Kempe equivalence classes of cubic graphs embedded on the projective plane

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Dedicated to Professor Katsuhiro Ota on the occasion of his 60th birthday

Abstract

A Kempe switch of a 3-edge-coloring of a cubic graph G on a bicolored cycle C swaps the colors on C and gives rise to a new 3-edge-coloring of G. Two 3-edge-colorings of G are Kempe equivalent if they can be obtained from each other by a sequence of Kempe switches. Fisk proved that any two 3-edge-colorings in a cubic bipartite planar graph are Kempe equivalent. In this paper, we obtain an analog of this theorem and prove that all 3-edge-colorings of a cubic bipartite projective-planar graph G are pairwise Kempe equivalent if and only if G has an embedding in the projective plane such that the chromatic number of the dual triangulation G^* is at least 5. As a by-product of the results in this paper, we prove that the list-edge-coloring conjecture holds for cubic graphs G embedded on the projective plane provided that the dual G^* is not 4-vertex-colorable.

Keywords: 4-Color Theorem, 3-edge-coloring, Kempe equivalence,
MSC 2010: 05C10, 05C15

1 Introduction

A *Kempe switch* of a 3-edge-coloring of a cubic graph G on a bicolored cycle C swaps the colors on C and gives rise to a new 3-edge-coloring of G. Two 3-edge-colorings of G are *Kempe equivalent* if they can be obtained from each other by a sequence of Kempe switches. This is indeed an equivalence relation on the set of all 3-edge-colorings of G. Note that if we perform a Kempe switch on each bicolored cycle of two particular colors, then we have the 3-edge-coloring obtained by the transposition of those colors.

This notion comes from Kempe's invalid "proof" [18] of the 4-Color Theorem, where an error was pointed out by Heawood. It became a very useful tool, however. For example, it was used in the proofs [4, 33] of the 4-Color Theorem. (To be exact, the dual form was used to prove the reducibility of configurations for the proof of the 4-Color Theorem. See Section 3.1 for the duality of 3-edge-colorings.)

Note that the statement of the 4-Color Theorem can be rephrased as "no planar snark exists", where a *snark* is a non-3-edge-colorable 2-edge-connected cubic graph. (Note that the definition of a snark sometimes requires more conditions on the connectivity and girth.) The class of snarks is known to be important, since it may contain counterexamples to several well-known conjectures, such as the cycle double cover conjecture, the nowhere zero 5-flow conjectures, the 4-connected line graph Hamiltonian conjecture, and so on. Therefore, properties of snarks are interesting in their own right. For example, Nedela and Škoviera

[30] used Kempe switches in subcubic graphs to obtain results about snarks. This method appears in several other papers as well [15, 23, 35].

As Mohar [26] explained, the number of Kempe equivalence classes has some applications in statistical physics (see [27, 42] for example) and Markov chains [41]. Furthermore, there are several related works on the Kempe equivalence classes: belcastro and Haas [5] constructed some families of cubic graphs with particular numbers of Kempe equivalence classes, and McDonald, Mohar and Scheide [24] considered Kempe equivalence classes for 4-edge-colorings in cubic graphs. The vertex-coloring version has been also considered in several papers [7, 9, 40].

Related to those works, it is important to study Kempe equivalence classes of 3-edge-colorings, and in particular, we would like to check whether a given 3-edge-colorable cubic graph has only one Kempe equivalence class. Fisk ([10, Theorem 6] and [12, Theorem 1]) proved the following theorem. As pointed out in [26], he originally expressed the statement in the dual version.

Theorem 1 (Fisk [10]) *Any two 3-edge-colorings in a cubic bipartite planar graph are Kempe equivalent.*

In this paper, we consider the case of the projective plane, and give several results on Kempe equivalence classes. The following, which is an analog of Theorem 1, can be obtained from our results.

Theorem 2 *All 3-edge-colorings of a cubic bipartite projective-planar graph G are pairwise Kempe equivalent if and only if G has an embedding in the projective plane such that the chromatic number of the dual triangulation G^* is at least 5.*

In those results, we focus on bipartite graphs. Note that the 3-edge-colorability problem for cubic graphs is well-known to be NP-complete [13], while every cubic bipartite graph admits a 3-edge-coloring. Therefore, it is reasonable to start with cubic bipartite graphs. In fact, Mohar [26] posed the following problem:

Problem 3 *Characterize all cubic bipartite graphs that have only one Kempe equivalence class.*

Since there are infinitely many cubic bipartite graphs G embedded on the projective plane such that the dual G^* is not 4-vertex-colorable, Theorem 2 gives a new family of cubic bipartite graphs having only one Kempe equivalence class.

This paper is organized as follows: In the next section, we give some notation. In Section 3, we define the *type* of a 3-edge-coloring, which can distinguish Kempe equivalence classes under certain conditions, and in Section 4, we introduce the *signature* of a 3-edge-coloring and show its relation to types. In Sections 5 and 6, we prove Theorem 2 establishing several properties on types of 3-edge-colorings. In the last section, we give some remarks, and as a by-product of the results in this paper, we prove that the list-edge-coloring conjecture holds for cubic graphs G embedded on the projective plane such that the dual G^* is not 4-vertex-colorable.

2 Preliminaries

A *surface* F^2 is a connected compact 2-dimensional manifold without boundary. A closed curve γ on F^2 is *essential* if γ does not bound a 2-cell region on F^2. Otherwise, γ is *contractible*. A closed curve γ on F^2 is *one-sided* if a tubular neighborhood of γ is a Möbius strip; otherwise it is *two-sided*. Two closed curves are *homotopic* if they can be continuously deformed into each other on the surface. When the surface F^2 is the projective plane, it is known that a closed curve is essential if and only if it is one-sided. In fact, the projective plane has the unique homotopy class of essential closed curves. The following well-known fact for the projective plane is used several times in this paper:

Lemma 4 *Let γ_1 and γ_2 be two closed curves on the projective plane. Then γ_1 and γ_2 are both essential (i.e. one-sided) if and only if they intersect transversally an odd number of times.*

In this paper, by a graph *embedded on a surface* F^2, we always mean a graph 2-cell embedded on F^2 (i.e., an embedding on F^2 such that each face is homeomorphic to an open disc).

A *triangulation* of a surface F^2 is a graph embedded on F^2 with each face triangular. In the standard terminology, intersecting triangles in a triangulation can share only a vertex or an edge together with its end vertices, but in this paper, we do not require the condition. Thus, some triangulations may contain multiple edges. A *facial walk* in a graph embedded on a surface is the boundary walk of some face. If it is a cycle, then it is called a *facial cycle* in particular. Note that a cycle of a graph embedded on a surface F^2 can be regarded as a closed curve on F^2. Thus, we say that a cycle is essential or contractible in the sense of closed curves.

For a graph G embedded on a surface, the *dual* of G is denoted by G^*. For $S \subseteq E(G)$, we denote by S^* the set of dual edges e^* in G^* taken over all edges e in S. In this paper, we regard a cycle also as the set of all edges in the cycle. For example, if D is a cycle in a graph embedded on a surface, then D stands for $E(D)$ and we write D^* for $\big(E(D)\big)^*$.

Let T be a 2-factor of a graph G embedded on the projective plane, where a 2-*factor* is a spanning subgraph in which every vertex has degree exactly 2. So T consists of pairwise vertex-disjoint cycles in G. Using the topology of the projective plane, it is easy to prove the following lemma.

Lemma 5 *Let G be a graph embedded on the projective plane, and let T be a 2-factor of G. Then T satisfies exactly one of the following:*

(I) All cycles in T is contractible, and T is 2-face-colorable.

(II) T contains exactly one essential cycle, and T is not 2-face-colorable.

3 Type of a 3-edge-coloring of a cubic graph

In this section, we give a method to distinguish a Kempe equivalence class of 3-edge-colorings. This plays the central role in this paper. To do that, we need some definitions.

3.1 The duality of 3-edge-colorings

Let G be a cubic plane graph, and let φ be a 3-edge-coloring of G. In this subsection, we use the elements of $\mathbb{Z}_2 \times \mathbb{Z}_2 - \big\{(0,0)\big\}$ as colors of φ, that is, φ is a mapping from $E(G)$ to $\mathbb{Z}_2 \times \mathbb{Z}_2 - \big\{(0,0)\big\}$. As explained by Tait [37], a 3-edge-coloring φ of G *produces* a mapping $f_\varphi : V(G^*) \to \mathbb{Z}_2 \times \mathbb{Z}_2$ of the dual triangulation G^* of G as follows. First, fix a vertex of G^* and color it by $(0,0)$, and then we extend the colors in the following rule: Let h and h' be two adjacent vertices of G^* such that h is already colored but h' has not been colored yet, and let e^* be the edge connecting them in G^*. Then we color h' so that $f_\varphi(h') = f_\varphi(h) + \varphi(e)$, where

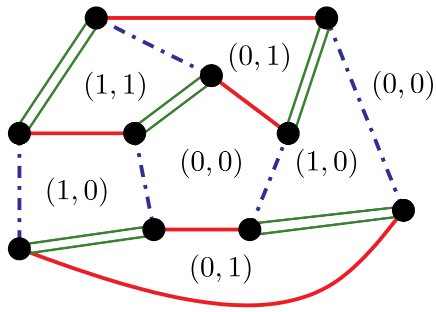

Figure 1: A 3-edge-coloring φ of a cubic plane graph G and a 4-vertex-coloring f_φ of G^*.

$+$ means the sum on $\mathbb{Z}_2 \times \mathbb{Z}_2$. Tait [37] proved that f_φ is well-defined and indeed a 4-vertex-coloring of G^*. Conversely, when we are given a 4-vertex-coloring $f : V(G^*) \to \mathbb{Z}_2 \times \mathbb{Z}_2$ in G^*, we can define the mapping $\varphi_f : E(G) \to \mathbb{Z}_2 \times \mathbb{Z}_2 - \{(0,0)\}$ in G by the opposite rules, and the obtained mapping φ_f is a 3-edge-coloring of G.

