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ON THE SUPPORT OF THE GROVER WALK ON HIGHER-DIMENSIONAL LATTICES

By

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Abstract. This paper presents the minimum supports of states for stationary measures of the Grover walk on the d-dimensional lattice by solving the corresponding eigenvalue problem. The numbers of the minimum supports for moving and flip-flop shifts are 2^d $(d \geq 1)$ and 4 $(d \geq 2)$, respectively.

1. Introduction

The quantum walk was introduced by Aharonov et al. [1] as a generalization of the random walk on graphs. On the one-dimensional lattice $\mathbb{Z}$, where $\mathbb{Z}$ is the set of integers, the properties of quantum walks are well studied, see Konno [6], for example. There are some results on the Grover walk on $\mathbb{Z}^2$, such as weak limit theorem by Watabe et al. [8] (moving shift case) and Higuchi et al. [2] (flip-flop shift case), and localization shown by Inui et al. [3] (moving shift case) and Higuchi et al. [2] (flip-flop shift case).

In this paper, we present the minimum support of states for the stationary measures of the Grover walk on $\mathbb{Z}^d$ by solving the corresponding eigenvlaue problem. As for the number of the support of the Grover walk on $\mathbb{Z}^d$ with moving shift, 2^2 ($\mathbb{Z}^2$ case) and 3^d ($\mathbb{Z}^d$ case with $d \geq 2$) were given in Stefanak et al. [7] and Komatsu and Konno [4] by the Fourier analysis, respectively. Compared with the above-mentioned previous results, the number of our minimum support for $\mathbb{Z}^d$ case with $d \geq 1$ is 2^d (Theorem 1). Moreover, concerning the number of the support of the Grover walk on $\mathbb{Z}^d (d \geq 2)$ with flip-flop shift, 4 was obtained in Higuchi et al. [2] by the spectral mapping theorem, which coincides with our result (Theorem 2). Remark that any finite support does not exist for $\mathbb{Z}$ case.

The rest of the paper is as follows. Section 2 is devoted to the definition of the discrete-time quantum walk on $\mathbb{Z}^d$. Section 3 deals with the stationary measure of the Grover walk on $\mathbb{Z}^d$. We give main results on minimum support for the Grover walk on $\mathbb{Z}^d$ with moving shift (Theorem 1) in Section 4 and flip-flop shift

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(Theorem 2) in Section 5, respectively. Section 6 summarizes our paper.

2. Discrete-time quantum walks on $\mathbb{Z}^d$

In this section, we give the definition of $2d$-state discrete-time quantum walks on $\mathbb{Z}^d$. The quantum walk is defined by using a shift operator and a unitary matrix. Let $\mathbb{C}$ be the set of complex numbers. For $i \in \{1, 2, \dots, d\}$, the shift operator τ_i is given by

$$(\tau_i f)(\boldsymbol{x}) = f(\boldsymbol{x} - \boldsymbol{e}_i) \quad (f : \mathbb{Z}^d \longrightarrow \mathbb{C}^{2d},\ \boldsymbol{x} \in \mathbb{Z}^d),$$

where $\{\boldsymbol{e}_1, \boldsymbol{e}_2, \dots, \boldsymbol{e}_d\}$ denotes the standard basis of $\mathbb{Z}^d$. Let $A = (a_{ij})_{i,j=1,2,\dots,2d}$ be a $2d \times 2d$ unitary matrix. We call this unitary matrix the coin matrix. To describe the time evolution of the quantum walk, decompose the unitary matrix A as

$$A = \sum_{i=1}^{2d} P_i A,$$

where P_i denotes an orthogonal projection onto the one-dimensional subspace $\mathbb{C}\eta_i$ in $\mathbb{C}^{2d}$. Here $\{\eta_1, \eta_2, \dots, \eta_{2d}\}$ denotes the standard basis on $\mathbb{C}^{2d}$. The walk associated with the coin matrix A for moving and flip-flop shifts are given by

$$\begin{aligned} U_A &= \sum_{i=1}^{d} \Big(P_{2i-1} A \tau_i^{-1} + P_{2i} A \tau_i \Big), \\ U_A &= \sum_{i=1}^{d} \Big(P_{2i} A \tau_i^{-1} + P_{2i-1} A \tau_i \Big), \end{aligned} \tag{2.1}$$

respectively.

Let $\mathbb{Z}_{\geq} = \{0, 1, 2, \dots\}$. The state at time $n \in \mathbb{Z}_{\geq}$ and location $\boldsymbol{x} \in \mathbb{Z}^d$ can be expressed by a $2d$-dimensional vector:

$$\Psi_n(\boldsymbol{x}) = {}^T \left[\Psi_n^1(\boldsymbol{x}), \Psi_n^2(\boldsymbol{x}), \cdots, \Psi_n^{2d}(\boldsymbol{x}) \right] \in \mathbb{C}^{2d},$$

where T denotes a transposed operator. For $\Psi_n : \mathbb{Z}^d \longrightarrow \mathbb{C}^{2d}$ $(n \in \mathbb{Z}_{\geq})$, it follows from Eq. (2.1) that

$$\Psi_{n+1}(\boldsymbol{x}) \equiv (U_A \Psi_n)(\boldsymbol{x}) = \sum_{i=1}^{d} \Big(P_{2i-1} A \Psi_n(\boldsymbol{x} + \boldsymbol{e}_i) + P_{2i} A \Psi_n(\boldsymbol{x} - \boldsymbol{e}_i) \Big),$$

with moving shift case and

$$\Psi_{n+1}(\boldsymbol{x}) \equiv (U_A \Psi_n)(\boldsymbol{x}) = \sum_{i=1}^{d} \Big(P_{2i} A \Psi_n(\boldsymbol{x} + \boldsymbol{e}_i) + P_{2i-1} A \Psi_n(\boldsymbol{x} - \boldsymbol{e}_i) \Big),$$

with flip-flop shift case. This equation means that, moving shift case for example, the particle moves at each step one unit to the x_i-axis direction with matrix $P_{2i}A$ or one unit to the $-x_i$-axis direction with matrix $P_{2i-1}A$. For time $n \in \mathbb{Z}_{\geq}$ and location $\boldsymbol{x} \in \mathbb{Z}^d$, we define the measure $\mu_n(\boldsymbol{x})$ by

$$\mu_n(\boldsymbol{x}) = \|\Psi_n(\boldsymbol{x})\|^2_{\mathbb{C}^{2d}},$$

where $\|\cdot\|_{\mathbb{C}^{2d}}$ denotes the standard norm on $\mathbb{C}^{2d}$. Let $\mathbb{R}_{\geq} = [0, \infty)$. Here we introduce a map $\phi : (\mathbb{C}^{2d})^{\mathbb{Z}^d} \longrightarrow (\mathbb{R}_{\geq})^{\mathbb{Z}^d}$ such that if $\Psi_n : \mathbb{Z}^d \longrightarrow \mathbb{C}^{2d}$ and $\boldsymbol{x} \in \mathbb{Z}^d$, thus we get

$$\phi(\Psi_n)(\boldsymbol{x}) = \sum_{j=1}^{2d} |\Psi_n^j(\boldsymbol{x})|^2 = \mu_n(\boldsymbol{x}),$$

namely this map ϕ has a role to transform from amplitudes to measures.

