Yokohama Mathematical
Journal Vol. 66, 2020

A MEAN CONVERGENCE THEOREM FINDING A COMMON ATTRACTIVE POINT OF TWO NONLINEAR MAPPINGS

By

Takanori Ibaraki and Yukio Takeuchi

(Received June 21, 2019; Revised February 27, 2020)

Abstract. In this article, we present a mean convergence theorem finding a common attractive point of commutative nonlinear self–mappings S and T on a bounded subset of a Hilbert apace, where S is λ–hybrid and T is μ–hybrid with real numbers λ, μ.

1. Introduction

In this article, N and N_0 denote the sets of all positive integers and all non–negative integers, respectively. $N(i,j)$ denotes the set $\{k \in N_0 : i \leq k \leq j\}$ for all $i, j \in N_0$ with $i \leq j$. R denotes the set of all real numbers. Unless otherwise noted, H always denotes a real Hilbert space with inner product $\langle \cdot, \cdot \rangle$ and norm $\| \cdot \|$ derived from $\langle \cdot, \cdot \rangle$, and C always denotes a non–empty subset of H.

Let T be a mapping from C into H. Then, T^0 denotes the identity mapping on C, and $F(T)$ denotes the set of all fixed points of T, that is, $F(T) = \{x \in C : x = Tx\}$. T is called nonexpansive if $\|Tx - Ty\| \leq \|x - y\|$ for all $x, y \in C$. P_C denotes the metric projection from H onto C when C is closed and convex.

In 1963, DeMarr [9] proved a common fixed point theorem for a family of commuting nonexpansive self–mappings in a Banach space; for an elementary proof, see Kubota and Takeuchi [16]. After DeMarr, many researchers studied for common fixed points of families of nonexpansive mappings; see Linhart [18], Bruck [7, 8], Ishikawa [11], Kuhfittig [17], Shimoji and Takahashi [20], Suzuki [22], and so on. On the other hand, in 1975, Baillon [6] proved the following mean convergence theorem which is well–known as the first nonlinear ergodic theorem in a Hilbert space.

Theorem B. *Let C be a bounded closed and convex subset of a Hilbert space H and let T be a nonexpansive self–mapping on C. Let $\{b_n\}$ be the sequence in*

2010 Mathematics Subject Classification: 47H09, 47H10, 41A65
Key words and phrases: Attractive point, λ–hybrid mappings, mean convergence theorem.

C defined by

$$v_1 \in C, \quad v_{n+1} = Tv_n, \quad b_n = \frac{1}{n}\sum_{k=1}^{n} v_k \qquad \text{for each } n \in N.$$

Then, the sequence $\{b_n\}$ *converges weakly to a fixed point of* T.

After Baillon, many mean convergence theorems appeared.

Recently, some wide classes of nonlinear mappings were introduced. Aoyama and co–authors [2] introduced the class of λ–hybrid mappings for $\lambda \in R$. Kocourek and co–authors [12] introduced the class of generalized hybrid mappings. In a different direction, Aoyama [1] and Kohsaka [13] presented convergence theorems for quasi–nonexpansive type mappings. In 2010, Takahashi and Takeuchi [25] introduced the notion of an attractive point of a mapping T. They denote by $A(T)$ the set of all attractive points of T. Then, they proved the following mean convergence theorem finding an attractive point of a generalized hybrid mapping without closedness and convexity of its domain.

Theorem TT. *Let* C *be a subset of a Hilbert space* H *and* T *be a generalized hybrid self–mapping on* C. *Let* $\{v_n\}$ *and* $\{b_n\}$ *be sequences defined by*

$$v_1 \in C, \qquad v_{n+1} = Tv_n, \quad b_n = \frac{1}{n}\sum_{k=1}^{n} v_k \qquad \text{for each } n \in N.$$

Suppose $\{v_n\}$ *is bounded. Then the following hold*:

(1) $A(T)$ *is non-empty, closed and convex.*
(2) $\{b_n\}$ *converges weakly to* $u \in A(T)$, *where* $u = \lim_n P_{A(T)} v_n$.

Remark. In the case when C is closed and convex, $u \in F(T)$ holds.

In 1997, Shimizu and Takahashi [19] considered for common fixed points of a finite family of commutative nonexpansive mappings. Then, they introduced an iteration scheme combined Halpern type and Baillon type, and proved a strong convergence theorem in Hilbert spaces. In 1998, Atsushiba and Takahashi [4] considered common fixed points of commutative two nonexpansive mappings. They introduced an iteration scheme combined Mann type and Baillon type, and proved a weak convergence theorem in uniformly convex Banach spaces. Motivated by [4], Suzuki [21] presented a result in general Banach spaces; also see Takeuchi [26]. Atsushiba and Takahashi [5] presented a mean convergence theorem finding a common attractive point of commutative two nonexpansive mappings in Hilbert spaces; also see Ibaraki and Takeuchi [10].

Very recently, Kohsaka [14] presented some extensions of main results in [19] and [4], in Hilbert spaces. Kohsaka [14] also presented the following theorem.

THEOREM K. *Let C be a bounded closed and convex subset of a Hilbert space H. Let S be a λ–hybrid self–mapping and T be a μ–hybrid self–mapping on C with $\lambda, \mu \in R$. Let $F = F(S) \cap F(T)$. Assume $ST = TS$. Let $\{x_n\}$ be the sequence defined by*

$$x_1 \in C, \qquad x_{n+1} = \frac{1}{(n+1)^2} \sum_{i=0}^{n} \sum_{j=0}^{n} S^i T^j x_1 \qquad \text{for each} \quad n \in N.$$

Then the following hold:

(1) $\{P_F S^i T^j x_1\}_{(i,j)\in N_0^2}$ *converges strongly to an element u of F.*
(2) $\{x_n\}$ *converges weakly to $u \in F$.*

Remark. Of course, we can replace the boundedness of C by $F \neq \emptyset$.

Motivated by the works as above, we present a mean convergence theorem finding a common attractive point of commutative nonlinear mappings S and T on a bounded subset of a Hilbert apace, where S is λ–hybrid and T is μ–hybrid with $\lambda, \mu \in R$.

2. Preliminaries

Let H be a Hilbert space. Then, we know the following:

(1) A bounded closed and convex subset C of H is weakly compact. A bounded sequence in H has a weakly convergent subsequence.

(2) Let $\{u_n\}$ be a sequence in H and z be a point in H. Then $\{u_n\}$ converges weakly to $z \in H$ if every weak cluster point of $\{u_n\}$ and z are the same.

(3) Let C be a closed and convex subset of H. Then, for each $x \in H$, there is a unique point z_x of C satisfying $\|x - z_x\| = \inf\{\|x - z\| : z \in C\}$. z_x is called the unique nearest point of C to x. Define a mapping P_C by $P_C x = z_x$ for $x \in H$. P_C is called the metric projection from H onto C. For each $x \in H$ and $y \in C$, the following holds:

$$0 \leq \langle x - P_C x, P_C x - y\rangle \quad \text{and} \quad \|x - P_C x\|^2 + \|P_C x - y\|^2 \leq \|x - y\|^2.$$

Of course, $P_C x = x$ for all $x \in C$. It is known that P_C is nonexpansive. We presented some basic facts needed in the sequel; for details, see Takahashi [23].