For example, consider a 3-edge-coloring φ of a cubic plane graph G in Figure 1, where the red thick edges, the blue dashed-dotted line edges, and the green double line edges in G represent those colored by $(0,1)$, $(1,0)$ and $(1,1)$, respectively. The 3-edge-coloring φ produces the 4-vertex-coloring f_φ of G^* as indicated by the element of $\mathbb{Z}_2 \times \mathbb{Z}_2$ in corresponding faces.

Similarly to the planar case, we may consider the duality between a 3-edge-coloring of a cubic graph G embedded on a non-spherical surface F^2 and a 4-vertex-coloring in the dual G^*. In fact, each 4-vertex-coloring of the dual G^* always produces a 3-edge-coloring of G, but the converse does not hold. For example, consider the 3-edge-coloring φ_1 on the left of Figure 2. Suppose that we put the color $(0,0)$ into the central hexagon. By the coloring rule, the top and the bottom faces must have color $(1,1)$ and $(1,0)$, respectively, but those faces coincide, a contradiction. On the other hand, the 3-edge-coloring φ_2 on the right of Figure 2 produces a 4-vertex-coloring in the dual G^*. The difference between these two colorings is important in this paper.

3.2 Type of a perfect matching

Let G be a cubic graph, and let X be a subset of the edge set of G. Then for a perfect matching M of G, define

$$\sigma_X(M) \equiv |X| - |X \cap M| \pmod 2$$

with $\sigma_X(M) \in \{0,1\}$. In particular, when G is embedded on a surface F^2 and $X = D$, where D is the dual of an essential cycle D^* in G^*, the type $\sigma_D(M)$ represents the topological properties of the perfect matching M, as follows.

Theorem 6 (See [20, 22, 29]) *Let G be a cubic graph embedded on a surface F^2, and let M be a perfect matching of G. Then the following hold:*

(I) *$\sigma_{D_1}(M) = \sigma_{D_2}(M)$ for any two homotopic cycles D_1^* and D_2^* in G^*.*

(II) *In particular, $\sigma_D(M) = 0$ if D^* is a contractible cycle in G^*.*

(III) *The following three statements are equivalent:*

- *For every essential cycle D^* in G^*, $\sigma_D(M) = 0$.*

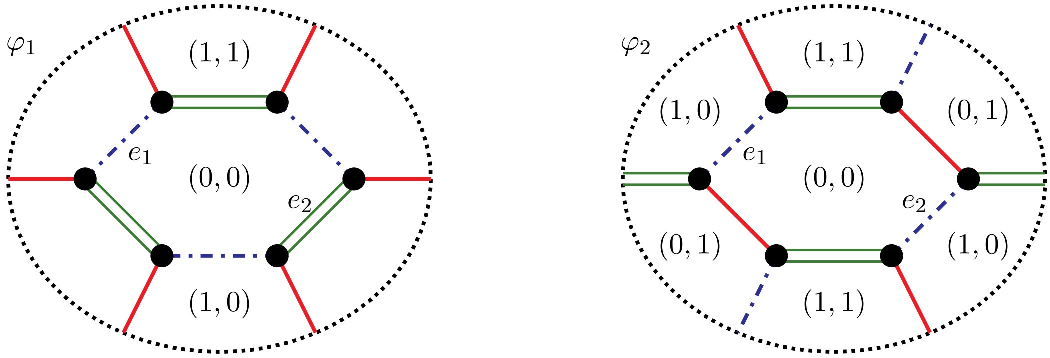

Figure 2: Two 3-edge-colorings of $K_{3,3}$ embedded on the projective plane, where the projective plane is obtained by identifying the antipodal points of the outer dotted circle.

- *$G - M$ is a 2-factor that is 2-face-colorable.*
- *$G^* - M^*$ is a bipartite quadrangulation on F^2.*

3.3 Type of a 3-edge-coloring

Using the type of a perfect matching, we define the type of a 3-edge-coloring. In the remainder of this paper, we use a, b and c for the colors of 3-edge-colorings for simplicity of notation. Recall that φ can be regarded as a partition of $E(G)$ into three perfect matchings M_a, M_b and M_c, where M_x is the set of edges of color x for $x \in \{a, b, c\}$. In all figures, red thick line edges, blue dashed-dotted line edges, and green double line edges represent those colored by a, b and c, respectively.

Let φ be a 3-edge-coloring of G and let X be a subset of the edge set of G. We define the *type with respect to* X, denoted by $\rho_X(\varphi)$, of φ as the triple of $\sigma_X(M_x)$ for $x \in \{a, b, c\}$. Namely,

$$\rho_X(\varphi) = \big(\sigma_X(M_a), \sigma_X(M_b), \sigma_X(M_c)\big) \in \{0, 1\}^3.$$

See Figure 2 for an example. Let $D = \{e_1, e_2\}$. For the 3-edge-coloring φ_1 on the left, we have $\sigma_D(M_a) \equiv 2 - 0 \equiv 0 \pmod 2$, $\sigma_D(M_b) \equiv 2 - 1 \equiv 1 \pmod 2$, and $\sigma_D(M_c) \equiv 2 - 1 \equiv 1 \pmod 2$, and hence $\rho_D(\varphi_1) = (0, 1, 1)$. Similarly, we see $\rho_D(\varphi_2) = (0, 0, 0)$.

For the type of 3-edge-colorings, the following holds.

Proposition 7 *For a 3-edge-coloring φ of a cubic graph G and a subset X of $E(G)$, we have*

$$\rho_X(\varphi) = (0, 0, 0),\ (1, 1, 0),\ (1, 0, 1),\ \textit{or}\ (0, 1, 1).$$

Proof. Since $\{M_a, M_b, M_c\}$ is a partition of $E(G)$, we see that $|X \cap M_a| + |X \cap M_b| + |X \cap M_c| = |X|$. Therefore,

$$\sigma_X(M_a) + \sigma_X(M_b) + \sigma_X(M_c) \equiv 3|X| - |X| \equiv 0 \pmod 2.$$

This directly implies Proposition 7. $\square$

Proposition 7 shows that $\rho_X(\varphi)$ attains only four types. In addition, if $\rho_X(\varphi) \neq (0, 0, 0)$, say $\rho_X(\varphi) = (1, 1, 0)$, then we obtain 3-edge-colorings φ' and φ'' with $\rho_X(\varphi') = (1, 0, 1)$ and $\rho_X(\varphi'') = (0, 1, 1)$ from φ by a permutation of colors. Therefore, the set of 3-edge-colorings φ can be divided into two sets according as $\rho_X(\varphi) = (0, 0, 0)$ or $\rho_X(\varphi) \neq (0, 0, 0)$ in some sense.

When G is embedded on a surface F^2 and $X = D$, where D is the dual of an essential cycle D^* in G^*, the type $\rho_D(\varphi)$ represents the topological property of the 3-edge-coloring φ, similarly to the type $\sigma_D(M)$ of a perfect matching M. Indeed, the following holds (see [11, Theorem 2]).

Theorem 8 *Let G be a cubic graph embedded on a surface. A 3-edge-coloring φ of G produces a 4-vertex-coloring of the dual G^* if and only if $\rho_D(\varphi) = (0, 0, 0)$ for every essential cycle D^* in G^*.*

3.4 Types and Kempe equivalence classes

Now we consider a cubic graph G embedded on the projective plane. Recall that the projective plane has the unique homotopy class of essential closed curves. Therefore, by Theorems 6 (I) and 8, there are only two possibilities for the type of a 3-edge-coloring φ:

- $\rho_D(\varphi) = (0, 0, 0)$ for an essential cycles D^* in G^*. Then φ is said to be *even*. In this case, φ produces a 4-vertex-coloring of the dual G^*.
- $\rho_D(\varphi) \neq (0, 0, 0)$ for an essential cycles D^* in G^*. Then φ is said to be *odd*. In this case, φ does not produce a 4-vertex-coloring of the dual G^*.

Indeed, these two types can distinguish Kempe equivalence classes. See Figure 2 for an example, where the 3-edge-coloring φ_1 on the left is odd type, while φ_2 on the right is even type. By the following theorem, the two 3-edge-colorings φ_1 and φ_2 in Figure 2 are not Kempe equivalent.

Theorem 9 *If two 3-edge-colorings of a cubic graph embedded on the projective plane are Kempe equivalent, then they have the same type.*

Proof of Theorem 9. Let G be a cubic graph embedded on the projective plane and let φ be an odd 3-edge-coloring of G. Then for every essential cycle D^* in G^*, we have $\rho_D(\varphi) \neq (0, 0, 0)$. By symmetry, we may assume that $\rho_D(\varphi) = (1, 1, 0)$. Let C be a bicolored cycle with respect to φ, and let φ' be the 3-edge-coloring of G obtained by the Kempe switch on C. It suffices to prove that $\rho_D(\varphi') \neq (0, 0, 0)$.

If C consists of edges of color a and c, then M_b is still the set of edges of color b by φ', so it follows from Proposition 7 that $\rho_D(\varphi') = (1, 1, 0)$ or $(0, 1, 1)$. The same also occurs if C consists of edges of color b and c. Therefore, we may assume that C consists of edges of color a and b.

Assume that C is contractible on the projective plane. By Lemma 4, C and D^* intersect transversally an even number of times. This implies that $|C \cap D|$ is even. Since $|C \cap D| = |C \cap D \cap M_a| + |C \cap D \cap M_b|$, we obtain $|C \cap D \cap M_a| \equiv |C \cap D \cap M_b| \pmod 2$. Therefore, for the matching M'_a obtained from M_a by the Kempe switch on C, we have

$$\begin{aligned}
\sigma_D(M'_a) &\equiv |D| - |D \cap M'_a| \\
&= |D| - \Big(|D \cap M_a| - |C \cap D \cap M_a| + |C \cap D \cap M_b|\Big) \\
&\equiv |D| - |D \cap M_a| \quad \equiv \sigma_D(M_a) \quad = 1 \qquad \pmod 2.
\end{aligned}$$

The same also holds for M_b, and we obtain $\rho_D(\varphi') = (1, 1, 0)$.