3. Stationary measure of the Grover walk on $\mathbb{Z}^d$

In this section, we give the definition of the stationary measure for the quantum walk. We define a set of measures, $\mathcal{M}_s(U_A)$, by

$$\mathcal{M}_s(U_A) = \Big\{\mu \in [0,\infty)^{\mathbb{Z}^d} \setminus \{\mathbf{0}\};\text{ there exists } \Psi_0 \in \left(\mathbb{C}^{2d}\right)^{\mathbb{Z}^d} \text{ such that } \phi(U_A^n\Psi_0) = \mu \quad (n \in \mathbb{Z}_{\geq})\Big\},$$

where $\mathbf{0}$ is the zero vector. Here U_A is the time evolution operator of quantum walk associated with a unitary matrix A. We call this measure $\mu \in \mathcal{M}_s(U_A)$ the stationary measure for the quantum walk defined by the unitary operator U_A. If $\mu \in \mathcal{M}_s(U_A)$, then $\mu_n = \mu$ for $n \in \mathbb{Z}_{\geq}$, where μ_n is the measure of quantum walk given by U_A at time n.

Next we consider the following eigenvalue problem of the quantum walk determined by U_A:

$$U_A\Psi = \lambda\Psi \quad (\lambda \in \mathbb{C},\ |\lambda| = 1). \tag{3.1}$$

We introduce the set of solutions of Eq. (3.1) for $\lambda \in \mathbb{C}$ with $|\lambda| = 1$ as follows.

$$W(\lambda) = \{\Psi \neq \mathbf{0} : U_A\Psi = \lambda\Psi\}.$$

Then for $\Psi \in W(\lambda)$, we see that $\phi(\Psi) \in \mathcal{M}_s(U_A)$. If the function Ψ satisfied with $\lambda = 1$ in Eq. (3.1), then Ψ is called the *stationary amplitude*. From now on, we focus on the Grover Walk on $\mathbb{Z}^d$ which is defined by the following $2d \times 2d$ coin matrix $G = (g_{ij})_{i,j=1,2,\ldots,2d}$ with

$$g_{ij} = \frac{1}{d} - \delta_{ij}.$$

Our purpose of this paper is to investigate the support of the $2d$-state Grover walk on $\mathbb{Z}^d$.

4. Grover walk on $\mathbb{Z}^d$ with moving shift

In this section, we present our main results on the support of the Grover walk on $\mathbb{Z}^d$ with moving shift. Remark that Komatsu and Tate [5] showed that the eigenvalue of Eq. (3.1) is only $\lambda = \pm 1$ for the d-dimensional Gorver walk with moving shift. We begin with the eigenvalue problem $U_G \Psi = \lambda \Psi$ ($\lambda \in \mathbb{C}$ with $|\lambda| = 1$), which is equivalent to

$$\begin{cases} \lambda\Psi^1(\boldsymbol{x}) = \dfrac{1-d}{d}\Psi^1(\boldsymbol{x}+\boldsymbol{e}_1) + \dfrac{1}{d}\Psi^2(\boldsymbol{x}+\boldsymbol{e}_1) + \cdots + \dfrac{1}{d}\Psi^{2d-1}(\boldsymbol{x}+\boldsymbol{e}_1) + \dfrac{1}{d}\Psi^{2d}(\boldsymbol{x}+\boldsymbol{e}_1), \\ \lambda\Psi^2(\boldsymbol{x}) = \dfrac{1}{d}\Psi^1(\boldsymbol{x}-\boldsymbol{e}_1) + \dfrac{1-d}{d}\Psi^2(\boldsymbol{x}-\boldsymbol{e}_1) + \cdots + \dfrac{1}{d}\Psi^{2d-1}(\boldsymbol{x}-\boldsymbol{e}_1) + \dfrac{1}{d}\Psi^{2d}(\boldsymbol{x}-\boldsymbol{e}_1), \\ \qquad \vdots \\ \lambda\Psi^{2d-1}(\boldsymbol{x}) = \dfrac{1}{d}\Psi^1(\boldsymbol{x}+\boldsymbol{e}_d) + \dfrac{1}{d}\Psi^2(\boldsymbol{x}+\boldsymbol{e}_d) + \cdots + \dfrac{1-d}{d}\Psi^{2d-1}(\boldsymbol{x}+\boldsymbol{e}_d) + \dfrac{1}{d}\Psi^{2d}(\boldsymbol{x}+\boldsymbol{e}_d), \\ \lambda\Psi^{2d}(\boldsymbol{x}) = \dfrac{1}{d}\Psi^1(\boldsymbol{x}-\boldsymbol{e}_d) + \dfrac{1}{d}\Psi^2(\boldsymbol{x}-\boldsymbol{e}_d) + \cdots + \dfrac{1}{d}\Psi^{2d-1}(\boldsymbol{x}-\boldsymbol{e}_d) + \dfrac{1-d}{d}\Psi^{2d}(\boldsymbol{x}-\boldsymbol{e}_d), \end{cases} \tag{4.1}$$

where $\Psi(\boldsymbol{x}) = {}^T\left[\Psi^1(\boldsymbol{x}), \Psi^2(\boldsymbol{x}), \cdots, \Psi^{2d}(\boldsymbol{x})\right]$ ($\boldsymbol{x} \in \mathbb{Z}^d$). Put $\Gamma(\boldsymbol{x}) = \sum_{j=1}^{2d} \Psi^j(\boldsymbol{x})$ for $\boldsymbol{x} \in \mathbb{Z}^d$. By using $\Gamma(\boldsymbol{x})$, Eq. (4.1) can be written as

$$\lambda\Psi^{2k-1}(\boldsymbol{x}-\boldsymbol{e}_k) + \Psi^{2k-1}(\boldsymbol{x}) = \frac{1}{d}\Gamma(\boldsymbol{x}), \tag{4.2}$$

$$\lambda\Psi^{2k}(\boldsymbol{x}+\boldsymbol{e}_k) + \Psi^{2k}(\boldsymbol{x}) = \frac{1}{d}\Gamma(\boldsymbol{x}), \tag{4.3}$$

for any $k = 1, 2, \ldots, d$ and $\boldsymbol{x} \in \mathbb{Z}^d$. From Eqs. (4.2) and (4.3), we get immediately

$$\lambda\Psi^{2k-1}(\boldsymbol{x}-\boldsymbol{e}_k) + \Psi^{2k-1}(\boldsymbol{x}) = \lambda\Psi^{2k}(\boldsymbol{x}+\boldsymbol{e}_k) + \Psi^{2k}(\boldsymbol{x}), \tag{4.4}$$

for any $k = 1, 2, \ldots, d$ and $\boldsymbol{x} \in \mathbb{Z}^d$. In order to state the following lemma, we introduce the support of $\Psi : \mathbb{Z}^d \to \mathbb{C}^{2d}$ as follows.

$$S(\Psi) = \{\boldsymbol{x} \in \mathbb{Z}^d : \Psi(\boldsymbol{x}) \neq \boldsymbol{0}\}.$$

Lemma 1. *Let $\Psi \in W(\lambda)$ with $\lambda = \pm 1$. Suppose $\#(S(\Psi)) < \infty$, where $\#(A)$ is the cardinality of a set A. If there exist $k \in \{1, 2, \cdots, d\}$ and $\boldsymbol{x} \in \mathbb{Z}^d$ such that*

$$\begin{bmatrix} \Psi^{2k-1}(\boldsymbol{x}) \\ \Psi^{2k}(\boldsymbol{x}) \end{bmatrix} \neq \begin{bmatrix} 0 \\ 0 \end{bmatrix},$$

then we have

$$\begin{bmatrix}\Psi^{2k-1}(\boldsymbol{x}-\boldsymbol{e}_k)\\ \Psi^{2k}(\boldsymbol{x}-\boldsymbol{e}_k)\end{bmatrix} \neq \begin{bmatrix}0\\0\end{bmatrix} \quad \textit{or} \quad \begin{bmatrix}\Psi^{2k-1}(\boldsymbol{x}+\boldsymbol{e}_k)\\ \Psi^{2k}(\boldsymbol{x}+\boldsymbol{e}_k)\end{bmatrix} \neq \begin{bmatrix}0\\0\end{bmatrix}.$$

