Yokohama Mathematical
Journal Vol. 66, 2020

# ON THE REGULARITY OF THE SOLUTION OF NON-HOMOGENEOUS BURGERS EQUATION

By

Yassine Benia and Boubaker-Khaled Sadallah

(Received May 25, 2020; Revised December 4, 2020)

**Abstract.** In this paper, we consider Cauchy-Dirichlet problem for non-homogeneous Burgers equation with some hypotheses on the right-hand side, and we give a new regularity result of the solution in an anisotropic Sobolev space using the Faedo-Galerkin method. This work is an extension of solvability results we obtained for a right-hand side $f$ in Lebesgue space, set in a non-regular domain [4, 5]. Here, $f$ is in an anisotropic Sobolev space.

## 1. Introduction

One of the most important partial differential equations of the nonlinear conservation laws theory, is the semilinear diffusion equation, called Burgers equation:

$$\partial_t u + u\partial_x u - \nu\partial_x^2 u = f, \tag{1}$$

where $u$ stands, generally, for a velocity, $t$ the time variable, $x$ the space variable and $\nu$ the constant of viscosity (or the diffusion coefficient). The mathematical structure of this equation includes a nonlinear convection term $u\partial_x u$ which makes the equation more interesting, and a viscosity term of higher order $\partial_x^2 u$ which regularizes the equation and produces a dissipation effect of the solution near a shock.

Historically, Forsyth treated an equation which converts by some variable changes to the Equation (1) in 1906. Later in 1915, Bateman [2] introduced the Equation (1): He was interested in the case when $\nu$ approaches zero, Equation (1) is reduced to the transport equation, which represents the inviscid Burgers equation $\partial_t u + u\partial_x u = f$. Burgers (1948) has published a study on the Equation (1) (which it owes his name), in his paper [7] about modeling the turbulence phenomena.

---

2010 Mathematics Subject Classification: 35K58, 35Q35.
Key words and phrases: Semilinear parabolic problem, Burgers equation, regularity, anisotropic Sobolev space.

The so called Hopf-Cole transformation has been discovered by Cole [10] and independently by Hopf [12] in 1951. It converts Equation (1), with $f = 0$ to the linear heat equation and then (1) is explicitly solved. Burgers continued his study of what he called "nonlinear diffusion equation". This study treated mainly the static aspects of the equation. The results of these works can be found in the book [6].

The objective of Burgers was to consider a simplified version of the incompressible Navier Stokes equations by neglecting the mass conservation law and the pressure term.

Among the most interesting applications of the homogeneous Burgers equation, we mention the phenomena of turbulence, supersonic flow, heat conduction, elasticity, fusion [9], traffic flow, growth of interfaces, and financial mathematics [13, 16]. Non-homogeneous Burgers equation is an effective model of the dynamics of nonlinear dissipative media of various physical nature [19, 20]. It was also applied to other physical phenomena, such as wind forcing the buildup of water waves, the electrohydrodynamic field in plasma physics, nonlinear standing waves in the cylindrical resonator, and design of feedback control [22].

Using the Hopf-Cole transformation $u = -2\nu \frac{\partial_x \varphi}{\varphi}$, Equation (1) can be transformed into the equation

$$\partial_t \varphi - \nu \partial_x^2 \varphi = -F(x,t)\frac{\varphi}{2\nu},$$

and explicitly solved, under some choices of the right-hand side of (1), where $F(x,t) = \int f(x,t)dx + c(t)$ and $c$ is an arbitrary function in $t$. For example, if the right-hand side depends only on time $f(x,t) = G(t)$, this equation can be transformed into an homogeneous Burgers equation, see [15]. The problems with $f(x,t) = kx$, $f(x,t) = \frac{kx}{(2\beta t+1)^2}$ and (an elastic forcing term) $f(x,t) = -k^2 x + f(t)$ considered in a half-space, where $k, \beta$ are some constants are discussed, and their analytical solutions are obtained in [21, 17]. Later, the problem with $f(x,t) = G(t)x$ has been solved in [11]. In [18], the authors consider a forced Burgers equation with variable coefficients in a half-space, and discuss different types of solutions such as shock solitary wave, triangular wave, N -wave and rational function solutions. In the work [1], the authors have explicit solutions when the right-hand side is successively of the form $f(t)x + g(t)$, $f(t)$, $g(x)$ and $e^{\alpha x+\beta t}$, and the numerical simulation is given.