Let C be a subset of H and T be a mapping from C into H. $A(T)$ denotes the set of all attractive points of T, that is, $A(T) = \{x \in H : \|Ty - x\| \leq \|x - y\| \text{ for all } y \in C\}$; see Takahashi and Takeuchi [25]. T is called quasi-nonexpansive if $F(T) \neq \emptyset$ and $\|Tx - y\| \leq \|x - y\|$ for all $x \in C$ and $y \in F(T)$, that is, $\emptyset \neq F(T) \subset A(T)$. Then T is quasi-nonexpansive if T is nonexpansive and $F(T) \neq \emptyset$.

Here, we show an example in Atsushiba and co–authors [3] which represents properties of the sets of attractive points for two typical nonexpansive mappings.

Example 2.1. Let D be the bounded subset $\{(x_1, x_2) \in R^2 : 1 < x_1^2 + x_2^2 < 4\}$ of the 2–dimensional Euclidean space R^2. Then D is neither closed nor convex. Let $\alpha \in (0, 2\pi)$. Let S and T be nonexpansive self–mappings on D such that, for each
$(x_1, x_2) \in D$,

$$S(x_1, x_2) = (-x_1, x_2), \quad T(x_1, x_2) = (x_1 \cos\alpha - x_2 \sin\alpha, x_1 \sin\alpha + x_2 \cos\alpha).$$

Then, we can easily see

$$\begin{aligned} F(S) &= \{(x_1, x_2) \in R^2 : x_1 = 0,\ 1 < |x_2| < 2\}, \qquad & F(T) &= \emptyset, \\ A(S) &= \{(x_1, x_2) \in R^2 : x_1 = 0\}, & A(T) &= \{(0, 0)\}. \end{aligned}$$

So, $F(S)$ consists of two line segments and $F(T) = \emptyset$. On the other hand, $A(S)$ is the symmetric axis of this transformation S and $A(T)$ is the center of this rotation T.

Consider sequences $\{v_n\}, \{u_n\}$ in C, and $\{b_n\}, \{c_n\}$ in H as below:

$$\begin{aligned} &v_1 = (y_1, y_2), u_1 = (y_1', y_2') \in D, \\ &v_{n+1} = S^n v_n, \quad b_n = \tfrac{1}{n}\textstyle\sum_{i=1}^n v_n, \quad u_{n+1} = T^n u_n, \quad c_n = \tfrac{1}{n}\textstyle\sum_{i=1}^n u_n \quad \text{for all} \quad n \in N. \end{aligned}$$

By simple calculations, we see that $\{b_n\}$ and $\{c_n\}$ converge strongly to $v = (0, y_2) \in A(S)$ and $u = (0, 0) \in A(T)$, respectively. Obviously, $v = (0, y_2)$ is not always a point in $F(S)$. Also, $u = (0, 0)$ is not in D, that is, $u = (0, 0) \notin F(T)$.

Aoyama and co–authors [2] introduced the class of λ–hybrid mappings for $\lambda \in R$. Let $\lambda \in R$. Then T is called λ–hybrid if

$$(\lambda_h) \quad \|Tx - Ty\|^2 \le \|x - y\|^2 + 2(1 - \lambda)\langle x - Tx, y - Ty\rangle \quad \text{for all} \quad x, y \in C.$$

For example, Kohsaka [14] use the following expression: Let S be a λ–hybrid self–mapping on C and T be a μ–hybrid self–mapping on C. However, to avoid confusion, we call T (λ)–hybrid if there is $\lambda \in R$ satisfying (λ_h). Then the expression becomes as below: Let S and T be (λ)–hybrid self–mappings on C with λ and μ. It is easy to confirm that a (λ)–hybrid mapping T is quasi–nonexpansive if $F(T) \neq \emptyset$.

Also, Kocourek and co–authors [12] introduced the class of generalized hybrid mappings. T is called generalized hybrid if there exist $\alpha, \beta \in R$ such that

$$\alpha\|Tx - Ty\|^2 + (1 - \alpha)\|x - Ty\|^2 \le \beta\|Tx - y\|^2 + (1 - \beta)\|x - y\|^2 \text{ for all } x, y \in C.$$

The class of generalized hybrid mappings is wider than the class of (λ)–hybrid mappings. Nevertheless, the class of (λ)–hybrid mappings contains some important classes of nonlinear mappings. For example, a nonexpansive mapping is 1–hybrid, that is, (λ)–hybrid. Also, a nonspreading mapping is 0–hybrid and a hybrid mapping is 1/2–hybrid; the class of nonspreading mappings was introduced by Kohsaka and Takahashi [15], and the class of hybrid mappings was introduced by Takahashi [24]. Furthermore, since the last term in (λ_h) is written by inner product, it is easy to deal with. From these reasons, in this article, we mainly consider (λ)–hybrid mappings.

3. Lemmas

The following lemmas are due to Takahashi and Takeuchi [25].

Lemma 3.1. *Let C be a subset of H and T be a mapping from C into H. Then, $A(T)$ is a closed and convex subset of H.*

Lemma 3.2. *Let C be a subset of H and T be a self mapping on C. Suppose x is a point in $A(T)$ and z_x is the unique nearest point of C to x. Then $z_x \in F(T)$. In particular, $A(T) \cap C \subset F(T)$. Furthermore, $A(T) \cap C = F(T)$ holds if $F(T) \subset A(T)$.*

Maybe the following lemma is well–known.

Lemma 3.3. *Let x, v, w be points in H. Then, the following equality holds*:

$$\langle (x-v)+(x-w), v-w\rangle = \|x-w\|^2 - \|x-v\|^2.$$

Proof. Fix any $x, v, w \in H$. Then we easily have

$$\begin{aligned}\langle (x-v)+(x-w), v-w\rangle &= \langle (x-v)+(x-w), (v-x)+(x-w)\rangle \\ &= \|x-w\|^2 - \|x-v\|^2 + \langle x-v, x-w\rangle + \langle x-w, v-x\rangle \\ &= \|x-w\|^2 - \|x-v\|^2.\end{aligned}$$

□

Remark 3.4. Let $\{z_i\}$ be a sequence in H and set $s_n = \frac{1}{n}\sum_{i=1}^{n} z_i$ for each $n \in N$. Then, for each $n \in N$, the following equality follows immediately from Lemma 3.3:

$$\langle (s_n-v)+(s_n-w), v-w\rangle = \frac{1}{n}\sum_{i=1}^{n}\|z_i-w\|^2 - \frac{1}{n}\sum_{i=1}^{n}\|z_i-v\|^2.$$

The following lemma is essentially due to Takahashi and Takeuchi [25].