Proof. First we assume that

$$\begin{bmatrix}\Psi^{2k-1}(\boldsymbol{x})\\ \Psi^{2k}(\boldsymbol{x})\end{bmatrix} \neq \begin{bmatrix}0\\0\end{bmatrix}, \tag{4.5}$$

for some $k \in \{1, 2, \cdots, d\}$ and $\boldsymbol{x} \in \mathbb{Z}^d$. Moreover we suppose

$$\begin{bmatrix}\Psi^{2k-1}(\boldsymbol{x}-\boldsymbol{e}_k)\\ \Psi^{2k}(\boldsymbol{x}-\boldsymbol{e}_k)\end{bmatrix} = \begin{bmatrix}0\\0\end{bmatrix} \quad \text{and} \quad \begin{bmatrix}\Psi^{2k-1}(\boldsymbol{x}+\boldsymbol{e}_k)\\ \Psi^{2k}(\boldsymbol{x}+\boldsymbol{e}_k)\end{bmatrix} = \begin{bmatrix}0\\0\end{bmatrix},$$

that is,

$$\Psi^{2k-1}(\boldsymbol{x}-\boldsymbol{e}_k) = 0, \tag{4.6}$$

$$\Psi^{2k}(\boldsymbol{x}-\boldsymbol{e}_k) = 0, \tag{4.7}$$

$$\Psi^{2k-1}(\boldsymbol{x}+\boldsymbol{e}_k) = 0,$$

$$\Psi^{2k}(\boldsymbol{x}+\boldsymbol{e}_k) = 0. \tag{4.8}$$

Combining Eq. (4.4) with Eqs. (4.6) and (4.8), we have

$$\Psi^{2k-1}(\boldsymbol{x}) = \Psi^{2k}(\boldsymbol{x}). \tag{4.9}$$

From the assumption Eqs. (4.5) and (4.9), we put

$$\Psi^{2k-1}(\boldsymbol{x}) = \Psi^{2k}(\boldsymbol{x}) = \eta, \tag{4.10}$$

where $\eta \in \mathbb{C}$ with $\eta \neq 0$. Furthermore, by Eq. (4.4) for $\boldsymbol{x}-\boldsymbol{e}_k$, we obtain

$$\lambda\Psi^{2k-1}(\boldsymbol{x}-2\boldsymbol{e}_k) + \Psi^{2k-1}(\boldsymbol{x}-\boldsymbol{e}_k) = \lambda\Psi^{2k}(\boldsymbol{x}) + \Psi^{2k}(\boldsymbol{x}-\boldsymbol{e}_k). \tag{4.11}$$

Combining Eq. (4.11) with Eqs. (4.6), (4.7) and (4.10) implies

$$\Psi^{2k-1}(\boldsymbol{x}-2\boldsymbol{e}_k) = \eta, \tag{4.12}$$

since $\lambda \neq 0$. In a similar way, Eq. (4.4) for $\boldsymbol{x}-2\boldsymbol{e}_k$ becomes

$$\lambda\Psi^{2k-1}(\boldsymbol{x}-3\boldsymbol{e}_k) + \Psi^{2k-1}(\boldsymbol{x}-2\boldsymbol{e}_k) = \lambda\Psi^{2k}(\boldsymbol{x}-\boldsymbol{e}_k) + \Psi^{2k}(\boldsymbol{x}-2\boldsymbol{e}_k). \tag{4.13}$$

From Eq. (4.13) with Eqs. (4.7) and (4.12), we have

$$\Psi^{2k-1}(\boldsymbol{x}-3\boldsymbol{e}_k) = \lambda\{\Psi^{2k}(\boldsymbol{x}-2\boldsymbol{e}_k) - \eta\}, \tag{4.14}$$

since $\lambda = \pm 1$. Similarly, Eq. (4.4) for $\boldsymbol{x} - 3\boldsymbol{e}_k$ becomes

$$\lambda \Psi^{2k-1}(\boldsymbol{x} - 4\boldsymbol{e}_k) + \Psi^{2k-1}(\boldsymbol{x} - 3\boldsymbol{e}_k) = \lambda \Psi^{2k}(\boldsymbol{x} - 2\boldsymbol{e}_k) + \Psi^{2k}(\boldsymbol{x} - 3\boldsymbol{e}_k). \quad (4.15)$$

From Eq. (4.15) with Eq. (4.14), we get

$$\Psi^{2k-1}(\boldsymbol{x} - 4\boldsymbol{e}_k) = \lambda \Psi^{2k}(\boldsymbol{x} - 3\boldsymbol{e}_k) + \eta.$$

Continuing this argument repeatedly, we finally abtain

$$\Psi^{2k-1}(\boldsymbol{x} - (j+1)\boldsymbol{e}_k) = \lambda \Psi^{2k}(\boldsymbol{x} - j\boldsymbol{e}_k) + (-\lambda)^{j+1}\eta, \quad (4.16)$$

for any $j = 0, 1, 2, \cdots$. Assumption $\#(S(\Psi)) < \infty$ implies that there exists J such that

$$\Psi^{2k-1}(\boldsymbol{x} - j'\boldsymbol{e}_k) = \Psi^{2k}(\boldsymbol{x} - j'\boldsymbol{e}_k) = 0, \quad (4.17)$$

for any $j' \geq J$. Combining Eq. (4.16) with Eq. (4.17) gives $\eta = 0$ since $\lambda \neq 0$. Therefore contradiction occurs, so the proof is complete.

Lemma 2. *Let $\Psi \in W(\lambda)$ with $\lambda = \pm 1$. Suppose $\#(S(\Psi)) < \infty$. If there exist $k \in \{1, 2, \cdots, d\}$ and $\boldsymbol{x} \in \mathbb{Z}^d$ such that*

$$\begin{bmatrix} \Psi^{2k-1}(\boldsymbol{x}) \\ \Psi^{2k}(\boldsymbol{x}) \end{bmatrix} \neq \begin{bmatrix} 0 \\ 0 \end{bmatrix},$$

then there exist $m^{(-)}(\leq 0)$ and $m^{(+)}(\geq 0)$ with $m^{(-)} < m^{(+)}$ and $\alpha, \beta \in \mathbb{C}$ with $\alpha\beta \neq 0$ such that

$$\begin{bmatrix} \Psi^{2k-1}(\boldsymbol{x} + m\boldsymbol{e}_k) \\ \Psi^{2k}(\boldsymbol{x} + m\boldsymbol{e}_k) \end{bmatrix} = \begin{cases} {}^T[0, 0] & (m < m^{(-)}) \\ {}^T[\alpha, 0] & (m = m^{(-)}) \\ {}^T[0, \beta] & (m = m^{(+)}) \\ {}^T[0, 0] & (m > m^{(+)}) \end{cases}. \quad (4.18)$$

Moreover, we have

$$\begin{bmatrix} \Psi^{2l-1}(\boldsymbol{x} + m^{(-)}\boldsymbol{e}_k) \\ \Psi^{2l}(\boldsymbol{x} + m^{(-)}\boldsymbol{e}_k) \end{bmatrix} \neq \begin{bmatrix} 0 \\ 0 \end{bmatrix},$$

and

$$\begin{bmatrix} \Psi^{2l-1}(\boldsymbol{x} + m^{(+)}\boldsymbol{e}_k) \\ \Psi^{2l}(\boldsymbol{x} + m^{(+)}\boldsymbol{e}_k) \end{bmatrix} \neq \begin{bmatrix} 0 \\ 0 \end{bmatrix},$$

for any $l \in \{1, 2, \cdots, d\} \backslash \{k\}$.