Thus, we may assume that C is essential on the projective plane. By Lemma 4, C and D^* intersect transversally an odd number of times, and hence $|C \cap D| \equiv 1 \pmod 2$. Since

$\sigma_D(M_c) = 0$,

$$\begin{aligned} |D \cap (M_a \cup M_b) - C| &= |D \cap (M_a \cup M_b)| - |C \cap D| \\ &= |D| - |D \cap M_c| - |C \cap D| \\ &\equiv \sigma_D(M_c) - 1 \equiv 1 \pmod 2. \end{aligned}$$

Since $M_a \cup M_b$ forms a set of vertex-disjoint cycles, this implies that there exists a cycle C' in $M_a \cup M_b$ such that $C' \neq C$ and $|C' \cap D|$ is odd. It follows from Lemma 4 that C' is also essential. This implies that C and C' are two vertex-disjoint essential cycles, contradicting Lemma 4. $\square$

3.5 Questions about the types

In Theorem 9, we have given a method to distinguish two 3-edge-colorings belonging to different Kempe equivalence classes for cubic graphs embedded on the projective plane. This naturally poses the following questions.

- Does every cubic graph embedded on the projective plane admit an odd 3-edge-coloring and an even 3-edge-coloring?
- Is it true that two 3-edge-colorings of a cubic graph embedded on the projective plane are always Kempe equivalent if they have the same type?

With Problem 3 in mind, we give answers to those questions for bipartite graphs in Sections 5 and 6, respectively.

4 Signature of 3-edge-colorings

Here we introduce the signature of 3-edge-colorings, which will be used for our proofs. The idea has been used for several purposes, e.g. for counting the number of 3-edge-colorings [14, 34], list-edge-colorings [2, 8] and Pfaffian labelings [31, 38]. In Section 7.3, we prove a corollary of our results for list-edge-colorings.

4.1 Definition of the signature

For two 3-edge-colorings φ_1 and φ_2 of a cubic graph G, we define the *signature* between them as follows. First, we focus on a vertex v in G, and let e_1, e_2 and e_3 be the three edges incident with v in G. Then some element, say π_v, in the symmetric group of degree three acts on the colors by φ_1 and φ_2. Namely, $\pi_v(\varphi_1(e_i)) = \varphi_2(e_i)$ for any $1 \leq i \leq 3$. Recall that the *signature* of π_v is defined as $\mathrm{sign}(\pi_v) = +1$ (resp. -1) if π_v is an even (resp. odd) permutation. See Figure 3 for an example. For the vertex v on the left, we have $\pi_v(a) = b$, $\pi_v(b) = c$ and $\pi_v(c) = a$, which is an even permutation (so $\mathrm{sign}(\pi_v) = +1$), while for the vertex u on the right, we have $\pi_u(a) = b$, $\pi_u(b) = a$ and $\pi_u(c) = c$, which is an odd permutation (so $\mathrm{sign}(\pi_u) = -1$). Then let

$$\mathrm{sign}(\varphi_1, \varphi_2) = \prod_{v \in V(G)} \mathrm{sign}(\pi_v),$$

which is called the *signature* between φ_1 and φ_2.

The following proposition suggests that signature can distinguish the Kempe equivalence classes. It appeared in [8, p. 346] without proofs. Since the proof is neither difficult, nor long, we include it here.

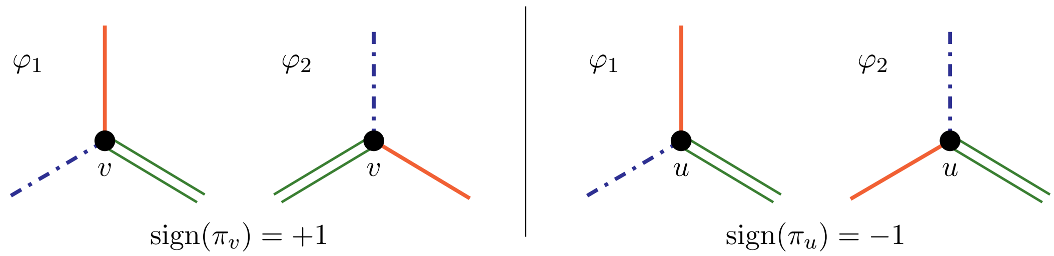

Figure 3: Two vertices v and u, where $\text{sign}(\pi_v) = +1$ and $\text{sign}(\pi_u) = -1$.

Proposition 10 *Let G be a cubic graph, and let φ_1 and φ_2 be two 3-edge-colorings of G. If φ_1 and φ_2 are Kempe equivalent, then sign$(\varphi_1, \varphi_2) = +1$.*

Proof. Let φ_1' be the 3-edge-coloring of G obtained from φ_1 by a Kempe switch on a bicolored cycle C. It is enough to prove that $\text{sign}(\varphi_1, \varphi_1') = +1$. For each vertex $v \in V(G) - V(C)$, π_v is the identity permutation, and hence $\text{sign}(\pi_v) = +1$. Let $u \in V(C)$. Since we swap the colors of the edges in C and exactly two edges in C are incident with u, we have $\text{sign}(\pi_u) = -1$. (See the right of Figure 3.) Since $|C|$ is even, we have

$$\text{sign}(\varphi_1, \varphi_1') = \prod_{v \in V(G)} \text{sign}(\pi_v) = (-1)^{|C|} = +1. \qquad \square$$

It was proven in [14, Proposition 1], [16, Theorem 3.2] and [17, Parity Lemma] that any two 3-edge-colorings of every cubic planar graph have the same signature. From this and Proposition 10, one may expect that any two 3-edge-colorings of a cubic planar graph are Kempe equivalent. This is generally false, but true for bipartite cubic planar graphs (see Theorem 1).

4.2 Relation between type and signature of 3-edge-colorings

As mentioned in the previous section, for two 3-edge-colorings, having the same type is a necessary condition to be Kempe equivalent. In this section, we explain the relation between the type and the signature. In fact, we prove the following, which together with Proposition 10 gives another proof of Theorem 9.

Theorem 11 *Let G be a cubic graph embedded on the projective plane, and let φ_1 and φ_2 be two 3-edge-colorings of G. Then φ_1 and φ_2 have the same type if and only if sign$(\varphi_1, \varphi_2) = +1$.*

We precede the proof by defining *singular* and *non-singular* edges. Note that they were defined by Fisk [10] in the dual version. Let G be a cubic graph embedded on a surface, and let φ be a 3-edge-coloring of G. An edge e is said to be *singular* (with respect to φ) if for a face h containing e, the two edges incident with h and adjacent to e have the same color by φ: Otherwise, e is said to be *non-singular*. See Figure 4 for an example. On the left, the 2-factors $G - M_a$ and $G - M_b$ intersect at e non-transversally, while on the right, they intersect at e transversally.

The following is easily obtained from the definition.

Lemma 12 *For a cycle C in a cubic graph embedded on a surface, any two of the following properties imply the third;*

- *C is a facial cycle.*

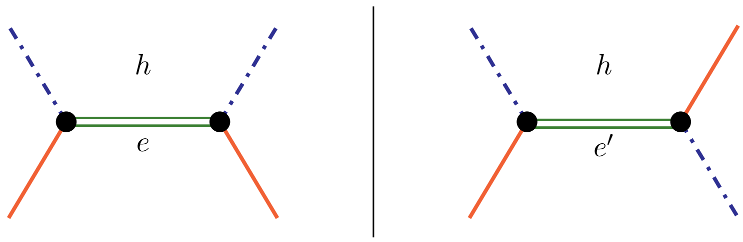

Figure 4: A singular edge e on the left, and a non-singular edge e' on the right.

- *C is a bicolored cycle.*
- *All edges in C are singular.*

See the cycle $v_2v_3v_5v_4$ in Figure 5, for an example of a cycle described in Lemma 12.

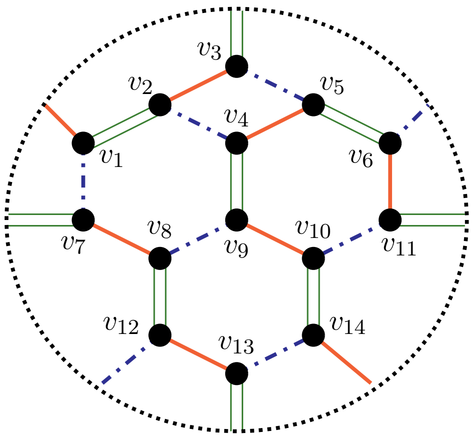

Figure 5: An essential cycle D^* in G^* represented by the outer dotted circle, and an odd 3-edge-coloring φ satisfying $\rho_D(\varphi) = (1, 1, 0)$.

Let G be a cubic graph embedded on the projective plane. For a 3-edge-coloring φ, we denote by $NS(\varphi)$ the set of non-singular edges with respect to φ, and furthermore, for each color x, denote by $NS_x(\varphi)$ the set of non-singular edges of color x for $x \in \{a, b, c\}$.

Let D^* be an essential cycle in G^*. Similarly to the signatures of two 3-edge-colorings, we define the *signature* of a 3-edge-coloring φ with respect to D, denoted by $\text{sign}_D(\varphi)$, as follows: By deleting all the edges of D, we obtain a spanning subgraph G' of G that is contained in a disk on the projective plane. On the disk, we can give a consistent clockwise rotation to each vertex of G. So, for an edge uv, u and v have the same consistent clockwise rotation if and only if $uv \notin D$. For a vertex v, if the colors a, b and c appear on the edges incident with v along the consistent clockwise rotation, then we set $\text{sign}_D(\pi_v) = +1$. Otherwise we set $\text{sign}_D(\pi_v) = -1$. Finally, we set

$$\text{sign}_D(\varphi) = \prod_{v \in V(G)} \text{sign}_D(\pi_v).$$

See Figure 5 for an example, where $\text{sign}_D(\pi_{v_i}) = +1$ for $i = 2, 5, 8, 9, 10, 12, 13, 14$, and $\text{sign}_D(\pi_{v_i}) = -1$ for $i = 1, 3, 4, 6, 7, 11$. These imply that $\text{sign}_D(\varphi) = (+1)^8(-1)^6 = +1$.

We now prove the following lemma.