Lemma 3.5. *Let C be a subset of H and T be a mapping from C into H. Let $\{u_n\}$ be a sequence in H which satisfies*

$$\limsup_n \sup_{y\in C}\langle (u_n - y) + (u_n - Ty), y - Ty\rangle \leq 0.$$

Suppose $\{u_n\}$ converges weakly to some point $u \in H$. Then, $u \in A(T)$.

Proof. Since $\{u_n\}$ converges weakly to $u \in H$, by Lemma 3.3, we have

$$\begin{aligned}\|u - Tx\|^2 - \|u - x\|^2 &= \langle (u - x) + (u - Tx), x - Tx\rangle \\ &= \limsup_n \langle (u_n - x) + (u_n - Tx), x - Tx\rangle \\ &\leq \limsup_n \sup_{y\in C}\langle (u_n - y) + (u_n - Ty), y - Ty\rangle \leq 0\end{aligned}$$

for every $x \in C$. This implies $u \in A(T)$. □

4. A mean convergence theorem

We need some lemmas to gain our end. Lemma 4.2 is a half of the proof of our main result; Lemma 4.5 is another half. We prepare Lemma 4.1 to prove Lemma 4.2.

Lemma 4.1. *Let C be a bounded subset of H. Set $L = \sup\{\|x-y\| : x, y \in C\}$. Let S be a (λ)–hybrid self–mapping on C with λ. Let T be a self–mapping on C. For each $n \in N$, define a mapping S_n from C into H by*

$$S_n = \frac{1}{n^2}\sum_{i=0}^{n-1}\sum_{j=0}^{n-1} S^i T^j.$$

Then, for each $n \in N$, the following holds:

$$\sup_{x,y\in C}\langle (S_n x - y) + (S_n x - Sy), y - Sy\rangle \leq \frac{1+2|1-\lambda|}{n} L^2.$$

Proof. Fix any $x, y \in C$ and $n \in N$. We easily have

$$\begin{aligned}\left|\sum_{i=1}^{n-1}\langle S^{i-1}x - S^i x, y - Sy\rangle\right| &= |\langle x - S^{n-1}x, y - Sy\rangle| \\ &\leq \|x - S^{n-1}x\|\|y - Sy\| \leq L^2.\end{aligned}$$

Since S is (λ)–hybrid with λ, we have

$$\begin{aligned}(4.1)\quad \frac{1}{n}\sum_{i=0}^{n-1}\|S^i x - Sy\|^2 &= \frac{1}{n}\|x - Sy\|^2 + \frac{1}{n}\sum_{i=1}^{n-1}\|S^i x - Sy\|^2 \\ &\leq \frac{1}{n}L^2 + \frac{1}{n}\sum_{i=0}^{n-2}\|S^i x - y\|^2 + \frac{2(1-\lambda)}{n}\sum_{i=1}^{n-1}\langle S^{i-1}x - S^i x, y - Sy\rangle \\ &\leq \frac{1}{n}L^2 + \frac{2|1-\lambda|}{n} \times L^2 + \frac{1}{n}\sum_{i=0}^{n-1}\|S^i x - y\|^2.\end{aligned}$$

In Remark 3.4, set $z_i = S^{i-1}x \in C$, $w = Sy$ and $v = y$. Then, by (4.1), we have

$$\begin{aligned}(4.2)\quad &\langle(\tfrac{1}{n}\textstyle\sum_{i=0}^{n-1} S^i x - y) + (\tfrac{1}{n}\sum_{i=0}^{n-1} S^i x - Sy), y - Sy\rangle \\ &= \tfrac{1}{n}\textstyle\sum_{i=0}^{n-1} \|S^i x - Sy\|^2 - \tfrac{1}{n}\sum_{i=0}^{n-1} \|S^i x - y\|^2 \leq \tfrac{1+2|1-\lambda|}{n} L^2.\end{aligned}$$

Fix any $j \in N(0, n-1)$ and replace x by $T^j x$ in (4.2). Then we have

$$(4.3)\qquad \langle(\tfrac{1}{n}\textstyle\sum_{i=0}^{n-1} S^i T^j x - y) + (\tfrac{1}{n}\sum_{i=0}^{n-1} S^i T^j x - Sy), y - Sy\rangle \leq \tfrac{1+2|1-\lambda|}{n} L^2.$$

Also, we know the following: $\frac{1}{n}\sum_{j=0}^{n-1}(\frac{1}{n}\sum_{i=0}^{n-1} S^i T^j x) = \frac{1}{n^2}\sum_{i=0}^{n-1}\sum_{j=0}^{n-1} S^i T^j x = S_n x$. Then, since (4.3) holds for any $j \in (0, n-1)$, we have

$$\langle(S_n x - y) + (S_n x - Sy), y - Sy\rangle \leq \tfrac{1+2|1-\lambda|}{n} L^2.$$

Finally, since x, y, n are arbitrary, we see that, for each $n \in N$,

$$\sup_{x,y\in C}\langle(S_n x - y) + (S_n x - Sy), y - Sy\rangle \leq \tfrac{1+2|1-\lambda|}{n} L^2.$$

□

LEMMA 4.2. *Let C be a bounded subset of H. Let S and T be self–mappings on C. For each $n \in N$, define a mapping S_n from C into H by*

$$S_n = \tfrac{1}{n^2}\textstyle\sum_{i=0}^{n-1}\sum_{j=0}^{n-1} S^i T^j.$$

Let x_1 be a point in C. Then the sequence $\{S_n x_1\}$ is bounded. Suppose further that S is (λ)–hybrid with λ. Then, the following hold:

(1) $\limsup_n \sup_{x,y\in C}\langle(S_n x - y) + (S_n x - Sy), y - Sy\rangle \leq 0$.
(2) *$A(S)$ is non–empty closed and convex.*
Every weak cluster point of $\{S_n x_1\}$ is a point in $A(S)$.

Furthermore, in the case when C is closed and convex, the following holds:

(3) *$F(S)$ is non–empty closed and convex.*
Every weak cluster point of $\{S_n x_1\}$ is a point in $F(S)$.

