,

$$v(S \cap T) + v(S \cup T) = p(S \cap T) + p(S \cup T) = p(S) + p(T) \geq v(S) + v(T),$$

which implies convexity of v. ■

3 Result

A game v is a *capacity* if it is non-negative ($v(S) \geq 0$ for all $S \in \mathcal{F}$), monotone ($v(S) \leq v(T)$ for $S, T \in \mathcal{F}$ with $S \subseteq T$), and normalized ($v(\Omega) = 1$). An additive capacity is called a *probability measure*.

For a capacity v, the Dempster-Shafer update for v given $E \in \mathcal{F}$ with $v'(E) > 0$ is a capacity v_E defined as follows:

$$v_E(S) = \frac{v(S \cup E^c) - v(E^c)}{v'(E)} \text{ for all } S \in \mathcal{F}. \tag{1}$$

The Dempster-Shafer update for a probability measure p is the Bayesian update:

$$p_E(S) = \frac{p(S \cup E^c) - p(E^c)}{p'(E)} = \frac{p(S \cap E)}{p(E)} \text{ for all } S \in \mathcal{F}. \tag{2}$$

[3]See Theorem 2 and Theorem 5 of Shapley (1971). See also Gibloa and Schmeidler (1993, p. 45).

Let P be a collection of probability measures over Ω. The maximum likelihood update for P given $E \in \mathcal{F}$ with $p(E) > 0$ for some $p \in P$ is a collection P_E of probability measures defined as follows:

$$P_E = \{p_E : \ p \in \arg\max_{q \in P} q(E)\}. \tag{3}$$

Let v be a capacity and let $P = C(v)$. We say that the Dempster-Shafer rule and the maximum likelihood rule are equivalent for v if

$$v_E(S) = \min_{p_E \in P_E} p_E(S) \text{ for all } E, S \in \mathcal{F} \text{ with } v'(E) > 0. \tag{4}$$

Note that it is part of the condition (4) that the set P_E is non-empty and the minimizer exists.

Gibloa and Schmeidler (1993) have shown the following result.

Proposition 3 *If v is convex, then the Dempster-Shafer rule and the maximum likelihood rule are equivalent for v.*

The following is the main result of this paper.

Proposition 4 *If the Dempster-Shafer rule and the maximum likelihood rule are equivalent for v, then v is convex.*

Proof. Assume that (4) holds. Observe that (4) implies $C_S(v) \neq \emptyset$ for all $S \in \mathcal{F}$. Indeed, since $v'(\Omega) = 1 > 0$ and $P_\Omega = P = C(v)$, (4) implies that

$$\min_{p \in P} p(S) = \min_{p \in P_\Omega} p(S) = v_\Omega(S) = v(S) \text{ for all } S \in \mathcal{F}, \tag{5}$$

which is equivalent to $C_S(v) \neq \emptyset$ for all $S \in \mathcal{F}$. In particular, setting $S = E^c$ in (5), we have

$$\max_{p \in P} p(E) = 1 - \min_{p \in P} p(E^c) = 1 - v(E^c) = v'(E).$$

This implies that, for $p \in P$,

$$p \in \arg\max_{q \in P} q(E) \Leftrightarrow p(E) = v'(E) \Leftrightarrow p(E^c) = v(E^c) \Leftrightarrow p \in C_{E^c}(v). \tag{6}$$

So (3) can be re-written as follows:

$$P_E = \{p_E : p \in C_{E^c}(v)\}. \tag{7}$$

Hence, for any $E, S \in \mathcal{F}$ with $v'(E) > 0$, we have

$$\begin{aligned}
\frac{v(S \cup E^c) - v(E^c)}{v'(E)} &= \min_{p_E \in P_E} p_E(S) \ \text{(by (1) and (4))} \\
&= \min_{p \in C_{E^c}(v)} \frac{p(S \cup E^c) - p(E^c)}{p(E)} \ \text{(by (2) and (7))} \\
&= \frac{\min_{p \in C_{E^c}(v)} p(S \cup E^c) - v(E^c)}{v'(E)} \ \text{(by (6))}.
\end{aligned}$$

Comparing the first and the last expressions, we have

$$v(S \cup E^c) = \min_{p \in C_{E^c}(v)} p(S \cup E^c).$$

This implies that there exists $p \in C(v)$ satisfying both $p(E^c) = v(E^c)$ and $p(S \cup E^c) = v(S \cup E^c)$, or equivalently, $C_{E^c}(v) \cap C_{S \cup E^c}(v) \neq \emptyset$ for all $E, S \in \mathcal{F}$ with $v'(E) > 0$. Now note that $v'(E) = 0$ holds if and only if $v(E^c) = 1$, which implies that $v(E^c \cup S) = 1$ and thus $C_{E^c}(v) \subseteq C_{E^c \cup S}(v)$ for all $S \in \mathcal{F}$. Consequently, we have $C_{E^c}(v) \cap C_{S \cup E^c}(v) \neq \emptyset$ for all $E, S \in \mathcal{F}$, or setting $S_1 = E^c$ and $S_2 = E^c \cup S$, we have $C_{S_1}(v) \cap C_{S_2}(v) \neq \emptyset$ for all $S_1, S_2 \in \mathcal{F}$ with $S_1 \subseteq S_2$. This implies convexity of v by Lemma 2. ■

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