Proof. From Lemma 1, we get $\#(S(\Psi)) \geq 2$. Therefore we see that there exist $m^{(-)}(\leq 0)$ and $m^{(+)}(\geq 0)$ with $m^{(-)} < m^{(+)}$ and $\alpha, \beta, \gamma, \delta \in \mathbb{C}$ with $|\alpha| + |\gamma| > 0$ and $|\beta| + |\delta| > 0$ such that

$$\begin{bmatrix} \Psi^{2k-1}(\boldsymbol{x} + m\boldsymbol{e}_k) \\ \Psi^{2k}(\boldsymbol{x} + m\boldsymbol{e}_k) \end{bmatrix} = \begin{cases} {}^T\left[0, 0\right] & (m < m^{(-)}) \\ {}^T\left[\alpha, \gamma\right] & (m = m^{(-)}) \\ {}^T\left[\delta, \beta\right] & (m = m^{(+)}) \\ {}^T\left[0, 0\right] & (m > m^{(+)}) \end{cases}. \tag{4.19}$$

By Eq. (4.4) for $\boldsymbol{x} + (m^{(-)} - 1)\boldsymbol{e}_k$, we have

$$\begin{aligned} &\lambda\Psi^{2k-1}(\boldsymbol{x} + (m^{(-)} - 2)\boldsymbol{e}_k) + \Psi^{2k-1}(\boldsymbol{x} + (m^{(-)} - 1)\boldsymbol{e}_k) \\ &\quad = \lambda\Psi^{2k}(\boldsymbol{x} + m^{(-)}\boldsymbol{e}_k) + \Psi^{2k}(\boldsymbol{x} + (m^{(-)} - 1)\boldsymbol{e}_k). \end{aligned} \tag{4.20}$$

Combining Eq. (4.19) with Eq. (4.20) gives

$$\Psi^{2k}(\boldsymbol{x} + m^{(-)}\boldsymbol{e}_k) = \gamma = 0, \tag{4.21}$$

since $\lambda \neq 0$. In a similar fashion, from Eq. (4.4) for $\boldsymbol{x} + (m^{(+)} + 1)\boldsymbol{e}_k$, we have

$$\Psi^{2k-1}(\boldsymbol{x} + m^{(+)}\boldsymbol{e}_k) = \delta = 0. \tag{4.22}$$

Thus combining Eqs. (4.19), (4.21) and (4.22) implies Eq. (4.18).

By Eq. (4.2) for $\boldsymbol{x} + m^{(-)}\boldsymbol{e}_k$, we have

$$\lambda\Psi^{2k-1}(\boldsymbol{x} + (m^{(-)} - 1)\boldsymbol{e}_k) + \Psi^{2k-1}(\boldsymbol{x} + m^{(-)}\boldsymbol{e}_k) = \frac{1}{d}\Gamma(\boldsymbol{x} + m^{(-)}\boldsymbol{e}_k). \tag{4.23}$$

Then combining Eq. (4.23) with Eq. (4.18) gives

$$\frac{1}{d}\Gamma(\boldsymbol{x} + m^{(-)}\boldsymbol{e}_k) = \alpha. \tag{4.24}$$

Similarly, by Eq. (4.3) for $\boldsymbol{x} + m^{(+)}\boldsymbol{e}_k$ and Eq. (4.18), we get

$$\frac{1}{d}\Gamma(\boldsymbol{x} + m^{(+)}\boldsymbol{e}_k) = \beta.$$

From now on, we assume that there exists $l \in \{1, 2, \cdots, d\} \backslash \{k\}$ such that

$$\begin{bmatrix} \Psi^{2l-1}(\boldsymbol{x} + m^{(-)}\boldsymbol{e}_k) \\ \Psi^{2l}(\boldsymbol{x} + m^{(-)}\boldsymbol{e}_k) \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}, \tag{4.25}$$

or

$$\begin{bmatrix} \Psi^{2l-1}(\boldsymbol{x} + m^{(+)}\boldsymbol{e}_k) \\ \Psi^{2l}(\boldsymbol{x} + m^{(+)}\boldsymbol{e}_k) \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}. \tag{4.26}$$

First we consider Eq. (4.25) case. We now use Eq. (4.2) with $k \to l$ and $\boldsymbol{x} \to \boldsymbol{x} + m^{(-)}\boldsymbol{e}_k$ to get

$$\lambda\Psi^{2l-1}((\boldsymbol{x} + m^{(-)}\boldsymbol{e}_k) - \boldsymbol{e}_l) + \Psi^{2l-1}(\boldsymbol{x} + m^{(-)}\boldsymbol{e}_k) = \frac{1}{d}\Gamma(\boldsymbol{x} + m^{(-)}\boldsymbol{e}_k).$$

Using the equation just derived and Eq. (4.24), we have

$$\lambda\Psi^{2l-1}((\boldsymbol{x} + m^{(-)}\boldsymbol{e}_k) - \boldsymbol{e}_l) + \Psi^{2l-1}(\boldsymbol{x} + m^{(-)}\boldsymbol{e}_k) = \alpha. \tag{4.27}$$

By assumption $\Psi^{2l-1}(\boldsymbol{x} + m^{(-)}\boldsymbol{e}_k) = 0$ in Eq. (4.25), we see that Eq. (4.27) becomes

$$\Psi^{2l-1}((\boldsymbol{x} + m^{(-)}\boldsymbol{e}_k) - \boldsymbol{e}_l) = \lambda\alpha, \tag{4.28}$$

since $\lambda = \pm 1$. Next we see Eq. (4.4) with $k \to l$ and $\boldsymbol{x} \to \boldsymbol{x} + m^{(-)}\boldsymbol{e}_k - \boldsymbol{e}_l$ to get

$$\begin{aligned}\lambda\Psi^{2l-1}&((\boldsymbol{x} + m^{(-)}\boldsymbol{e}_k) - 2\boldsymbol{e}_l) + \Psi^{2l-1}((\boldsymbol{x} + m^{(-)}\boldsymbol{e}_k) - \boldsymbol{e}_l) \\ &= \lambda\Psi^{2l}(\boldsymbol{x} + m^{(-)}\boldsymbol{e}_k) + \Psi^{2l}((\boldsymbol{x} + m^{(-)}\boldsymbol{e}_k) - \boldsymbol{e}_l).\end{aligned}$$

Combining this equation with Eq. (4.28) and assumption $\Psi^{2l}(\boldsymbol{x}+m^{(-)}\boldsymbol{e}_k) = 0$ in Eq. (4.25) gives

$$\Psi^{2l-1}((\boldsymbol{x} + m^{(-)}\boldsymbol{e}_k) - 2\boldsymbol{e}_l) = \lambda\Psi^{2l}((\boldsymbol{x} + m^{(-)}\boldsymbol{e}_k) - \boldsymbol{e}_l) - \lambda^2\alpha,$$

since $\lambda = \pm 1$. By the similar argument repeatedly, we obtain,

$$\Psi^{2l-1}((\boldsymbol{x} + m^{(-)}\boldsymbol{e}_k) - (j+1)\boldsymbol{e}_l) = \lambda\Psi^{2l}((\boldsymbol{x} + m^{(-)}\boldsymbol{e}_k) - j\boldsymbol{e}_l) - (-\lambda)^{j+1}\alpha, \tag{4.29}$$

for any $j = 1, 2, \cdots$. Assumption $\#(S(\Psi)) < \infty$ implies that there exists J such that

$$\Psi^{2l-1}((\boldsymbol{x} + m^{(-)}\boldsymbol{e}_k) - j'\boldsymbol{e}_l) = \Psi^{2l}((\boldsymbol{x} + m^{(-)}\boldsymbol{e}_k) - j'\boldsymbol{e}_l) = 0, \tag{4.30}$$

for any $j' \geq J$. Combining Eq. (4.29) with Eq. (4.30) gives $\alpha = 0$ since $\lambda \neq 0$. Thus we have a contradiction.