Proof. Set $L = \sup\{\|x - y\| : x, y \in C\}$. Consider the sequence $\{S_n x_1\}$. Fix any $y \in C$ and $n \in N$. By $S^i T^j x_1 \in C$ for $i, j \in N(0, n-1)$, we see that

$$\|S_n x_1 - y\| \leq \tfrac{1}{n^2}\textstyle\sum_{i=0}^{n-1}\sum_{j=0}^{n-1}\|S^i T^j x_1 - y\| \leq \tfrac{1}{n^2}\sum_{i=0}^{n-1}\sum_{j=0}^{n-1} L = L.$$

Then $\{S_n x_1\}$ is bounded. We show (1). By $\limsup_n \frac{1+2|1-\lambda|}{n} L^2 = 0$ and Lemma 4.1, we immediately have the result. We show (2). We know that $\{S_n x_1\}$

has a weakly convergent subsequence. Let $\{S_{n_k}x_1\}$ be a subsequence of $\{S_n x_1\}$ which converges weakly to some $u \in H$. By (1), we know

$$\limsup_k \sup_{y\in C}\langle (S_{n_k}x_1 - y) + (S_{n_k}x_1 - Sy), y - Sy\rangle \leq 0.$$

By Lemma 3.5, we see $u \in A(S)$; $A(S) \neq \emptyset$. By Lemma 3.1, $A(S)$ is closed and convex. We show (3). We know that $A(S)$ is closed and convex. Let $\{S_{n_k}x_1\}$ be a subsequence of $\{S_n x_1\}$ which converges weakly to some $u \in H$. We also know $u \in A(S)$. Since C is closed and convex, C is weakly closed and $\{S_n x_1\}$ is a sequence in C. Then, we see $u \in A(S) \cap C$; $A(S) \cap C \neq \emptyset$. By Lemma 3.2, we know $A(S) \cap C \subset F(S)$. Since S is (λ)–hybrid, we also know $F(S) \subset A(S)$. So, $A(S) \cap C = F(S)$. Since $A(S)$ and C are closed and convex, we see that (3) holds. □

We know that $N_0^2 = \{(i,j) : i,j \in N_0\}$ is a directed set by the binary relation:

$$(k,l) \leq (i,j) \quad \text{if} \quad k \leq i \quad \text{and} \quad l \leq j.$$

Let C be a subset of H and $x_1 \in C$. Let S and T be self–mappings on C. For example, $\{S^iT^jx_1\}_{(i,j)\in N_0^2}$ is a net in C; we denote $\{S^iT^jx_1\}_{(i,j)\in N_0^2}$ by $\{S^iT^jx_1\}$.

Remark 4.3. In Lemma 4.2, $\{S_n x_1\}$ is bounded if $\{S^iT^jx_1\}$ is bounded.

The proof of Lemma 4.4 is referred to Kohsaka [14]; also refer to Aoyama [1].

Lemma 4.4. *Let C be a subset of H and x_1 be a point in C. Let S and T be self–mappings on C. For each $n \in N$, define a mapping S_n from C into H by $S_n = \frac{1}{n^2}\sum_{i=0}^{n-1}\sum_{j=0}^{n-1} S^iT^j$. Suppose $A = A(S) \cap A(T) \neq \emptyset$ and every weak cluster point of the sequence $\{S_n x_1\}$ is a point in A. For simplicity, we denote $S^iT^jx_1$ by $u_{i,j}$ for all $(i,j) \in N_0^2$. Then the following hold:*

(1) *There is $c \in [0,\infty)$ satisfying $\lim_{(i,j)} \|P_A u_{i,j} - u_{i,j}\| = c$.*
(2) *There is an $M \in [0,\infty)$ such that $\|P_A u_{i,j} - u_{i,j}\| \leq M$ for all $(i,j) \in N_0^2$.*
(3) *There is $u_0 \in A$ satisfying $\lim_{(i,j)} \|P_A u_{i,j} - u_0\| = 0$ and*

$$\langle w - u_0, u_{i,j} - P_A u_{i,j}\rangle \leq \|P_A u_{i,j} - u_0\| M \quad \text{for all } (i,j) \in N_0^2 \text{ and } w \in A.$$

(4) $\lim_n \frac{1}{n^2}\sum_{i=0}^{n-1}\sum_{j=0}^{n-1} \|P_A u_{i,j} - u_0\| = 0.$
(5) $\{\frac{1}{n^2}\sum_{i=0}^{n-1}\sum_{j=0}^{n-1} P_A u_{i,j}\}$ *converges strongly to $u_0 \in A$.*
(6) $\{S_n x_1\}$ *converges weakly to $u_0 \in A$.*
(7) *In the case when C is closed and convex, $u_0 \in F = F(S) \cap F(T)$.*

Proof. It is obvious that $\{u_{i,j}\}$ is a net in C satisfying

$$(4.4) \qquad \|u_{i,j} - u\| \leq \|u_{k,l} - u\| \quad \text{whenever} \quad u \in A, \quad (k,l) \leq (i,j).$$

Since A is closed and convex, we can consider the metric projection P_A from H onto A. Recall properties of P_A. Reconfirm the following: For each $x \in H$ and $y \in A$,

$$\langle x - P_A x, y - P_A x \rangle \leq 0 \quad \text{and} \quad \|P_A x - y\|^2 \leq \|x - y\|^2 - \|x - P_A x\|^2.$$

We show (1). Fix any $(i,j),(k,l) \in N_0^2$ with $(k,l) \leq (i,j)$. By $P_A u_{i,j}, P_A u_{k,l} \in A$, the definition of P_A and (4.4), we have

$$\|P_A u_{i,j} - u_{i,j}\| \leq \|P_A u_{k,l} - u_{i,j}\| \leq \|P_A u_{k,l} - u_{k,l}\|. \tag{4.5}$$

From this, $\{\|P_A u_{i,j} - u_{i,j}\|\}$ converges. Then, there is a real number $c \in [0,\infty)$ satisfying $\lim_{(i,j)} \|P_A u_{i,j} - u_{i,j}\| = c$. We show (2). Fix any $(i,j) \in N_0^2$ and $u \in A$. By (4.4), we know $\|u_{i,j} - u\| \leq \|u_{0,0} - u\|$. Set $M = 2\|u_{0,0} - u\|$. Then we have

$$\|P_A u_{i,j} - u_{i,j}\| \leq \|P_A u_{i,j} - P_A u\| + \|u - u_{i,j}\| \leq 2\|u_{i,j} - u\| \leq M.$$

We show (3). Fix any $(i,j),(k,l) \in N_0^2$ with $(k,l) \leq (i,j)$. By $u_{i,j} \in H$, $P_A u_{k,l} \in A$ and properties of P_A, we know

$$\|P_A u_{i,j} - P_A u_{k,l}\|^2 \leq \|u_{i,j} - P_A u_{k,l}\|^2 - \|u_{i,j} - P_A u_{i,j}\|^2.$$

By (4.5), we have

$$\|P_A u_{i,j} - P_A u_{k,l}\|^2 \leq \|u_{k,l} - P_A u_{k,l}\|^2 - \|u_{i,j} - P_A u_{i,j}\|^2.$$

Then, by $\lim_{(i,j)} \|P_A u_{i,j} - u_{i,j}\| = c$, we see that $\{P_A u_{i,j}\}$ is a Cauchy net in A. Since A is closed, there is $u_0 \in A$ satisfying $\lim_{(i,j)} \|P_A u_{i,j} - u_0\| = 0$.