Lemma 13 *Let G be a cubic graph embedded on the projective plane, let φ be a 3-edge-coloring of G, and let D^* be an essential cycle in G^*. Then*

$$\mathrm{sign}_D(\varphi) = \begin{cases} (-1)^{\frac{1}{2}|G|+|D|+1} & \text{if } \varphi \text{ is odd type,} \\ (-1)^{\frac{1}{2}|G|+|D|} & \text{if } \varphi \text{ is even type.} \end{cases}$$

Since $\mathrm{sign}(\varphi_1, \varphi_2) = \mathrm{sign}_D(\varphi_1) \cdot \mathrm{sign}_D(\varphi_2)$, Theorem 11 directly follows from Lemma 13. Thus, it suffices to prove this lemma.

Proof. Let φ be a 3-edge-coloring of G. We first prove the case when φ is odd type.

By Proposition 7 and symmetry, we may assume that $\rho_D(\varphi) = (1, 1, 0)$. Denote $T_a = G - M_a$ (resp. $T_b = G - M_b$), which is a 2-factor of G such that each cycle is bicolored by the colors b and c (resp. by the colors a and c) in φ. Since

$$|D \cap T_a| \;=\; |D| - |D \cap M_a| \;\equiv\; \sigma_D(M_a) \;=\; 1 \pmod 2,$$

T_a transversally intersects with D^* an odd number of times. Lemma 4 then implies that T_a is essential. By symmetry, T_b is also essential. Therefore, again by Lemma 4, T_a and T_b intersect transversally an odd number of times. Note that T_a and T_b only share edges in M_c. It is easy to see that T_a and T_b intersect transversally at $e \in M_c$ if and only if e is non-singular. (See Figure 4.) Therefore, we have $|NS_c(\varphi)| \equiv 1 \pmod 2$.

Now we focus on each edge uv in M_c. Assume first that $uv \notin D$. Then by the definition, u and v have the same consistent rotations. This directly implies that $\mathrm{sign}_D(\pi_u) \neq \mathrm{sign}_D(\pi_v)$ if and only if the edge uv is singular. (See Figure 4.) On the other hand, if $uv \in D$, then the same argument implies that $\mathrm{sign}_D(\pi_u) \neq \mathrm{sign}_D(\pi_v)$ if and only if the edge uv is non-singular. Therefore, for an edge $uv \in M_c$, $\mathrm{sign}_D(\pi_u) \cdot \mathrm{sign}_D(\pi_v) = -1$ if and only if one of the following hold:

- $uv \notin D$ and $uv \notin NS_c(\varphi)$.
- $uv \in D$ and $uv \in NS_c(\varphi)$.

Since M_c is a perfect matching of G, we obtain the following:

$$\begin{aligned} \mathrm{sign}_D(\varphi) \;&=\; \prod_{uv \in M_c} \Big(\mathrm{sign}_D(\pi_u) \cdot \mathrm{sign}_D(\pi_v) \Big) \\ &=\; (-1)^{|M_c - (D \cup NS_c(\varphi))|} \cdot (-1)^{|D \cap NS_c(\varphi)|}. \end{aligned}$$

Since $|D| - |M_c \cap D| \equiv \sigma_D(M_c) = 0$, $|NS_c(\varphi)| \equiv 1 \pmod 2$ and $|M_c| = \frac{1}{2}|G|$, we have the following, which completes the proof for an odd 3-edge-coloring.

$$\begin{aligned} &|M_c - (D \cup NS_c(\varphi))| + |D \cap NS_c(\varphi)| \\ &\quad= \Big(|M_c| - |M_c \cap D| - |NS_c(\varphi) - D| \Big) + \Big(|NS_c(\varphi)| - |NS_c(\varphi) - D| \Big) \\ &\quad\equiv |M_c| - \big(|D| - \sigma_D(M_c) \big) + |NS_c(\varphi)| \\ &\quad\equiv \frac{1}{2}|G| + |D| + 1 \pmod 2. \end{aligned}$$

The case when φ is an even 3-edge-coloring can be proven similarly. In this case, we see that both T_a and T_b are contractible. This implies that they intersect transversally an even number of times, and hence $|NS_c(\varphi)| \equiv 0 \pmod 2$. On the other hand, we have

$$\begin{aligned} |M_c - (D \cup NS_c(\varphi))| + |D \cap NS_c(\varphi)| \;&\equiv\; |M_c| - \big(|D| - \sigma_D(M_c) \big) + |NS_c(\varphi)| \\ &\equiv\; \frac{1}{2}|G| + |D| \pmod 2. \end{aligned}$$

By the same calculation as in the previous case, this verifies the case when φ is an even 3-edge-coloring. $\square$

We explain the above argument using an example shown in Figure 5. Recall that an essential cycle D^* in G^* is represented by the outer dotted circle. The 3-edge-coloring φ is odd type and satisfies $\rho_D(\varphi) = (1, 1, 0)$. The 2-factor $T_a = G - M_a$ consists of the cycle $v_1v_2v_4v_9v_8v_{12}v_6v_5v_3v_{13}v_{14}v_{10}v_{11}v_7$, which is essential. Thus, T_a satisfies (II) in Lemma 5. Similarly, the 2-factor $T_b = G - M_b$ satisfies (II) in Lemma 5. Within edges in M_c, the edges v_1v_2, v_4v_9, v_5v_6 and v_7v_{11} are singular, while the edges v_3v_{13}, v_8v_{12} and $v_{10}v_{14}$ are non-singular. Thus, $NS_c(\varphi) = \{v_3v_{13}, v_8v_{12}, v_{10}v_{14}\}$. We have $M_c \cap D = \{v_3v_{13}, v_7v_{11}\}$. Note that an edge $v_iv_j \in M_c$ satisfies $\mathrm{sign}_D(\pi_{v_i}) \cdot \mathrm{sign}_D(\pi_{v_j}) = -1$ if and only if $v_iv_j = v_1v_2, v_4v_9, v_5v_6$ or v_3v_{13}. Therefore,

$$\mathrm{sign}_D(\varphi) = (-1)^4 = (-1)^{14/2+4+1} = (-1)^{\frac{1}{2}|G|+|D|+1}.$$

As mentioned in Section 4.1, two 3-edge-colorings of every cubic planar graph have the same signature. Note that Lemma 13 corresponds to the projective-planar analog of this result. Actually, the proof of Lemma 13 is based on the idea of the proofs in [14, Proposition 1] and [16, Theorem 3.2].

5 The existence of a 3-edge-coloring of prescribed type

In this section, we answer the first question in Section 3.5 for bipartite projective-planar graphs, by proving the following theorem.

Theorem 14 *Every cubic bipartite graph G embedded on the projective plane has an odd 3-edge-coloring. Moreover, it has an even 3-edge-coloring if and only if the dual triangulation G^* has chromatic number at most 4.*

We prove Theorem 14 by using the following result, which was obtained in several papers, see [20, 22, 29]. Note that the following theorem was in [20, 29] stated in the context of spanning bipartite quadrangulations as in Theorem 6 (III).

Theorem 15 ([20, 22, 29]) *Let G be a cubic bipartite graph embedded on the projective plane, and let D^* be an essential cycle in G^*. Then G admits a perfect matching M with $\sigma_D(M) = 1$.*

Proof of Theorem 14. It follows from Theorem 15 that G admits a perfect matching M_a such that $\sigma_D(M_a) = 1$. Since $G - M_a$ is a 2-regular bipartite graph, $E(G) - M_a$ can be naturally colored by two colors b and c and hence we can extend M_a to the 3-edge-coloring of G, say φ. Since $\sigma_D(M_a) = 1$, we see that $\rho_D(\varphi) = (1, 1, 0)$ or $(1, 0, 1)$, that is, φ is an odd 3-edge-coloring.

The second half of Theorem 14 directly follows from Theorem 8. $\square$

We also give another proof of the first half of Theorem 14, using a *generating theorem* established in [19, Theorem 10] (see also [36] for a related work). To do that, we define some terminology. An *even map* is a graph embedded on a surface such that every facial walk has even length. Note that any bipartite graph embedded on a surface is an even map, while the converse does not generally hold. The two cubic even maps T_1^* and T_2^* and the family $\left\{(T_3^k)^* : k \geq 1\right\}$ of cubic even maps on the projective plane are depicted in Figure 6. A 2-*bridging* for a cubic even map on a surface is the operation for two edges contained in a same facial cycle to subdivide them twice and add two new edges as on the left of Figure

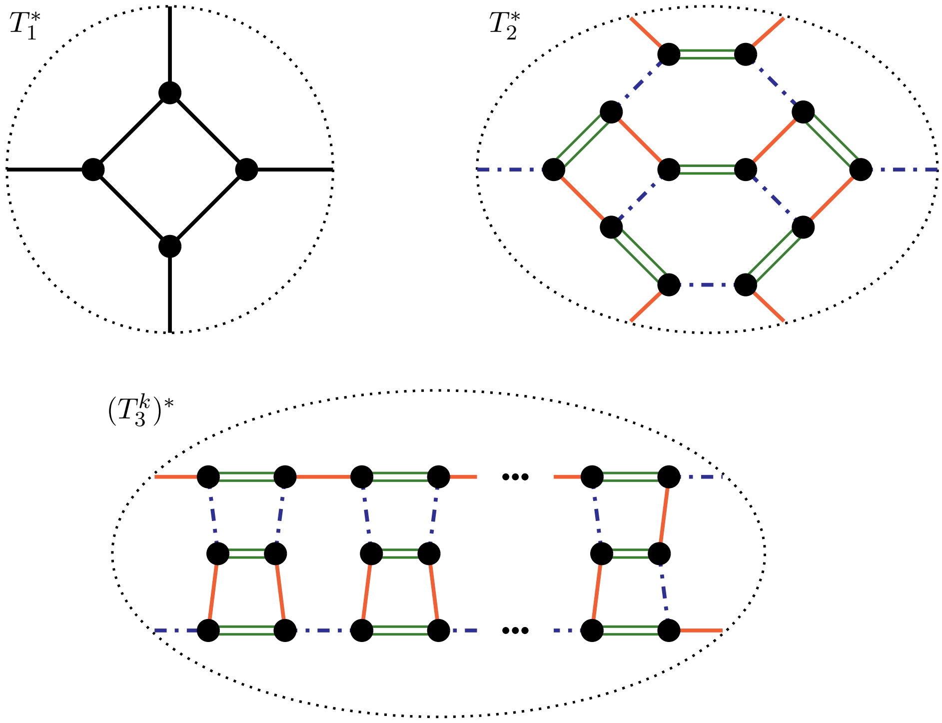

Figure 6: The two cubic even maps T_1^* and T_2^* on the projective plane and the family $\{(T_3^k)^* : k \geq 1\}$ of cubic even maps, where k is the number of hexagons in $(T_3^k)^*$. Each of T_2^* and $(T_3^k)^*$ contains a 3-edge-coloring, as depicted.