Next we consider Eq. (4.26) case. In a simiar fashion, we get $\beta = 0$ and have a contradiction. Therefore the proof of Lemma 2 is complete.

THEOREM 1. *For the Grover walk on* $\mathbb{Z}^d$ *with moving shift, we have*

$$\#(S(\Psi)) \geq 2^d,$$

for any $\Psi \in W(\lambda)$ *with* $\lambda = \pm 1$. *In particular, there exists* $\Psi_\star^{(\lambda)} \in W(\lambda)$ *such that*

$$\#(S(\Psi_\star^{(\lambda)})) = 2^d,$$

for $\lambda = \pm 1$. *In fact, we obtain*

$$\Psi_{\star}^{(\lambda)}(\boldsymbol{x}) = \lambda^{x_1+x_2+\cdots+x_d} \times \ {}^T\left[|x_1\rangle,\ |x_2\rangle,\ \cdots,\ |x_d\rangle\right] \quad (\boldsymbol{x} \in S(\Psi_{\star}^{(\lambda)})),$$

where

$$S(\Psi_{\star}^{(\lambda)}) = \{\boldsymbol{x} = (x_1, x_2, \cdots, x_d) \in \mathbb{Z}^d : x_k \in \{0, 1\}\ (k = 1, 2, \cdots, d)\}.$$

Here $|0\rangle = {}^T[1,\ 0]$ *and* $|1\rangle = {}^T[0,\ 1]$.

Proof. For $\Psi \in W(\lambda)$ with $\lambda = \pm 1$, there exist $k \in \{1, 2, \cdots, d\}$ and $\boldsymbol{x} \in \mathbb{Z}^d$ such that

$$\begin{bmatrix} \Psi^{2k-1}(\boldsymbol{x}) \\ \Psi^{2k}(\boldsymbol{x}) \end{bmatrix} \neq \begin{bmatrix} 0 \\ 0 \end{bmatrix}.$$

Thus, we have $\boldsymbol{x} \in S(\Psi)$.

First we consider $d = 1$ case. From Lemma 1, we see that $\boldsymbol{x} - \boldsymbol{e}_1 \in S(\Psi)$ or $\boldsymbol{x} + \boldsymbol{e}_1 \in S(\Psi)$, so $\#(S(\Psi)) \geq 2$. If fact, we can construct a $\Psi_{\star}^{(\lambda)} \in W(\lambda)$ with $\lambda = \pm 1$ satisfying $\#(S(\Psi_{\star}^{(\lambda)})) = 2$ as follows.

$$\begin{bmatrix} \Psi^1(\boldsymbol{x} + m_1\boldsymbol{e}_1) \\ \Psi^2(\boldsymbol{x} + m_1\boldsymbol{e}_1) \end{bmatrix} = \begin{cases} {}^T\left[0, 0\right] & (m_1 < 0) \\ \lambda^{m_1} \times {}^T\left[1, 0\right] & (m_1 = 0) \\ \lambda^{m_1} \times {}^T\left[0, 1\right] & (m_1 = 1) \\ {}^T\left[0, 0\right] & (m_1 > 1) \end{cases},$$

where $m_1 \in \mathbb{Z}$.

Next we deal with $d = 2$ case. Considering the argument for $d = 1$ case, we can assume

$$\begin{bmatrix} \Psi^1(\boldsymbol{x}) \\ \Psi^2(\boldsymbol{x}) \end{bmatrix} \neq \begin{bmatrix} 0 \\ 0 \end{bmatrix} \text{ and } \begin{bmatrix} \Psi^1(\boldsymbol{x} + \boldsymbol{e}_1) \\ \Psi^2(\boldsymbol{x} + \boldsymbol{e}_1) \end{bmatrix} \neq \begin{bmatrix} 0 \\ 0 \end{bmatrix}. \tag{4.31}$$

By Lemma 2 with Eq. (4.31), we can also assume $m^{(-)} = 0$ and $m^{(+)} = 1$ to minimize the $\#(S(\Psi))$, then we have

$$\begin{bmatrix} \Psi^3(\boldsymbol{x}) \\ \Psi^4(\boldsymbol{x}) \end{bmatrix} \neq \begin{bmatrix} 0 \\ 0 \end{bmatrix}, \tag{4.32}$$

and

$$\begin{bmatrix} \Psi^3(\boldsymbol{x} + \boldsymbol{e}_1) \\ \Psi^4(\boldsymbol{x} + \boldsymbol{e}_1) \end{bmatrix} \neq \begin{bmatrix} 0 \\ 0 \end{bmatrix}. \tag{4.33}$$

From Lemma 1 with Eqs. (4.32) and (4.33), we obtain "$\boldsymbol{x} - \boldsymbol{e}_2 \in S(\Psi)$ or $\boldsymbol{x} + \boldsymbol{e}_2 \in S(\Psi)$" and "$\boldsymbol{x} + \boldsymbol{e}_1 - \boldsymbol{e}_2 \in S(\Psi)$ or $\boldsymbol{x} + \boldsymbol{e}_1 + \boldsymbol{e}_2 \in S(\Psi)$" respectively, so

$\#(S(\Psi)) \geq 4$. In fact, we can construct a $\Psi_\star^{(\lambda)} \in W(\lambda)$ with $\lambda = \pm 1$ satisfying $\#(S(\Psi_\star^{(\lambda)})) = 4$ as follows.

$$\Psi(\boldsymbol{x} + m_1\boldsymbol{e}_1 + m_2\boldsymbol{e}_2) = \begin{cases} \lambda^{m_1+m_2} \times {}^T\left[1,0,1,0\right] & (m_1,m_2) = (0,0) \\ \lambda^{m_1+m_2} \times {}^T\left[0,1,1,0\right] & (m_1,m_2) = (1,0) \\ \lambda^{m_1+m_2} \times {}^T\left[1,0,0,1\right] & (m_1,m_2) = (0,1) \\ \lambda^{m_1+m_2} \times {}^T\left[0,1,0,1\right] & (m_1,m_2) = (1,1) \\ {}^T\left[0,0,0,0\right] & (otherwise) \end{cases}, \quad (4.34)$$

for $m_1, m_2 \in \mathbb{Z}$. Remark that Eq. (4.34) has been introduced in Stefanak et al. [7]. Continuing a similar argument for $d = 3, 4, \cdots$, we have the desired conclusion.