Fix any $w \in A$. By (2) and $\langle w - P_A u_{i,j}, u_{i,j} - P_A u_{i,j} \rangle \leq 0$, we have

$$\begin{aligned}
&\langle w - u_0, u_{i,j} - P_A u_{i,j} \rangle \\
&\quad = \langle w - P_A u_{i,j}, u_{i,j} - P_A u_{i,j} \rangle + \langle P_A u_{i,j} - u_0, u_{i,j} - P_A u_{i,j} \rangle \\
&\quad \leq \langle P_A u_{i,j} - u_0, u_{i,j} - P_A u_{i,j} \rangle \leq \|P_A u_{i,j} - u_0\| \|u_{i,j} - P_A u_{i,j}\| \\
&\quad \leq \|P_A u_{i,j} - u_0\| M.
\end{aligned}$$

We show (4). Fix any $\varepsilon > 0$. By (3), there is a $(k,l) \in N_0^2$ satisfying $\|P_A u_{i,j} - u_0\| < \varepsilon/2$ for all $(i,j) \in N_0^2$ with $(k,l) \leq (i,j)$. For each $n \in N$ satisfying $(k,l) < (n,n)$, set

$$\begin{aligned}
&B_n = \{(i,j) \in N_0^2 : i,j \in N(0,n-1)\}, \quad B_{(k,l)\leq} = \{(i,j) \in B_n : (k,l) \leq (i,j)\}, \\
&B_{<k} = \{(i,j) \in B_n : i \in N(0,k-1)\}, \quad B_{<l} = \{(i,j) \in B_n : j \in N(0,l-1)\}.
\end{aligned}$$

Then, it is obvious that

$$\begin{aligned}&\tfrac{1}{n^2}\textstyle\sum_{i=0}^{n-1}\sum_{j=0}^{n-1}\|P_A u_{i,j}-u_0\|\\ &\quad\le \tfrac{1}{n^2}\big(\textstyle\sum_{B_{(k,l)\le}}\|P_A u_{i,j}-u_0\|+\sum_{B_{<k}}\|P_A u_{i,j}-u_0\|+\sum_{B_{<l}}\|P_A u_{i,j}-u_0\|\big)\\ &\quad< \tfrac{\varepsilon}{2}+\tfrac{nk}{n^2}\|u_{0,0}-u_0\|+\tfrac{nl}{n^2}\|u_{0,0}-u_0\|.\end{aligned}$$

For sufficiently large $n \in N$, we know $\frac{k}{n}\|u_{0,0}-u_0\|+\frac{l}{n}\|u_{0,0}-u_0\|<\varepsilon/2$, that is,

$$0 \le \tfrac{1}{n^2}\textstyle\sum_{i=0}^{n-1}\sum_{j=0}^{n-1}\|P_A u_{i,j}-u_0\|<\varepsilon.$$

We show (5). It is obvious that the following holds:

$$\|\tfrac{1}{n^2}\textstyle\sum_{i=0}^{n-1}\sum_{j=0}^{n-1}P_A u_{i,j}-u_0\| \le \tfrac{1}{n^2}\textstyle\sum_{i=0}^{n-1}\sum_{j=0}^{n-1}\|P_A u_{i,j}-u_0\|.$$

Then, by (4), we have the result.

We show (6). In the proof of (2), we already know that $\{u_{i,j}\}$ is bounded. Then, by Remark 4.3, $\{S_n x_1\}$ is bounded and has a weakly convergent subsequence.

Let $\{S_{n_k} x_1\}$ be a subsequence of $\{S_n x_1\}$ converging weakly to some $w' \in H$. By our assumptions, $w' \in A$ holds. Then, by (3), we see that, for each $k \in N$,

$$\begin{aligned}&\langle w'-u_0, S_{n_k}x_1-\tfrac{1}{n_k^2}\textstyle\sum_{i=0}^{n_k-1}\sum_{j=0}^{n_k-1}P_A u_{i,j}\ \rangle\\ &\quad=\Big\langle w'-u_0, \tfrac{1}{n_k^2}\textstyle\sum_{i=0}^{n_k-1}\sum_{j=0}^{n_k-1}u_{i,j}-\tfrac{1}{n_k^2}\sum_{i=0}^{n_k-1}\sum_{j=0}^{n_k-1}P_A u_{i,j}\ \Big\rangle\\ &\quad=\tfrac{1}{n_k^2}\textstyle\sum_{i=0}^{n_k-1}\sum_{j=0}^{n_k-1}\langle w'-u_0, u_{i,j}-P_A u_{i,j}\ \rangle\\ &\quad\le\tfrac{1}{n_k^2}\textstyle\sum_{i=0}^{n_k-1}\sum_{j=0}^{n_k-1}\|P_A u_{i,j}-u_0\|M.\end{aligned}$$

Since $\{S_{n_k} x_1\}$ converges weakly to w', by (4) and (5), this inequality asserts

$$\|w'-u_0\|^2=\lim_k\langle w'-u_0, S_{n_k}x_1-\tfrac{1}{n_k^2}\textstyle\sum_{i=0}^{n_k-1}\sum_{j=0}^{n_k-1}P_A u_{i,j}\ \rangle\le 0.$$

Thus we see that every weak cluster point of $\{S_n x_1\}$ and u_0 are the same. This implies that $\{S_n x_1\}$ itself converges weakly to $u_0 \in A$.

We show (7). Since C is closed and convex, C is weakly closed and $S_n x_1 \in C$ for all $n \in N$. Then, $u_0 \in A(S) \cap A(T) \cap C$. By Lemma 3.2, we see $u_0 \in F(S) \cap F(T) = F$. □

Lemma 4.5 is an abstract of Lemma 4.4.

Lemma 4.5. *Let C be a subset of H and x_1 be a point in C. Let S and T be self–mappings on C. For each $n \in N$, define a mapping S_n from C into H by $S_n = \frac{1}{n^2}\sum_{i=0}^{n-1}\sum_{j=0}^{n-1}S^iT^j$. Suppose $A = A(S) \cap A(T) \neq \emptyset$ and every weak cluster point of $\{S_n x_1\}$ is a point in A. Then $\{S_n x_1\}$ converges weakly to $u_0 \in A$, where $u_0 = \lim_{(i,j)} P_A S^i T^j x_1$. When C is closed and convex, $u_0 \in F = F(S) \cap F(T)$ holds.*

The following is our main result.

THEOREM 4.6. *Let C be a bounded subset of H and x_1 be a point in C. Let S and T be (λ)–hybrid self–mappings on C with λ and μ which satisfy $ST = TS$. For each $n \in N$, define a mapping S_n by $S_n = \frac{1}{n^2}\sum_{i=0}^{n-1}\sum_{j=0}^{n-1} S^iT^j$. Then, the following hold:*

(1) *$A = A(S) \cap A(T)$ is non–empty, closed and convex.*
(2) *$\{S_n x_1\}$ converges weakly to $u_0 \in A$, where $u_0 = \lim_{(i,j)} P_A S^iT^jx_1$.*
(3) *In the case when C is closed and convex, $u_0 \in F = F(S) \cap F(T)$.*

Remark. In (2), $u_0 \in F = F(S) \cap F(T)$ holds if $u_0 \in C$.