7. A *cube addition* is the operation to replace a vertex with seven vertices and nine edge as on the right of Figure 7. A graph G on a non-spherical surface F^2 is *k-representative* if all essential closed curves on F^2 intersect with G at least k times.

Theorem 16 (Kobayashi, Nakamoto and Yamaguchi [19]) *Every 3-edge-connected 2-representative cubic even map on the projective plane can be obtained from T_1^*, T_2^* or $(T_3^k)^*$ for some $k \geq 1$ by a sequence of 2-bridgings and cube additions.*

For the even maps T_1^*, T_2^* and $(T_3^k)^*$ and the two operations, we observe the following.

Lemma 17 (i) *The even map T_1^* is not bipartite, while both T_2^* and $(T_3^k)^*$ for $k \geq 1$ are bipartite.*

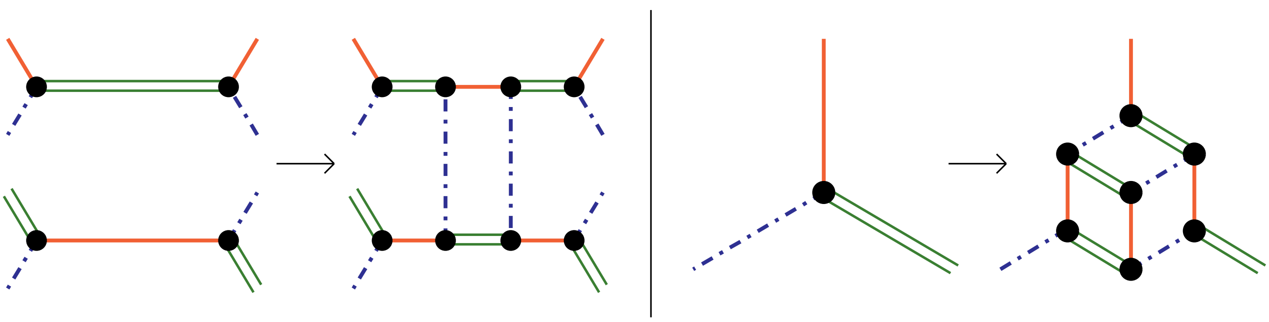

Figure 7: A 2-bridging (left) and a cube addition (right), together with extensions of a 3-edge-coloring.

(ii) *Both T_2^* and $(T_3^k)^*$ for $k \geq 1$ admit an odd 3-edge-coloring. (See Figure 6.)*

(iii) *Let G be a 3-edge-connected 2-representative cubic even map on the projective plane and let G' be the graph obtained from G by a 2-bridging or a cube addition. Then G is bipartite if and only if so is G'.*

(iv) *Let G and G' be as in (iii). If G admits an odd 3-edge-coloring, then so does G'.*

For Lemma 17 (iv), we provide an example in Figure 7, which shows how to obtain an odd 3-edge-coloring in G'. For a 2-bridging, two middle edges in the original map G might have the same color, but even in such a case, we can easily find an odd 3-edge-coloring in the new map G'.

Using Theorem 16 and Lemma 17, we have another proof of Theorem 14 (I).

Another proof of the first half of Theorem 14. Assume first that G is 2-representative and 3-edge-connected. By Theorem 16, G is obtained from T_1^*, T_2^* or $(T_3^k)^*$ for some $k \geq 1$ by a sequence of 2-bridgings and cube additions. Since G is bipartite, it follows from Lemma 17 (i) and (iii) that G is obtained from T_2^* or $(T_3^k)^*$ for some $k \geq 1$ by a sequence of 2-bridgings and cube additions. Then by Lemma 17 (ii) and (iv), G admits an odd 3-edge-coloring, and we are done.

Assume next that G is not 2-representative. Then there exist an edge e and an essential cycle D^* in G^* that consists only e^* (which is the dual edge of e and is a loop in G^*). Since G is bipartite and cubic, G admits a 3-edge-coloring φ. If we let $\varphi(e) = a$ by symmetry, then $\sigma_D(M_a) = |D| - |D \cap M_a| = 1 - 1 = 0$ and $\sigma_D(M_b) = \sigma_D(M_c) = 1$. Therefore, we have $\rho_D(\varphi) = (0, 1, 1)$, and hence φ is an odd 3-edge-coloring of G.

Assume finally that G is not 3-edge-connected. Since G is bipartite and cubic, G cannot have a bridge. Therefore, there exists a 2-edge-cut $\{u_1u_2, v_1v_2\}$ in G, where $G - \{u_1u_2, v_1v_2\}$ has two components, say G_1 and G_2, with $u_1, v_1 \in V(G_1)$ and $u_2, v_2 \in V(G_2)$. Then for $i \in \{1, 2\}$, let G_i' be the graph obtained from G_i by adding the edge u_iv_i. Since G is bipartite and embedded on the projective plane, it is easy to see that for some $i \in \{1, 2\}$, say $i = 1$, G_1' is a cubic bipartite graph embedded on the projective plane and G_2' is a cubic bipartite graph contained in a disk on the projective plane. Note that every 3-edge-coloring of G is obtained from a 3-edge-coloring of G_1' and that of G_2' with u_iv_i having the same color. Furthermore, it is easy to see that G admits an odd 3-edge-coloring if and only if G_1' admits an odd 3-edge-coloring. Since G_1' is smaller than G, we can prove by an inductive argument that G_1' (and hence G) admits an odd 3-edge-coloring. (The base case was done by the arguments in the previous paragraphs.) $\square$

6 Kempe equivalence of odd 3-edge-colorings

Considering the second question in Section 3.5, the main purpose of this section is to prove that any two odd 3-edge-colorings of a cubic bipartite graph embedded on the projective plane are Kempe equivalent. Note that this is the projective-planar analog of Theorem 1. We prove this theorem in Section 6.4, after several arguments in Sections 6.1–6.3. The argument in Section 6.1 is based on the proof of Theorem 1 in [10, 12], and then we need to deal with the global structure on the projective plane in Sections 6.2 and 6.3. In addition, we prove Theorem 2 in Section 6.5.

6.1 Canonical 3-edge-colorings

We first prepare the following important lemma on non-singular edges.

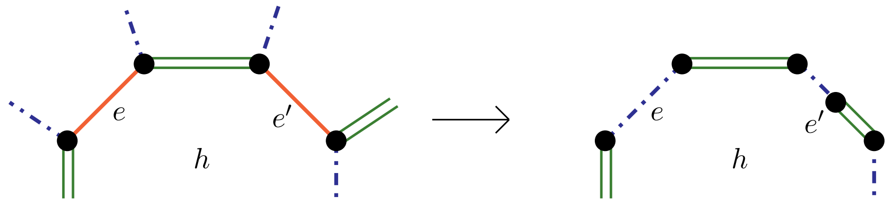

Figure 8: The face h in the proof of Lemma 18, where e is singular, while e' is non-singular.

Lemma 18 (Fisk [10]) *Let G be a cubic graph embedded on a surface and let φ be a 3-edge-coloring of G. Then for any face h and any color x,*

$$|E(h) \cap NS_x(\varphi)| \equiv |E(h)| \pmod 2,$$

where $E(h)$ is the set of all edges in G contained in the facial cycle of h.

A proof of Lemma 18 can be found in [10, Lemma 5] and [12, Lemma 2], but we prove it here.

Proof. Let h be a face and let x be a color. We may assume that $x = a$. Now we modify the facial cycle of h together with its colors. See Figure 8 for an example of the following argument.

Let e be an edge in $E(h)$ colored by a. If e is singular, then the two edges in $E(h)$ adjacent to e have the same color, say c by symmetry. In this case, we recolor e by the color b. On the other hand, if e is non-singular, then the two edges in $E(h)$ adjacent to e have different colors. In this case, subdivide e and recolor the two obtained edges by b and c so that edges with the same color are not consecutive. After recoloring all edges in $E(h)$ colored by a, we finally obtain a bicolored cycle colored by b and c, and hence its length is even. On the other hand, since the length of the new cycle is $|E(h)| + |E(h) \cap NS_a(\varphi)|$, we have the desired equality. $\square$

A *canonical* 3-edge-coloring of a cubic graph embedded on a surface is a 3-edge-coloring such that for each color x, either $\big(NS_x(\varphi)\big)^*$ consists of only essential cycles or is the empty set. (See [10, p.331].) We prove the following lemma.

Lemma 19 *Let G be a cubic even map on a surface. For every 3-edge-coloring φ of G, there exists a canonical 3-edge-coloring φ' such that φ' is Kempe equivalent to φ and for every cycle D^* in G^*, $\rho_D(\varphi) = (0,0,0)$ if and only if $\rho_D(\varphi') = (0,0,0)$.*

Proof. Let G be a cubic even map on a surface F^2, and let φ be a 3-edge-coloring. Then $|E(h)|$ is even for each face h, and hence Lemma 18 implies that for each color x, $\big(NS_x(\varphi)\big)^* \subseteq E(G^*)$ forms an Eulerian subgraph of G^*. Thus, $\big(NS_x(\varphi)\big)^*$ can be divided into edge-disjoint cycles in G^*. Assume that some cycle C^* in $\big(NS_x(\varphi)\big)^*$, say $x = a$, is contractible. Then C^* divides the surface F^2 into the interior, which is homeomorphic to a disk, and the exterior. Note that no edges in $M_b \cup M_c$ intersect with C^*, and hence the edges in $M_b \cup M_c$ that are contained in the interior of C^* form a set of vertex-disjoint contractible cycles. Then let φ' be the 3-edge-coloring obtained from φ by the Kempe switches on all such cycles. Let $C = (C^*)^* \subseteq NS_a(\varphi)$. It is easy to see the following:

- Since each edge in C is incident with exactly one vertex in the interior of C^* and the colors b and c are all swapped in the interior of C^*, any edge in C will be singular.
- For any other edges of color a, Kempe switches happened at both end vertices or at neither. Therefore, such edges do not change the property (singular or non-singular).