From Eq. (4.34), we obtain the following equation as one of a stationary measure of Grover walk on $\mathbb{Z}^2$ when $\lambda = 1$.

$$\Psi = \begin{bmatrix}1\\0\\1\\0\end{bmatrix}\delta_{(x,y)} + \begin{bmatrix}0\\1\\1\\0\end{bmatrix}\delta_{(x+1,y)} + \begin{bmatrix}1\\0\\0\\1\end{bmatrix}\delta_{(x,y+1)} + \begin{bmatrix}0\\1\\0\\1\end{bmatrix}\delta_{(x+1,y+1)},$$

for any $(x, y) \in \mathbb{Z}^2$. Let $x, y \in \{0, -1\}$, then we easily get $\#(S(\Psi)) = 9$ such that

$$\begin{aligned} \Psi = & \begin{bmatrix}2\\2\\2\\2\end{bmatrix}\delta_{(0,0)} + \begin{bmatrix}1\\1\\0\\2\end{bmatrix}\delta_{(0,1)} + \begin{bmatrix}0\\2\\1\\1\end{bmatrix}\delta_{(1,0)} + \begin{bmatrix}1\\1\\2\\0\end{bmatrix}\delta_{(0,-1)} + \begin{bmatrix}2\\0\\1\\1\end{bmatrix}\delta_{(-1,0)} \\ & + \begin{bmatrix}0\\1\\0\\1\end{bmatrix}\delta_{(1,1)} + \begin{bmatrix}0\\1\\1\\0\end{bmatrix}\delta_{(1,-1)} + \begin{bmatrix}1\\0\\1\\0\end{bmatrix}\delta_{(-1,-1)} + \begin{bmatrix}1\\0\\0\\1\end{bmatrix}\delta_{(-1,1)}. \end{aligned} \quad (4.35)$$

Remark that Eq. (4.35) has been introduced in Komatsu and Konno [4].

5. Grover walk on $\mathbb{Z}^d$ with flip-flop shift

In this section, we consider the case of the d-dimensional Grover walk with flip-flop shift. The eigenvalue problem $U_G\Psi = \lambda\Psi$ ($\lambda \in \mathbb{C}$ with $|\lambda| = 1$) is equivalent to

$$\begin{cases} \lambda\Psi^1(\boldsymbol{x}) = \frac{1}{d}\Psi^1(\boldsymbol{x}+\boldsymbol{e}_1) + \frac{1-d}{d}\Psi^2(\boldsymbol{x}+\boldsymbol{e}_1) + \cdots + \frac{1}{d}\Psi^{2d-1}(\boldsymbol{x}+\boldsymbol{e}_1) + \frac{1}{d}\Psi^{2d}(\boldsymbol{x}+\boldsymbol{e}_1), \\ \lambda\Psi^2(\boldsymbol{x}) = \frac{1-d}{d}\Psi^1(\boldsymbol{x}-\boldsymbol{e}_1) + \frac{1}{d}\Psi^2(\boldsymbol{x}-\boldsymbol{e}_1) + \cdots + \frac{1}{d}\Psi^{2d-1}(\boldsymbol{x}-\boldsymbol{e}_1) + \frac{1}{d}\Psi^{2d}(\boldsymbol{x}-\boldsymbol{e}_1), \\ \vdots \\ \lambda\Psi^{2d-1}(\boldsymbol{x}) = \frac{1}{d}\Psi^1(\boldsymbol{x}+\boldsymbol{e}_d) + \frac{1}{d}\Psi^2(\boldsymbol{x}+\boldsymbol{e}_d) + \cdots + \frac{1}{d}\Psi^{2d-1}(\boldsymbol{x}+\boldsymbol{e}_d) + \frac{1-d}{d}\Psi^{2d}(\boldsymbol{x}+\boldsymbol{e}_d), \\ \lambda\Psi^{2d}(\boldsymbol{x}) = \frac{1}{d}\Psi^1(\boldsymbol{x}-\boldsymbol{e}_d) + \frac{1}{d}\Psi^2(\boldsymbol{x}-\boldsymbol{e}_d) + \cdots + \frac{1-d}{d}\Psi^{2d-1}(\boldsymbol{x}-\boldsymbol{e}_d) + \frac{1}{d}\Psi^{2d}(\boldsymbol{x}-\boldsymbol{e}_d), \end{cases} \tag{5.1}$$

where $\Psi(\boldsymbol{x}) = {}^T\left[\Psi^1(\boldsymbol{x}), \Psi^2(\boldsymbol{x}), \cdots, \Psi^{2d}(\boldsymbol{x})\right]$ $(\boldsymbol{x} \in \mathbb{Z}^d)$. Put $\Gamma(\boldsymbol{x}) = \sum_{j=1}^{2d} \Psi^j(\boldsymbol{x})$ for $\boldsymbol{x} \in \mathbb{Z}^d$. By using $\Gamma(\boldsymbol{x})$, Eq. (5.1) can be written as

$$\lambda\Psi^{2k-1}(\boldsymbol{x}-\boldsymbol{e}_k) + \Psi^{2k}(\boldsymbol{x}) = \frac{1}{d}\Gamma(\boldsymbol{x}), \tag{5.2}$$

$$\lambda\Psi^{2k}(\boldsymbol{x}+\boldsymbol{e}_k) + \Psi^{2k-1}(\boldsymbol{x}) = \frac{1}{d}\Gamma(\boldsymbol{x}), \tag{5.3}$$

for any $k = 1, 2, \ldots, d$ and $\boldsymbol{x} \in \mathbb{Z}^d$. From Eqs. (5.2) and (5.3), we get immediately

$$\lambda\Psi^{2k-1}(\boldsymbol{x}-\boldsymbol{e}_k) + \Psi^{2k}(\boldsymbol{x}) = \lambda\Psi^{2k}(\boldsymbol{x}+\boldsymbol{e}_k) + \Psi^{2k-1}(\boldsymbol{x}), \tag{5.4}$$

for any $k = 1, 2, \ldots, d$ and $\boldsymbol{x} \in \mathbb{Z}^d$.

Lemma 3. *Let $\Psi \in W(\lambda)$ with $\lambda = \pm 1$. Suppose $\#(S(\Psi)) < \infty$. If there exist $k \in \{1, 2, \cdots, d\}$ and $\boldsymbol{x} \in \mathbb{Z}^d$ such that*

$$\begin{bmatrix} \Psi^{2k-1}(\boldsymbol{x}) \\ \Psi^{2k}(\boldsymbol{x}) \end{bmatrix} \neq \begin{bmatrix} 0 \\ 0 \end{bmatrix},$$

then we have

$$\begin{bmatrix} \Psi^{2k-1}(\boldsymbol{x}-\boldsymbol{e}_k) \\ \Psi^{2k}(\boldsymbol{x}-\boldsymbol{e}_k) \end{bmatrix} \neq \begin{bmatrix} 0 \\ 0 \end{bmatrix} \quad \text{or} \quad \begin{bmatrix} \Psi^{2k-1}(\boldsymbol{x}+\boldsymbol{e}_k) \\ \Psi^{2k}(\boldsymbol{x}+\boldsymbol{e}_k) \end{bmatrix} \neq \begin{bmatrix} 0 \\ 0 \end{bmatrix}.$$

Proof. First we assume that

$$\begin{bmatrix} \Psi^{2k-1}(\boldsymbol{x}) \\ \Psi^{2k}(\boldsymbol{x}) \end{bmatrix} \neq \begin{bmatrix} 0 \\ 0 \end{bmatrix},$$

for some $k \in \{1, 2, \cdots, d\}$ and $\boldsymbol{x} \in \mathbb{Z}^d$. Furthermore we suppose

$$\begin{bmatrix} \Psi^{2k-1}(\boldsymbol{x}-\boldsymbol{e}_k) \\ \Psi^{2k}(\boldsymbol{x}-\boldsymbol{e}_k) \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \quad \text{and} \quad \begin{bmatrix} \Psi^{2k-1}(\boldsymbol{x}+\boldsymbol{e}_k) \\ \Psi^{2k}(\boldsymbol{x}+\boldsymbol{e}_k) \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}.$$

By a similar calculation as in Lemma 1, we get the following equation corresponding to Eq. (4.16).

$$\Psi^{2k-1}(\boldsymbol{x}-(j+1)\boldsymbol{e}_k) = -\lambda\Psi^{2k}(\boldsymbol{x}-j\boldsymbol{e}_k) + \lambda^{j+1}\eta, \tag{5.5}$$

where $\eta = \Psi^{2k-1}(\boldsymbol{x}) = \Psi^{2k}(\boldsymbol{x})$ for any $j = 0, 1, 2, \cdots$. Assumption $\#(S(\Psi)) < \infty$ implies that there exists J such that

$$\Psi^{2k-1}(\boldsymbol{x}-j'\boldsymbol{e}_k) = \Psi^{2k}(\boldsymbol{x}-j'\boldsymbol{e}_k) = 0, \tag{5.6}$$

for any $j' \geq J$. Combining Eq. (5.5) with Eq. (5.6) gives $\eta = 0$ since $\lambda \neq 0$. Therefore contradiction occurs, so the proof is complete.