Proof. By $ST = TS$, we know $S_n = \frac{1}{n^2}\sum_{i=0}^{n-1}\sum_{j=0}^{n-1} S^iT^j = \frac{1}{n^2}\sum_{j=0}^{n-1}\sum_{i=0}^{n-1} T^jS^i$ for all $n \in N$. By Lemma 4.2, $\{S_n x_1\}$ is bounded. Let $\{S_{n_k} x_1\}$ be a subsequence of $\{S_n x_1\}$ which converges weakly to some $w \in H$. By $ST = TS$, Lemma 4.2 (2) asserts $w \in A = A(S) \cap A(T)$. So, A is non–empty, closed and convex. Lemma 4.2 (2) also asserts that every weak cluster point of $\{S_n x_1\}$ is a point in A.

Thus, by Lemma 4.5, we have the desired results. In (2), despite of the absence of closedness and convexity of C, $u_0 \in F$ is guaranteed if $u_0 \in C$. Because, by Lemma 3.2, we know $A \cap C = (A(S) \cap A(T)) \cap C \subset F(S) \cap F(T) = F$. □

Theorem 4.6 is an existence and weak convergence theorem. In section 1, we presented Theorem K due to Kohsaka [14] and Theorem TT due to Takahashi and Takeuchi [25]. We may regard Theorem 4.6 as an extension of Theorem K. However, we do not know whether Theorem 4.6 (3) and Theorem K are exactly the same. Because $u_0 = \lim_{(i,j)} P_A S^iT^jx_1$ does not automatically mean $u_0 = \lim_{(i,j)} P_F S^iT^jx_1$. In Theorem 4.6, let T be the identity mapping. Then, we have a mean convergence theorem for a (λ)–hybrid self–mapping S on C. We know that Theorem TT does not follow from this theorem. So, Theorem 4.6 is not an extension of Theorem TT. Nevertheless, the class of (λ)–hybrid mappings also contains some important classes of nonlinear mappings.

5. Examples

In this section, we present some examples to support the main issue. In advance, recall the following: a nonexpansive mapping, a nonspreading mapping, and a hybrid mapping are (λ)–hybrid, in the Hilbert space setting. We note that the class of nonspreading mappings was first defined in a smooth, strictly convex and reflexive Banach space.

Let C be a subset of a Hilbert space H and U be a mapping from C into H. Then, from Kohsaka and Takahashi [15], U is called nonspreading if

$$2\|Ux - Uy\|^2 \leq \|Ux - y\|^2 + \|Uy - x\|^2 \quad \text{for all} \;\; x, y \in C. \tag{5.1}$$

Also, from Takahashi [24], U is called hybrid if

$$3\|Ux - Uy\|^2 \leq \|x - y\|^2 + \|Ux - y\|^2 + \|Uy - x\|^2 \quad \text{for all} \;\; x, y \in C. \tag{5.2}$$

Example 5.1.

Let C be the bounded subset $\{(x_1, x_2) \in R^2 : |x_1| \in [0, \frac{1}{2}),\ |x_2| \in [0, \frac{1}{4}|x_1| + \frac{3}{8})\}$ of the Euclidean space R^2. Then, C is neither closed nor convex.

Let S and T be self–mappings on C such that, for each $(x_1, x_2) \in C$,

$$S(x_1, x_2) = (-x_1, x_2), \qquad T(x_1, x_2) = (-x_1, -x_2).$$

It is easy to see that S and T are nonexpansive. We confirm that S and T are neither nonspreading nor hybrid. Let $x = (0.2, 0.1)$, $y = (-0.2, 0.1)$, $\overline{x} = (0.2, -0.1)$ and $\overline{y} = (-0.2, -0.1)$. Then, $x, \overline{x}, y, \overline{y} \in C$, $Sx = y$, $Sy = x$, $Tx = \overline{y}$ and $Ty = \overline{x}$. We see

$$\begin{aligned}
\|Sx - Sy\|^2 &= \|x - y\|^2 = \|(0.4, 0)\|^2 = 0.16, \\
\|Sx - y\|^2 &= \|y - y\|^2 = 0 = \|x - x\|^2 = \|Sy - x\|^2, \\
\|Tx - Ty\|^2 &= \|\overline{y} - \overline{x}\| = \|(-0.4, 0)\|^2 = 0.16 = \|(0.4, 0)\|^2 = \|x - y\|^2, \\
\|Tx - y\|^2 &= \|\overline{y} - y\|^2 = \|(0, -0.2)\|^2 = 0.04 = \|\overline{x} - x\|^2 = \|Ty - x\|^2.
\end{aligned}$$

These imply that S and T satisfy neither (5.1) nor (5.2).

Consider the self–mapping U on C such that, for each $(x_1, x_2) \in C$,

$$U(x_1, x_2) = (x_1, |x_1| x_2).$$

Obviously, U is not nonexpansive, $SU = US$ and $TU = UT$. Also, we see

$$\begin{aligned}
A(S) &= \{(x_1, x_2) \in R^2 : x_1 = 0\}, & F(S) &= \{(x_1, x_2) \in C : x_1 = 0\}, \\
A(T) &= \{(0, 0)\}, & F(T) &= \{(0, 0)\}, \\
A(U) &= \{(x_1, x_2) \in R^2 : x_2 = 0\}, & F(U) &= \{(x_1, x_2) \in C : x_2 = 0\}.
\end{aligned}$$

We confirm that U is nonspreading; we are not interested in whether U is hybrid here. Let $x = (x_1, x_2)$ and $y = (y_1, y_2)$ be points in C. In the case of $x_2 y_2 < 0$, by considering the positional relation of x, y, Ux and Uy, obviously

(5.1) holds. There remains the case of $x_2y_2 \geq 0$. Assume $x_2y_2 \geq 0$. By $Ux = (x_1, |x_1|x_2)$ and $Uy = (y_1, |y_1|y_2)$, we have

$$\begin{aligned} \|Ux - Uy\|^2 &= \|(x_1 - y_1, |x_1|x_2 - |y_1|y_2)\|^2 = (x_1 - y_1)^2 + (|x_1|x_2 - |y_1|y_2)^2, \\ \|Ux - y\|^2 &= \|(x_1 - y_1, |x_1|x_2 - y_2)\|^2 = (x_1 - y_1)^2 + (|x_1|x_2 - y_2)^2, \\ \|Uy - x\|^2 &= \|(y_1 - x_1, |y_1|y_2 - x_2)\|^2 = (y_1 - x_1)^2 + (|y_1|y_2 - x_2)^2. \end{aligned}$$

Set $k, l, m \in R$ as below:

$$\begin{aligned} k &= (|x_1|x_2 - |y_1|y_2)^2 = x_1^2x_2^2 + y_1^2y_2^2 - 2|x_1||y_1|x_2y_2, \\ l &= (|x_1|x_2 - y_2)^2 = y_2^2 + x_1^2x_2^2 - 2|x_1|x_2y_2, \\ m &= (|y_1|y_2 - x_2)^2 = x_2^2 + y_1^2y_2^2 - 2|y_1|x_2y_2. \end{aligned}$$