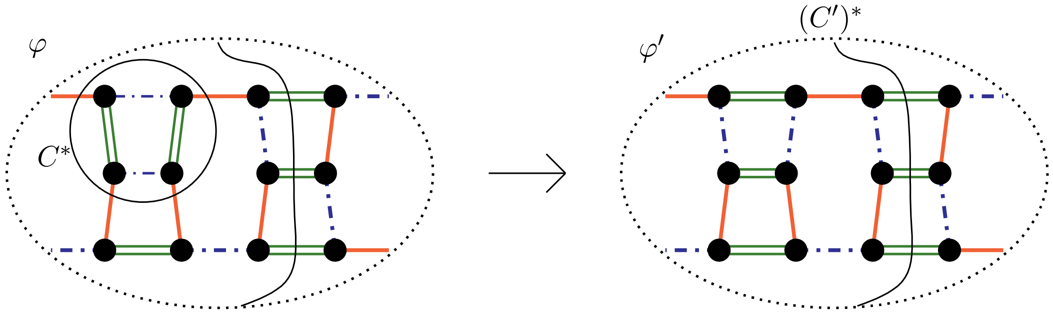

Figure 9: Two 3-edge-colorings of a cubic even map G on the projective plane. The right one is canonical.

- Take an essential cycle D^* in G^*. Since C^* is contractible, C^* and D^* intersect an even number of times. Then it is easy to see that $\rho_D(\varphi) = (0,0,0)$ if and only if $\rho_D(\varphi') = (0,0,0)$.

Therefore, the new 3-edge-coloring φ' has fewer non-singular edges such that for every cycle D^* in G^*, $\rho_D(\varphi) = (0,0,0)$ if and only if $\rho_D(\varphi') = (0,0,0)$.

This operation can be performed as long as there exists a contractible cycle in $\big(NS_x(\varphi)\big)^*$ for some color x, and as a result, we obtain a canonical 3-edge-coloring φ' with desired properties. This completes the proof of Lemma 19. □

In Figure 9, we provide an example of the proof of Lemma 19. The left has two cycles in $\big(NS(\varphi)\big)^*$, represented by thin lines: C^* is one of them, which is of color a and contractible, while the other is of color c and essential. If we perform Kempe switches on the cycles of $M_b \cup M_c$ inside C^*, then we obtain the 3-edge-coloring on the right with fewer non-singular edges.

Let G be a cubic bipartite plane graph. Since no essential cycles exist on the plane, a canonical 3-edge-coloring φ of G has no non-singular edges. It is easy to see that for the 4-vertex-coloring f_φ of G^* produced from φ, we only use three colors. (In fact, since all edges are singular, it follows from Lemma 12 that every facial cycle in G is bicolored by φ. The three color classes in f_φ correspond to the (a,b)-faces, the (a,c)-faces, and the (b,c)-faces, where an (x,y)-face is a face whose facial cycle is bicolored by the colors x and y.) Since such a 3-vertex-coloring of G^* is unique except for the permutation of colors, Lemma 19 directly implies Theorem 1.

For the case of the projective plane, Lemma 19 states that any 3-edge-coloring φ of a cubic even map G on the projective plane is Kempe equivalent to some canonical 3-edge-coloring φ' with the same type as φ.

6.2 Topological properties of non-singular edges

Now we consider a canonical 3-edge-coloring φ of a cubic bipartite graph embedded on the projective plane and the topological property on $\big(NS_x(\varphi)\big)^*$. We prove the following.

Lemma 20 *Let G be a cubic bipartite graph embedded on the projective plane, let D^* be an essential cycle in G^*, and let φ be a canonical 3-edge-coloring of G. Then for $x \in \{a,b,c\}$, $\big(NS_x(\varphi)\big)^*$ consists of only one essential cycle if and only if $\sigma_D(M_x) = 0$*

Proof. We first prove the "only if" part for $x = a$. Assume that $\big(NS_a(\varphi)\big)^*$ consists of only one essential cycle. Then cutting the projective plane along the cycle in $\big(NS_a(\varphi)\big)^*$

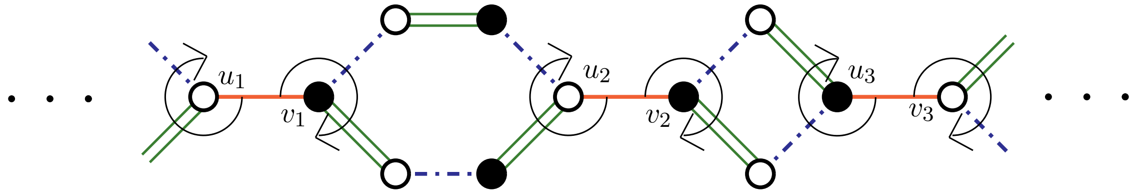

Figure 10: Rotation on the vertices $u_1, v_1, u_2, \ldots$ in the proof of Lemma 20.

results in a disk. Since all cycles in $M_b \cup M_c = E(G) - M_a$ are contained in this disk, they are contractible, and hence $M_b \cup M_c$ is 2-face-colorable by Lemma 5 (I). Then it follows from Theorem 6 (III) that $\sigma_D(M_a) = 0$. The cases for $x = b$ and $x = c$ are symmetric, and hence the "only if" part holds.

Next we prove the "if" part. Assume that $\sigma_D(M_a) = 0$, but $\big(NS_a(\varphi)\big)^*$ does not consist of only one essential cycle. Since φ is canonical, we have $\big(NS_a(\varphi)\big)^* = \emptyset$, so all edges of color a are singular. Since $\sigma_D(M_a) = 0$, it follows from Theorem 6 (III) that $G - M_a$ is 2-face-colorable. By Lemma 5, this implies that all cycles in $M_b \cup M_c$ are contractible.

We color each vertex in one partite set of G by black, and each vertex in the other partite set by white. Let C be an essential cycle in G, and let $u_1v_1, u_2v_2, \ldots, u_rv_r$ be the sequence of the edges in $M_a \cap E(C)$ such that they appear in C in that order and v_i and u_{i+1} are contained in the same cycle in $M_b \cup M_c$ for $1 \le i \le r$, where $u_{r+1} = u_1$. We take such an essential cycle C so that r is as small as possible. By this choice, each cycle in $M_b \cup M_c$ is incident with either exactly two edges in $M_a \cap E(C)$ or no edges in $M_a \cap E(C)$.

For each vertex w in $\{u_i, v_i : 1 \le i \le r\}$, we assign a local rotation to w in the following manner.

- If w is a black vertex, then the colors a, b and c appear in this order on the edges incident with w along the rotation.
- If w is a white vertex, then the colors a, b and c appear in the opposite order (the order of a, c, b) on the edges incident with w along the rotation.

See Figure 10 for an example, where all vertices $u_1, v_1, u_2, \ldots, u_r, v_r$ have the consistent rotation.

Let $1 \le i \le r$. Since u_i and v_i belong to different partite sets of G and the edge u_iv_i is singular, both u_i and v_i have the same rotation. On the other hand, note that v_i and u_{i+1} are contained in the same cycle in $M_b \cup M_c$. Since all cycles in $M_b \cup M_c$ are contractible, both edges u_iv_i and $u_{i+1}v_{i+1}$ leave from the cycle in $M_b \cup M_c$. Thus, both v_i and u_{i+1} have the same rotation, regardless of the partite set to which they belong. These give the consistent rotations along the sequence $u_1, v_1, \ldots, u_r, v_r, u_1$. However, this is impossible, since the sequence corresponds to an essential cycle on the projective plane, which is one-sided. This completes the proof of the "if" part. $\square$

6.3 The color factor and canonical odd 3-edge-colorings

Before proceeding to the proof, we introduce an important notation, called a *color factor*. Let H be an even map on the projective plane. Put a new vertex in each face of H and join it to all the vertices on the corresponding facial walk. Then the resulting map K is an Eulerian triangulation, which is called the *face subdivision* of H and denoted by $K = \mathrm{FS}(H)$. For an Eulerian triangulation K, a vertex set U is called a *color factor* of K if there exists an even map H such that $K = \mathrm{FS}(H)$ and $U = V(K) - V(H)$. It is easy to see that U is a color factor of K if and only if it satisfies both of the following conditions:

(CF1) $U \neq \emptyset$ and U is independent.

(CF2) For any two triangular facial cycles uxy and vxy in K sharing an edge xy, $u \in U$ if and only if $v \in U$.

We use the following theorem.

Theorem 21 (Mohar [25]) *Every Eulerian triangulation K of the projective plane has a color factor. Moreover, if K is not 3-vertex-colorable, the color factor is uniquely determined.*

The next lemma is well-known. We include a short proof here.

Lemma 22 *Let G be a cubic bipartite graph embedded on the projective plane. Then the dual G^* is an Eulerian triangulation that is not 3-vertex-colorable. Thus, G^* has the unique color factor.*

Proof. Let G be a cubic bipartite graph embedded on the projective plane. Since each facial cycle in G has even length, each vertex in G^* has even degree, that is, G^* is an Eulerian triangulation. Assume that G^* is 3-vertex-colorable, that is, the faces of G can be colored by three colors, say A, B and C, such that any two adjacent faces have different colors. We color each vertex in one partite set of G by black, and each vertex in the other partite set by white. Then for each vertex v in G, we give a rotation around v so that

- the colors A, B and C appear in the faces incident with v in this order along the rotation if v is a black vertex,
- and the colors A, B and C appear in the faces incident with v in the opposite order along the rotation if v is a white vertex.

This choice implies that each vertex admits a consistent rotation, which is a contradiction since G is a graph embedded on the projective plane. Therefore, G^* is not 3-vertex-colorable. By Theorem 21, G^* has the unique color factor. $\square$

Now, we prove the next lemma, which is the main result in this subsection.