Lemma 4. *Let $\Psi \in W(\lambda)$ with $\lambda = \pm 1$. Suppose $\#(S(\Psi)) < \infty$. If there exist $k \in \{1, 2, \cdots, d\}$ and $\boldsymbol{x} \in \mathbb{Z}^d$ such that*

$$\begin{bmatrix} \Psi^{2k-1}(\boldsymbol{x}) \\ \Psi^{2k}(\boldsymbol{x}) \end{bmatrix} \neq \begin{bmatrix} 0 \\ 0 \end{bmatrix},$$

then there exist $m^{(-)}(\leq 0)$ and $m^{(+)}(\geq 0)$ with $m^{(-)} < m^{(+)}$ and $\alpha, \beta \in \mathbb{C}$ with $\alpha\beta \neq 0$ such that

$$\begin{bmatrix} \Psi^{2k-1}(\boldsymbol{x}+m\boldsymbol{e}_k) \\ \Psi^{2k}(\boldsymbol{x}+m\boldsymbol{e}_k) \end{bmatrix} = \begin{cases} {}^T[0,0] & (m < m^{(-)}) \\ {}^T[\alpha,0] & (m = m^{(-)}) \\ {}^T[0,\beta] & (m = m^{(+)}) \\ {}^T[0,0] & (m > m^{(+)}) \end{cases}. \tag{5.7}$$

Moreover, we have

$$\Gamma(\boldsymbol{x}+m^{(-)}\boldsymbol{e}_k) = 0\ , \tag{5.8}$$

and

$$\Gamma(\boldsymbol{x}+m^{(+)}\boldsymbol{e}_k) = 0\ .$$

Proof. From Lemma 3, we get $\#(S(\Psi)) \geq 2$. Therefore we see that there exist $m^{(-)}(\leq 0)$ and $m^{(+)}(\geq 0)$ with $m^{(-)} < m^{(+)}$ and $\alpha, \beta, \gamma, \delta \in \mathbb{C}$ with $|\alpha| + |\gamma| > 0$ and $|\beta| + |\delta| > 0$ such that

$$\begin{bmatrix} \Psi^{2k-1}(\boldsymbol{x}+m\boldsymbol{e}_k) \\ \Psi^{2k}(\boldsymbol{x}+m\boldsymbol{e}_k) \end{bmatrix} = \begin{cases} {}^T[0,0] & (m < m^{(-)}) \\ {}^T[\alpha,\gamma] & (m = m^{(-)}) \\ {}^T[\delta,\beta] & (m = m^{(+)}) \\ {}^T[0,0] & (m > m^{(+)}) \end{cases}. \tag{5.9}$$

By Eq. (5.4) for $\boldsymbol{x}+(m^{(-)}-1)\boldsymbol{e}_k$, we have

$$\begin{aligned}\lambda\Psi^{2k-1}(\boldsymbol{x}+(m^{(-)}-2)\boldsymbol{e}_k)+\Psi^{2k}(\boldsymbol{x}+(m^{(-)}-1)\boldsymbol{e}_k)\\ =\lambda\Psi^{2k}(\boldsymbol{x}+m^{(-)}\boldsymbol{e}_k)+\Psi^{2k-1}(\boldsymbol{x}+(m^{(-)}-1)\boldsymbol{e}_k).\end{aligned} \tag{5.10}$$

Combining Eq. (5.9) with Eq. (5.10) gives

$$\Psi^{2k}(\boldsymbol{x}+m^{(-)}\boldsymbol{e}_k)=\gamma=0, \tag{5.11}$$

since $\lambda\neq 0$. In a similar fashion, from Eq. (5.4) for $\boldsymbol{x}+(m^{(+)}+1)\boldsymbol{e}_k$, we have

$$\Psi^{2k-1}(\boldsymbol{x}+m^{(+)}\boldsymbol{e}_k)=\delta=0. \tag{5.12}$$

Thus combining Eqs. (5.9), (5.11) and (5.12) implies Eq. (5.7).

By Eq. (5.2) for $\boldsymbol{x}+m^{(-)}\boldsymbol{e}_k$, we have

$$\lambda\Psi^{2k-1}(\boldsymbol{x}+(m^{(-)}-1)\boldsymbol{e}_k)+\Psi^{2k}(\boldsymbol{x}+m^{(-)}\boldsymbol{e}_k)=\frac{1}{d}\Gamma(\boldsymbol{x}+m^{(-)}\boldsymbol{e}_k). \tag{5.13}$$

Then combining Eq. (5.13) with Eq. (5.7) gives

$$\frac{1}{d}\Gamma(\boldsymbol{x}+m^{(-)}\boldsymbol{e}_k)=0.$$

Similarly, by Eq. (5.3) for $\boldsymbol{x}+m^{(+)}\boldsymbol{e}_k$ and Eq. (5.7), we get

$$\frac{1}{d}\Gamma(\boldsymbol{x}+m^{(+)}\boldsymbol{e}_k)=0.$$

Therefore the proof of Lemma 4 is complete.

Theorem 2. *For the Grover walk on $\mathbb{Z}^d$ with flip-flop shift, we have*

$$\begin{cases}\#(S(\Psi))=0 & (d=1)\\ \#(S(\Psi))\geq 4 & (d\geq 2)\end{cases},$$

for any $\Psi\in W(\lambda)$ with $\lambda=\pm1$. In particular, there exists $\Psi_\star^{(\lambda)}\in W(\lambda)$ such that

$$\#(S(\Psi_\star^{(\lambda)}))=4 \quad (d\geq 2)$$

for $\lambda=\pm1$. In fact, we obtain

$$\Psi_\star^{(\lambda)}(\boldsymbol{x})=\lambda^{x_1+x_2}\times {}^T\left[(-1)^{x_1+x_2}|x_1\rangle,\ (-1)^{x_1+x_2+1}|x_2\rangle,\mathbf{0},\cdots,\mathbf{0}\right] \quad (\boldsymbol{x}\in S(\Psi_\star^{(\lambda)})),$$

where

$$S(\Psi_\star^{(\lambda)})=\{\boldsymbol{x}=(x_1,x_2,\cdots,x_d)\in\mathbb{Z}^d: x_1,x_2\in\{0,1\},\ x_3=x_4=\cdots=x_d=0\}.$$

Here $|0\rangle={}^T[1,\ 0]$, $|1\rangle={}^T[0,\ 1]$ *and* $\mathbf{0}={}^T[0,\ 0]$.