Recall $|x_1|, |y_1| \in [0, 1/2)$. Then, we see $|x_1| + |y_1| - 2|x_1||y_1| < 1/2$ by

$$\begin{aligned} &|x_1| + |y_1| - 2|x_1||y_1| - \tfrac{1}{2} \\ &\qquad = \tfrac{1}{2}(2|x_1| - 1) + |y_1|(1 - 2|x_1|) = (1 - 2|x_1|)(|y_1| - \tfrac{1}{2}) < 0. \end{aligned}$$

Thus, by $x_1^2, y_1^2 < 1/4 < 1/2$ and $0 \leq x_2y_2$, we see that U satisfies (5.1):

$$\begin{aligned} &l + m - 2k \\ &\quad = x_2^2 + y_2^2 - x_1^2x_2^2 - y_1^2y_2^2 - 2x_2y_2(|x_1| + |y_1| - 2|x_1||y_1|) \\ &\quad \geq x_2^2 + y_2^2 - \tfrac{1}{2}x_2^2 - \tfrac{1}{2}y_2^2 - x_2y_2 = \tfrac{1}{2}x_2^2 + \tfrac{1}{2}y_2^2 - x_2y_2 = \tfrac{1}{2}(x_2 - y_2)^2 \geq 0, \\ &2\|Ux - y\|^2 \\ &\quad = 2(x_1 - y_1)^2 + 2k \leq 2(x_1 - y_1)^2 + l + m = \|Ux - y\|^2 + \|Uy - x\|^2. \end{aligned}$$

Let x be a point in C. We know the following:

- ∘ $F(S) \cap F(U) = A(S) \cap A(U) = \{(0, 0)\} \subset C$.
- ∘ $F(T) \cap F(U) = A(T) \cap A(U) = \{(0, 0)\} \subset C$.

Then, from the argument so far, Theorem 4.6 asserts the following:

- ∘ $\{\frac{1}{n^2} \sum_{i=0}^{n-1} \sum_{j=0}^{n-1} S^iU^jx\}$ converges strongly to $u_0 = (0, 0) \in F(S) \cap F(U)$.
- ∘ $\{\frac{1}{n^2} \sum_{i=0}^{n-1} \sum_{j=0}^{n-1} T^iU^jx\}$ converges strongly to $v_0 = (0, 0) \in F(T) \cap F(U)$.

Note that the strong topology and the weak topology are coincide in our setting.

Example 5.2.

Let D be the bounded subset $\{(x_1, x_2) \in R^2 : 1 \leq \max\{|x_1|, |x_2|\} < 2\}$ of the Euclidean space R^2. Define subsets D_1, D_2, D_3 of D by

$$D_1 = \{(x_1, x_2) \in D : |x_1| < 1\}, \quad D_2 = \{(x_1, x_2) \in D : |x_2| < 1\},$$
$$D_3 = \{(x_1, x_2) \in D : |x_1| \geq 1,\ |x_2| \geq 1\}.$$

D is neither closed nor convex. D and the disjoint union of $\{D_1, D_2, D_3\}$ are coincide.

Let S and T be self–mappings on D such that, for each $(x_1, x_2) \in D$,

$$S(x_1, x_2) = (-x_1, x_2), \qquad T(x_1, x_2) = (-x_1, -x_2).$$

It is easy to see that S and T are nonexpansive, and

$$A(S) = \{(x_1, x_2) \in R^2 : x_1 = 0\}, \qquad A(T) = \{(0, 0)\},$$
$$F(S) = \{(x_1, x_2) \in D : x_1 = 0,\ 1 \leq |x_2| < 2\}, \qquad F(T) = \emptyset.$$

Consider the following self–mapping U on D:

$$U(x_1, x_2) = (x_1, \tfrac{x_2}{2} + \tfrac{x_2}{2|x_2|}) \qquad \text{when } (x_1, x_2) \in D_1,$$
$$U(x_1, x_2) = (\tfrac{x_1}{2} + \tfrac{x_1}{2|x_1|}, x_2) \qquad \text{when } (x_1, x_2) \in D_2,$$
$$U(x_1, x_2) = (\tfrac{x_1}{|x_1|}, \tfrac{x_2}{|x_2|}) \qquad \text{when } (x_1, x_2) \in D_3.$$

Then, we can easily confirm

$$A(U) = \{(x_1, x_2) \in R^2 : \max\{|x_1|, |x_2|\} \leq 1\},$$
$$F(U) = \{(x_1, x_2) \in D : \max\{|x_1|, |x_2|\} = 1\}.$$

It is also easy to see that $SU = US$, $TU = UT$ and the following:

$$A(S) \cap A(U) = \{(x_1, x_2) \in R^2 : x_1 = 0,\ |x_2| \leq 1\}, \quad A(T) \cap A(U) = \{(0, 0)\},$$
$$F(S) \cap F(U) = \{(0, 1), (0, -1)\}, \qquad F(T) \cap F(U) = \emptyset.$$

We confirm that U is nonspreading. Let $x = (x_1, x_2)$ and $y = (y_1, y_2)$ be points in D. By considering the positional relation of x, y, Ux and Uy, it is obvious that U satisfies (5.1) in the following cases: $x, y \in D_1 \cup D_2$, $x, y \in D_3$, $x \in D_1$ and $y \in D_3$ with $x_2 y_2 < 0$, $x \in D_2$ and $y \in D_3$ with $x_1 y_1 < 0$. Then, there remain the following cases: $x \in D_1$ and $y \in D_3$ with $x_2 y_2 \geq 0$, $x \in D_2$ and $y \in D_3$ with $x_1 y_1 \geq 0$.

A little thought will tell us that we may consider only the case of $x \in D_1$ with $x_2 \geq 1$ and $y \in D_3$ with $y_1, y_2 \geq 1$. In this case, by $Ux = (x_1, \frac{1}{2}x_2 + \frac{1}{2})$, $Uy = (1,1)$, and $y_1 - x_1 \geq 1 - x_1$, we see that U satisfies (5.1):

$$\begin{aligned}
\|Ux - Uy\|^2 &= \|(x_1 - 1, \tfrac{1}{2}x_2 - \tfrac{1}{2})\|^2 = (x_1 - 1)^2 + (\tfrac{1}{2})^2(x_2 - 1)^2, \\
\|Ux - y\|^2 &= \|(x_1 - y_1, \tfrac{1}{2}x_2 + \tfrac{1}{2} - y_2)\|^2 \\
&= (x_1 - y_1)^2 + (\tfrac{1}{2}x_2 + \tfrac{1}{2} - y_2)^2 \geq (x_1 - y_1)^2 \geq (x_1 - 1)^2, \\
\|Uy - x\|^2 &= \|(1 - x_1, 1 - x_2)\|^2 = (1 - x_1)^2 + (1 - x_2)^2, \\
2\|Ux - Uy\|^2 &= 2(x_1 - 1)^2 + \tfrac{1}{2}(x_2 - 1)^2 \\
&\leq 2(x_1 - 1)^2 + (x_2 - 1)^2 \leq \|Ux - y\|^2 + \|Uy - x\|^2.
\end{aligned}$$

We confirm that U is not nonexpansive. Let $y = (1, 1.8) \in D_3$. Let $\{a_n\}$ be a sequence in $(0,1)$ converging to 1. For each $n \in N$, set $z_n = (a_n, 1.8) \in D_1$. It is obvious that $\{z_n\}$ converges strongly to y. On the other hand, we see the following:

$$\|Uz_n - Uy\|^2 = (a_n - 1)^2 + (1.4 - 1)^2 \geq (0.4)^2 \quad \text{for all} \quad n \in N.$$

Then, U is not continuous. So, we see that U is not nonexpansive. Furthermore, we confirmed that a nonspreading mapping need not be continuous.