Lemma 23 *Let G be a cubic bipartite graph embedded on the projective plane, let D^* be an essential cycle in G^*, and let φ be a canonical odd 3-edge-coloring in G satisfying $\rho_D(\varphi) = (1, 1, 0)$. Furthermore, let U^* be the color factor of G^*, which is obtained by Lemma 22. Then both of the following hold:*

(i) *U^* corresponds to the set of the cycles in $M_a \cup M_b$.*

(ii) *$V(C^*) \cap U^* = \emptyset$, where C^* is the unique essential cycle in $\big(NS_c(\varphi)\big)^*$.*

Proof. Let G be a cubic bipartite graph embedded on the projective plane, let D^* be an essential cycle in G^*, let φ be a canonical odd 3-edge-coloring satisfying $\rho_D(\varphi) = (1, 1, 0)$, and let U^* be the color factor of G^*. It follows from Lemma 20 that $NS_a(\varphi) = NS_b(\varphi) = \emptyset$ and $\big(NS_c(\varphi)\big)^*$ consists of only one essential cycle, say C^*. Note that all edges colored by a or b are singular. Thus, by Lemma 12, each cycle in $M_a \cup M_b$ is facial. Let U^* be the set of vertices in G^* that correspond to each cycle in $M_a \cup M_b$. Then it is easy to see that U^* satisfies both conditions (CF1) and (CF2). Therefore, it follows from Theorem 21 that U^* is the unique color factor of G^*. In particular, since $E(C^*) \subseteq \big(NS_c(\varphi)\big)^*$, C^* does not pass

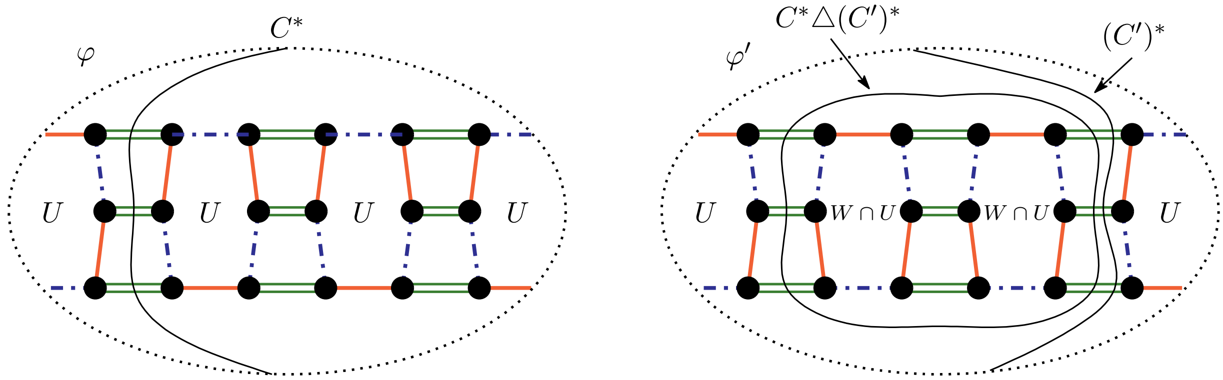

Figure 11: Two canonical odd 3-edge-colorings φ and φ' in a cubic bipartite graph G embedded on the projective plane.

through any face bounded by a cycle in $M_a \cup M_b$, and hence $V(C^*) \cap U^* = \emptyset$. This completes the proof of Lemma 23. $\square$

See the left of Figure 11 for an example, where the cycles in $M_a \cup M_b$, indicated by the symbol U, correspond to the color factor U^* in the dual G^*. The thin curve represents the unique cycle C^* in $\big(NS_c(\varphi)\big)^*$.

6.4 Kempe equivalence of odd 3-edge-colorings

In this section, we prove the following theorem, which is the main result in Section 6.

Theorem 24 *Let G be a cubic bipartite graph embedded on the projective plane. Then any two odd 3-edge-colorings in G are Kempe equivalent.*

Proof of Theorem 24. Let G be a cubic bipartite graph embedded on the projective plane, let D^* be an essential cycle in G^*, and let φ and φ' be two odd 3-edge-colorings in G. By Lemma 19, it suffices to prove the case where both φ and φ' are canonical odd 3-edge-colorings. By swapping the colors (which can be expressed as Kempe switches) if necessary, we may assume that $\rho_D(\varphi) = \rho_D(\varphi') = (1, 1, 0)$.

Let C^* and $(C')^*$ be the unique essential cycles in $\big(NS_c(\varphi)\big)^*$ and the one in $\big(NS_c(\varphi')\big)^*$, respectively. Since both C^* and $(C')^*$ are essential cycles on the projective plane, $C^*\triangle(C')^*$ bounds some disks on the projective plane. Let W^* be the vertices of G^* that appear inside the disks, excluding its boundary. (So, W^* does not contain any vertices in $C^* \cup (C')^*$).

Let U^* be the color factor of G^*. By Lemma 23 (ii), $V(C^*)\cap U^* = \emptyset$ and $V\big((C')^*\big)\cap U^* = \emptyset$. This implies that $W^* \cap U^* \neq \emptyset$. By Lemma 23 (i), each vertex in $W^* \cap U^*$ corresponds to a cycle in $M_a \cup M_b$. Then performing Kempe switches to φ' on all cycles corresponding to $W^* \cap U^*$, we obtain the following.

- For each edge in $C\triangle C'$, Kempe switches are performed at exactly one end vertex.
- For any other edges, Kempe switches are performed at both or neither end vertices.

Let φ'' be the obtained 3-edge-coloring. Then, φ'' satisfies that C^* is the unique essential cycle in $NS_c(\varphi'')$, and hence φ'' is indeed equivalent to φ, up to the permutation of the colors. This completes the proof of Theorem 24. $\square$

See the right of Figure 11 for example. From the unique cycle C^* in $\big(NS_c(\varphi)\big)^*$ and $(C')^*$ in $\big(NS_c(\varphi')\big)^*$, we take the contractible curve $C^*\triangle(C')^*$ obtained by their symmetric difference. The set of all faces inside $C^* \triangle (C')^*$ is W. From φ', by Kempe switches on the cycles corresponding to $W \cap U$, we obtain the 3-edge-coloring φ.

6.5 Proof of Theorem 2

Let G be a cubic bipartite projective-planar graph.

We first prove the "only if" part. Assume that G has an embedding in the projective plane such that the dual triangulation G^* is 4-vertex-colorable. Then it follows from Theorem 14 that G admits both an odd 3-edge-coloring and an even 3-edge-coloring. By Theorem 9, they are not Kempe equivalent, and hence there are at least two Kempe equivalence classes. This completes the proof of the "only if" part.

We next prove the "if" part. Assume that G has an embedding in the projective plane such that the chromatic number of the dual triangulation G^* is at least 5. Then it follows from Theorem 14 that G admits an odd 3-edge-coloring, but no even 3-edge-colorings. By Theorem 24, all odd 3-edge-colorings are Kempe equivalent. This completes the proof of the "if" part. $\square$

7 Remarks

7.1 Discussion for even 3-edge-colorings

In Section 6, we have explained several properties on odd 3-edge-colorings in a cubic bipartite graph embedded on the projective plane. In this subsection, we explain that similar (but not same) properties hold with odd 3-edge-colorings replaced by even 3-edge-colorings. Recall that Lemmas 19 and 20 hold also for even 3-edge-colorings. Then similarly to the proof of Lemma 23, we can prove the following lemma, which shows the relation to the color factor of G^*. We leave the details for the readers.

Lemma 25 *Let G be a cubic bipartite graph embedded on the projective plane, let φ be a canonical even 3-edge-coloring in G, and furthermore, let U^* be the color factor of G^*, which is obtained by Lemma 22. By Lemma 20, for each color x in $\{a, b, c\}$, $\big(NS_x(\varphi)\big)^*$ forms an essential cycle, say C_x^*. Then the following hold:*

(U1) *For each $x \in \{a, b, c\}$, C_x^* alternates a vertex in U^* and not in U^*.*

(U2) *There exists a unique vertex v_0^* in U^* such that the three essential cycles C_a^*, C_b^* and C_c^* pairwise transversally intersect at v_0^*.*

In contrast to odd 3-edge-colorings, two even 3-edge-colorings in a cubic bipartite graph embedded on the projective plane are not necessarily Kempe equivalent. For example, see the two even 3-edge-colorings in Figure 12. The elements of $\mathbb{Z}_2 \times \mathbb{Z}_2$ on faces represent the produced 4-vertex-coloring of G^*, and $(0, 0)$-faces form a color factor of G^*. The central hexagon corresponds to the vertex v_0^* in (U2) in Lemma 25. Since in each 3-edge-coloring, the edges of any two colors form a Hamiltonian cycle, any Kempe switch only gives rise to a 3-edge-coloring obtained by the permutation of colors. Note that the graph in Figure 12 contains exactly three Kempe equivalence classes, two of which consist of even 3-edge-colorings and the other consists of odd ones.

After submitting the first version of this paper, Nozaki [32] pointed out that if we replace the central hexagon of the cubic bipartite graph G in Figure 12 with G itself, see Figure 13, then the resultant graph has four Kempe equivalence classes consisting of even 3-edge-colorings. (Both the inner side and the outer side have two even 3-edge-colorings as in Figure

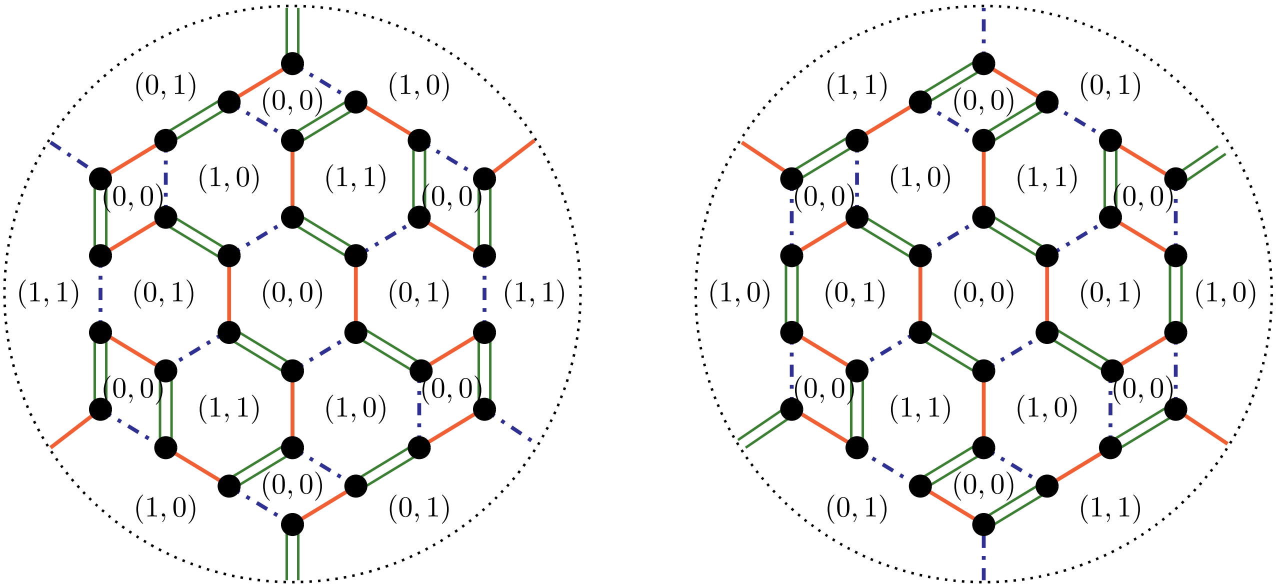

Figure 12: Two even 3-edge-colorings in a cubic bipartite graph G on the projective plane such that they are not Kempe equivalent.