Proof. First, we consider $d = 1$ case. For $\Psi \in W(\lambda)$ with $\lambda = \pm 1$, there exists $x \in \mathbb{Z}$ such that

$$\begin{bmatrix} \Psi^1(x) \\ \Psi^2(x) \end{bmatrix} \neq \begin{bmatrix} 0 \\ 0 \end{bmatrix}.$$

From Lemma 4, we have $m_1^{(-)}(\leq 0)$ and $m_1^{(+)}(\geq 0)$ with $m_1^{(-)} < m_1^{(+)}$ and $\alpha, \beta \in \mathbb{C}$ with $\alpha\beta \neq 0$ such that

$$\begin{bmatrix} \Psi^1(x+m_1) \\ \Psi^2(x+m_1) \end{bmatrix} = \begin{cases} {}^T\begin{bmatrix} 0, 0 \end{bmatrix} & (m_1 < m_1^{(-)}) \\ {}^T\begin{bmatrix} \alpha, 0 \end{bmatrix} & (m_1 = m_1^{(-)}) \\ {}^T\begin{bmatrix} 0, \beta \end{bmatrix} & (m_1 = m_1^{(+)}) \\ {}^T\begin{bmatrix} 0, 0 \end{bmatrix} & (m_1 > m_1^{(+)}) \end{cases}, \tag{5.14}$$

and

$$\begin{cases} \Gamma(x + m_1^{(-)}) = 0 \\ \Gamma(x + m_1^{(+)}) = 0 \end{cases}. \tag{5.15}$$

By definition of Γ and Eq. (5.14), we have

$$\begin{cases} \Gamma(x + m_1^{(-)}) = \alpha \\ \Gamma(x + m_1^{(+)}) = \beta \end{cases}. \tag{5.16}$$

Combining Eq. (5.15) with Eq. (5.16), we get $\alpha = \beta = 0$. So we see that the finite support for d=1 does not exist.

Next we deal with $d = 2$ case. For $\Psi \in W(\lambda)$ with $\lambda = \pm 1$, we assume that there exists $\boldsymbol{x} \in \mathbb{Z}^2$ such that

$$\begin{bmatrix} \Psi^1(\boldsymbol{x}) \\ \Psi^2(\boldsymbol{x}) \end{bmatrix} \neq \begin{bmatrix} 0 \\ 0 \end{bmatrix}, \tag{5.17}$$

and we put

$$\begin{cases} m^{(-)} = 0 \\ m^{(+)} = 1 \end{cases}, \tag{5.18}$$

for Eq. (5.7) on Lemma 4 to minimize $\#(S(\Psi))$. By using (5.7) with Eq. (5.18), we have

$$\begin{bmatrix} \Psi^1(\boldsymbol{x}) \\ \Psi^2(\boldsymbol{x}) \end{bmatrix} = \begin{bmatrix} \alpha \\ 0 \end{bmatrix}. \tag{5.19}$$

By definition of Γ with Eqs. (5.8), (5.18) and (5.19), we get

$$\begin{bmatrix} \Psi^3(\boldsymbol{x}) \\ \Psi^4(\boldsymbol{x}) \end{bmatrix} \neq \begin{bmatrix} 0 \\ 0 \end{bmatrix}, \tag{5.20}$$

since $\alpha \neq 0$.

Similarly, from Lemma 3 with Eq. (5.17), we can assume

$$\begin{bmatrix} \Psi^1(\boldsymbol{x}+\boldsymbol{e}_1) \\ \Psi^2(\boldsymbol{x}+\boldsymbol{e}_1) \end{bmatrix} \neq \begin{bmatrix} 0 \\ 0 \end{bmatrix},$$

and we obtain

$$\begin{bmatrix} \Psi^3(\boldsymbol{x}+\boldsymbol{e}_1) \\ \Psi^4(\boldsymbol{x}+\boldsymbol{e}_1) \end{bmatrix} \neq \begin{bmatrix} 0 \\ 0 \end{bmatrix}, \tag{5.21}$$

since $\beta \neq 0$. From Lemma 3 with Eqs. (5.20) and (5.21), we obtain "$\boldsymbol{x}-\boldsymbol{e}_2 \in S(\Psi)$ or $\boldsymbol{x}+\boldsymbol{e}_2 \in S(\Psi)$" and "$\boldsymbol{x}+\boldsymbol{e}_1-\boldsymbol{e}_2 \in S(\Psi)$ or $\boldsymbol{x}+\boldsymbol{e}_1+\boldsymbol{e}_2 \in S(\Psi)$" respectively, so $\#(S(\Psi)) \geq 4$. In fact, we can construct a $\Psi_\star^{(\lambda)} \in W(\lambda)$ with $\lambda = \pm 1$ satisfying $\#(S(\Psi_\star^{(\lambda)})) = 4$ as follows.

$$\Psi(\boldsymbol{x}+m_1\boldsymbol{e}_1+m_2\boldsymbol{e}_2) = \begin{cases} {}^T\left[1,0,-1,0\right] & (m_1,m_2)=(0,0) \\ {}^T\left[0,-\lambda,\lambda,0\right] & (m_1,m_2)=(1,0) \\ {}^T\left[-\lambda,0,0,\lambda\right] & (m_1,m_2)=(0,1) \\ {}^T\left[0,1,0,-1\right] & (m_1,m_2)=(1,1) \\ {}^T\left[0,0,0,0\right] & (otherwise) \end{cases}, \tag{5.22}$$

for $m_1, m_2 \in \mathbb{Z}$.

Finally, we consider $d \geq 3$ case by continuing the argument on $d = 2$ case. To expand Eq. (5.22) to $d \geq 3$, we focus on the fact that $\Gamma(\boldsymbol{x}+m_1\boldsymbol{e}_1+m_2\boldsymbol{e}_2) = 0$ for any $\boldsymbol{x} \in \mathbb{Z}^2$ and $m_1, m_2 \in \mathbb{Z}$ in Eq. (5.22). By assuming $\Psi^{2k-1}(\boldsymbol{x}+m_1\boldsymbol{e}_1+m_2\boldsymbol{e}_2) = \Psi^{2k}(\boldsymbol{x}+m_1\boldsymbol{e}_1+m_2\boldsymbol{e}_2) = 0$ for any $k \in \{3,4,\cdots,d\}$, we can contruct a $\Psi_\star^{(\lambda)} \in W(\lambda)$ with $\lambda = \pm 1$ satisfying $\#(S(\Psi_\star^{(\lambda)})) = 4$ as follows.

$$\Psi(\boldsymbol{x}+m_1\boldsymbol{e}_1+m_2\boldsymbol{e}_2) = \begin{cases} {}^T\left[1,0,-1,0,0,\cdots,0\right] & (m_1,m_2)=(0,0) \\ {}^T\left[0,-\lambda,\lambda,0,0,\cdots,0\right] & (m_1,m_2)=(1,0) \\ {}^T\left[-\lambda,0,0,\lambda,0,\cdots,0\right] & (m_1,m_2)=(0,1) \\ {}^T\left[0,1,0,-1,0,\cdots,0\right] & (m_1,m_2)=(1,1) \\ {}^T\left[0,0,0,0,0,\cdots,0\right] & (otherwise) \end{cases}.$$

Theorem 2 can be derived from another approach based on the spectral mapping theorem, see Corollary 2 in Higuchi et al. [2].

6. Summary

We presented the minimum supports of states for the Grover walk on $\mathbb{Z}^d$ with moving and flip-flop shifts, respectively, by solving the eigenvalue problem $U_G\Psi = \lambda\Psi$. Results on the moving shift model was obtained by Theorem 1 which coincides with result in Stefanak et al. [7] ($\mathbb{Z}^2$ case) and improves result in Komatsu and Konno [4] ($\mathbb{Z}^d$ case). Moreover, results on the flip-flop shift model shown by Higuchi et al. [2] was given by Theorem 2. One of the interesting future problems might be to clarify a relationship between the stationary measure and the time-averaged limit measure of the Grover walk on $\mathbb{Z}^d$.

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