Let x be a point in D. From the argument so far, Theorem 4.6 asserts the following:

- ○ $\{\frac{1}{n^2}\sum_{i=0}^{n-1}\sum_{j=0}^{n-1} S^i U^j x\}$ converges strongly to some $u_0 \in A(S) \cap A(U)$.
- ○ $\{\frac{1}{n^2}\sum_{i=0}^{n-1}\sum_{j=0}^{n-1} T^i U^j x\}$ converges strongly to $(0,0) \in A(T) \cap A(U)$.

However, by the absence of closedness and convexity of D, we do not know whether $u_0 \in F(S) \cap F(U)$, even if we know $\emptyset \neq F(S) \cap F(U) \subset A(S) \cap A(U)$.

Acknowledgements. The authors are grateful to the referees for their helpful reviews and suggestions.

This work was supported by JSPS KAKENHI Grant Number JP19K03632.

References

[1] K. Aoyama, Halpern's iteration for a sequence of quasinonexpansive type mappings, In *Non–linear Mathematics for Uncertainly and its Applications*, Springer–Verlag, Berlin Heidelberg, 2011, 387–394.

[2] K. Aoyama, S. Iemoto, F. Kohsaka and W. Takahashi, Fixed point and ergodic theorems for λ-hybrid mappings in Hilbert spaces, *J. Nonlinear Convex Anal.* **11** (2010), 335–343.

[3] S. Atsushiba, S. Iemoto, R. Kubota and Y. Takeuchi, Convergence theorems for some classes of nonlinear mappings in Hilbert spaces, *Linear and Nonlinear Anal.* **2** (2016), 125–153.

[4] S. Atsushiba and W. Takahashi, Approximating common fixed points of two nonexpansive mappings in Banach spaces, *Bull. Austral. Math. Soc.* **57** (1998), 117–127.

[5] S. Atsushiba and W. Takahashi, Nonlinear ergodic theorems without convexity for nonexpansive semigroups in Hilbert spaces, *J. Nonlinear and Convex Anal.* **14** (2013), 209–219.

[6] J.-B. Baillon, Un theoreme de type ergodique pour les contractions non lineaires dans un espace de Hilbert, *C. R. Acad. Sci. Paris Ser. A-B* **280** (1975), 1511–1514.

[7] R. E. Bruck, A common fixed point theorem for a commuting family of nonexpansive mappings, *Pacific J. Math.* **53** (1974), 59–71.

[8] R. E. Bruck, On the convex approximation property and the asymptotic behavior of nonlinear contractions in Banach spaces, *Israel J. Math.* **38** (1981), 304–314.

[9] R. DeMarr, Common fixed points for commuting contraction mappings, *Pacific J. Math.* **13** (1963), 1139–1141.

[10] T. Ibaraki and Y. Takeuchi, New convergence theorems for common fixed points of a wide range of nonlinear mappings, *J. Nonlinear Anal. and Optim.* **9** (2018), 95–114.

[11] S. Ishikawa, Common fixed points and iteration of commuting nonexpansive mappings, *Pacific J. Math.* **80** (1979), 493–501.

[12] P. Kocourek, W. Takahashi and J.-C. Yao, Fixed point theorems and weak convergence theorems for generalized hybrid mappings in Hilbert spaces, *Taiwanese J. Math.* **14** (2010), 2497–2511.

[13] F. Kohsaka, Weak convergence theorem for a sequence of quasinonexpansive type mappings, *Nonlinear analysis and Convex Analysis* (Hirosaki, 2013), Yokohama Publishers, 2015, 289–300.

[14] F. Kohsaka, Existence and approximation of common fixed points of two hybrid mappings in Hilbert spaces, *J. Nonlinear and Convex Anal.* **16** (2015), 2193–2205.

[15] F. Kohsaka and W. Takahashi, Fixed point theorems for a class of nonlinear mappings related to maximal monotone operators in Banach spaces, *Arch. Math.* **91** (2008), 166–177.

[16] R. Kubota and Y. Takeuchi, On DeMarr's common fixed point theorem, *Nonlinear Analysis and Convex Analysis* (Chiang Rai, 2015), Yokohama Publishers, 2016, 207–209.

[17] P. K. F. Kuhfittig, Common fixed points of nonexpansive mappings by iteration, *Pacific J. Math.* **97** (1981), 137–139.

[18] J. Linhart, Beiträge zur Fixpunkttheorie nichtexpandierender Operatoren, *Monatsh. Math.* **76** (1972), 239–249 (German).

[19] T. Shimizu and W. Takahashi, Strong convergence to common fixed points of families of nonexpansive mappings, *J. Math. Anal. Appl.* **211** (1997), 71–83.

[20] K. Shimoji and W. Takahashi, Strong convergence to common fixed points of infinite nonexpansive mappings and applications, *Taiwanese J. Math.* **5** (2001), 387–404.

[21] T. Suzuki, Strong convergence theorem to common fixed points of two nonexpansive mappings in general Banach spaces, *J. Nonlinear Convex Anal.* **3** (2002), 381–391.

[22] T. Suzuki, Strong convergence theorems for infinite families of nonexpansive mappings in general Banach spaces, *Fixed Point Theory and Applications* (2005), 103–123.

[23] W. Takahashi, *Nonlinear Functional Analysis*, Yokohama Publishers, Yokohama, 2000.

[24] W. Takahashi, Fixed point theorems for new nonlinear mappings in a Hilbert space, *J. Nonlinear Convex Anal.* **11** (2010), 79–88.

[25] W. Takahashi and Y. Takeuchi, Nonlinear ergodic theorem without convexity for generalized hybrid mappings in a Hilbert space, *J. Nonlinear Convex Anal.* **12** (2011), 399–406.

[26] Y. Takeuchi, An iteration scheme finding a common fixed point of commuting two nonexpansive mappings in general Banach spaces, *Linear and Nonlinear Anal.* **2** (2016), 317–327.

Takanori Ibaraki
Department of Mathematics Education,
Yokohama National University,
79-2 Tokiwadai, Hodogaya, Yokohama 240-8501,
Japan
E-mail: ibaraki@ynu.ac.jp

Yukio Takeuchi
Takahashi Institute for Nonlinear Analysis
1-11-11 Nakazato, Minami, Yokohama 232-0063,
Japan
E-mail: aho314159@yahoo.co.jp