12, and their combination produces four even 3-edge-colorings of the whole graph. In each of them, the edges of any two colors form a Hamiltonian cycle, and hence they are not Kempe equivalent.) Repeating the replacement of the central hexagon with G t times, we can construct a cubic bipartite graph embedded on the projective plane having 2^{t+1} Kempe equivalence classes consisting of even 3-edge-colorings.

7.2 Other surfaces

In this paper, we have focused on cubic graphs embedded on the projective plane. Then it is natural to consider other surfaces. However, there do not seem to exist nice properties on the Kempe equivalence classes even for cubic bipartite graphs embedded on the torus.

Consider the two 3-edge-colorings in Figure 14. Since the Kempe switch on the cycle depicted by the thin black line on the left gives rise to the 3-edge-coloring on the right, they are Kempe equivalent. Note that the left one does not produce a 4-vertex-coloring in the dual, while the right one does. Therefore, Theorem 9 does not hold for a cubic graph embedded on the torus, even if it is bipartite.

Furthermore, it is natural to ask how many Kempe equivalence classes a cubic bipartite graph embedded on the torus can have. To answer this question, consider the graph in Figure 15, which is the dual of the complete graph K_7 embedded on the torus, and is called the *Heawood graph*. It has exactly forty-eight 3-edge-colorings all of which have the property that the edges of any two colors form a Hamiltonian cycle. Because of this property, the number of 3-edge-colorings contained in each Kempe equivalence class is exactly $3! = 6$, and hence the graph in Figure 15 has exactly eight Kempe equivalence classes. In addition, repeating the replacement of one hexagon of the graph in Figure 15 with the cubic bipartite graph G in Figure 12, we can construct cubic bipartite graphs embedded on the torus with arbitrary many Kempe equivalence classes.

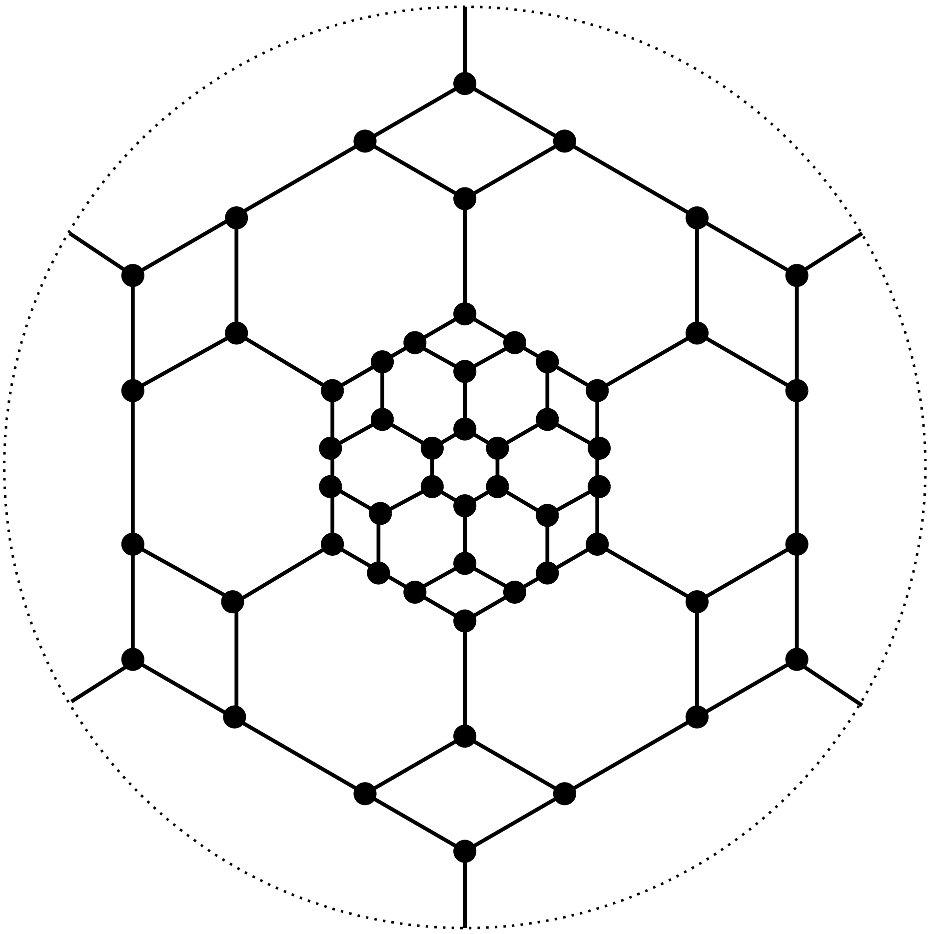

Figure 13: A cubic bipartite graph embedded on the projective plane such that there are four Kempe equivalence classes consisting of even 3-edge-colorings.

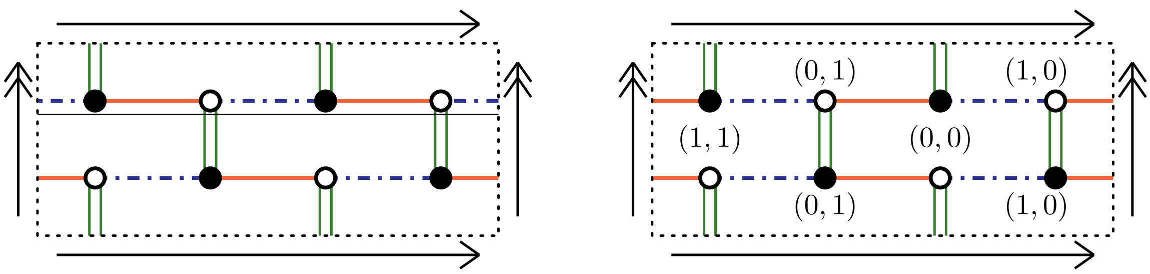

Figure 14: Two 3-edge-colorings of a cubic bipartite graphs embedded on the torus. We obtain the torus by identifying the top and the bottom, and the left and the right, respectively.

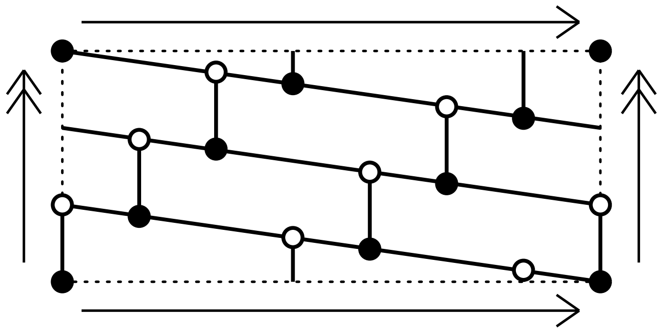

Figure 15: The Heawood graph embedded on the torus.

7.3 Application to list-edge-colorings

We prove a corollary of our results for list-edge-coloring conjecture. For a positive integer k, an *edge-list* $L : E(G) \to 2^{\mathbb{N}}$ of a graph G is a mapping that assigns a set of colors to each edge. An *L-edge-coloring* of G is an edge-coloring f such that $f(e) \in L(e)$ for any edge $e \in E(G)$. If G admits an L-edge-coloring for any edge-list L with $|L(e)| \geq k$ for each edge $e \in E(G)$, then G is said to be *k-list-edge-colorable*. Since a k-edge-coloring is an L-edge-coloring with $L(e) = [k]$ for any edge $e \in E(G)$, any k-list-edge-colorable graph is k-edge-colorable. The converse is an open problem, which is known as list-edge-coloring conjecture:

Conjecture 26 (List-edge-coloring conjecture, see [6]) *Any k-edge-colorable graph is k-list-edge-colorable.*

To attack Conjecture 26, Alon [2] posed a method using the signature of 3-edge-colorings.

Lemma 27 (Alon [2]) *For a 3-edge-colorable cubic graph G, if all pairs of two 3-edge-colorings φ_1 and φ_2 in G satisfy sign$(\varphi_1, \varphi_2) = +1$, then G is 3-list-edge-colorable.*

Thus, Theorem 11 and Lemma 27, together with Theorem 8, imply the next corollary. This gives a new class of graphs for which Conjecture 26 hold.

Corollary 28 *Let G be a 3-edge-colorable cubic graph embedded on the projective plane. If the dual G^* is not 4-vertex-colorable, then G is 3-list-edge-colorable.*

Note that the signature can be similarly defined for k-edge-colorings in k-regular graphs with $k \geq 4$. To be exact, Alon [2] established that Lemma 27 holds even for k-edge-colorable k-regular graph under a suitable definition of sign(φ_1, φ_2). Ellingham and Goddyn [8] proved that for every k-regular plane graph, all k-edge-colorings have the same signature. This together with Lemma 27 implies that every k-edge-colorable k-regular planar graph is k-list-edge-colorable. The similar situations for the projective-planar case was discussed by Abe and the author [1].

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