As far as we know, there is no work about the regularity of solution for the Burgers equation in Sobolev spaces (depending on the regularity of the right-hand side of the equation), except our works [4, 5]. This is the reason why the present work is interested in proving a result on a maximal regularity (in Sobolev spaces) for the non-homogeneous Burgers problem

$$\begin{cases} \partial_t u(t,x) + u(t,x)\partial_x u(t,x) - \nu\partial_x^2 u(t,x) = f(t,x) \quad (t,x) \in R, \\ u(0,x) = \psi(x) \quad x \in \Gamma_0, \\ u(t,0) = u(t,1) = 0 \quad t \in (0,T), \end{cases} \tag{2}$$

in the rectangle $R = (0,T)\times I$ with $I = (0,1)$ and $\Gamma_0 = \{0\}\times I$, $T$ is finite and $\nu$ is a positive constant; $\psi \in H^3(\Gamma_0) \cap H_0^1(\Gamma_0)$ i.e., $\psi \in H^3(\Gamma_0)$ and $\psi(0) = \psi(1) = 0$, and $f \in H^{1,2}(R)$ are given functions, where $H^3(\Gamma_0)$, $H_0^1(\Gamma_0)$ are usual Sobolev spaces and $H^{1,2}(R)$ is the anisotropic Sobolev space defined by

$$H^{1,2}(R) = \left\{u \in L^2(R) : \partial_t u \in L^2(R),\ \partial_x u \in L^2(R),\ \partial_x^2 u \in L^2(R)\right\}.$$

In previous works (see [4, 5]) we have studied Burgers equation $\partial_t u + u\partial_x u - \partial_x^2 u = f$ (with Dirichlet boundary conditions) in the polygonal domain $\Omega \subset \mathbb{R}^2$

$$\begin{aligned} \Omega &= \{(t,x) \in \mathbb{R}^2 : 0 < t < T,\ x \in I_t\}, \\ I_t &= \{x \in \mathbb{R} : \varphi_1(t) < x < \varphi_2(t),\ t \in (0,T)\}. \end{aligned} \tag{3}$$

When the right-hand side lies in the Lebesgue space $L^2(\Omega)$, the initial condition is in the space $H_0^1(\Gamma_0)$, we have established the existence of a unique solution in $H^{1,2}(\Omega)$.

In this paper, we suppose that $f \in H^{1,2}(R)$. Our main result is as follows:

**Theorem 1.1.** *Let $f \in H^{1,2}(R)$ and $\psi \in H^3(\Gamma_0) \cap H_0^1(\Gamma_0)$. Assume that $f$ and $\psi$ satisfy the compatibility condition $f_{|\Gamma_0} + \psi'' \in H_0^1(\Gamma_0)$. Then Problem (2) admits a unique solution $u$ lies in*

$$H^{2,4}(R) = \left\{u \in L^2(R) : \partial_t^i \partial_x^j u \in L^2(R),\ 2i + j \leq 4\right\}.$$

## 2. Preliminaries

**Lemma 2.1.** *Assume that $s_1$, $s_2$ and $s$ are real numbers such that $s_1, s_2 \geq s \geq 0$. If $u \in H^{s_1}(R)$ and $v \in H^{s_2}(R)$ then $uv \in H^s(R)$ where $s < s_1 + s_2 - 1$.*

This lemma is a special case of Theorem 7.5, [3].

**Lemma 2.2.** *For any $u \in H^1(I)$, $I = (0,1)$ and $2 \leq q < \infty$, we have*

$$\|u\|_{L^q(I)}^2 \leq C_1 \left(\|u\|_{L^2(I)}^2 + \|\partial_x u\|_{L^2(I)}^{2\alpha} \|u\|_{L^2(I)}^{2-2\alpha}\right), \tag{4}$$

$$\|u\|_{L^q(I)} \leq C_2 \left(\|u\|_{L^2(I)} + \|\partial_x u\|_{L^2(I)}^{\alpha} \|u\|_{L^2(I)}^{1-\alpha}\right), \tag{5}$$

*where $\alpha = \frac{1}{2} - \frac{1}{q}$, and $C_1$, $C_2$ are positive constants.*

*Proof.* For any $u \in H^1(a,b)$, with $u(a)=0$, we have the following inequality

$$\|u\|_{L^q(a,b)} \le 2^\alpha \|\partial_x u\|^\alpha_{L^2(a,b)} \|u\|^{1-\alpha}_{L^2(a,b)}, \tag{6}$$

where $2 \le q$ and $\alpha = \frac{1}{2} - \frac{1}{q}$, (see Theorem 2.2, [14]).

As in [8], let $u$ be in $H^1(I)$ and its extension

$$\tilde{u}(x) = \begin{cases} u(x) & ; \quad x \in [0,1], \\ (x+1)u(-x) & ; \quad x \in [-1,0]. \end{cases}$$

We have $\tilde{u} \in H^1(-1,1)$ and $\tilde{u}(-1)=0$, then

$$\|\tilde{u}\|_{L^q(-1,1)} \le 2^\alpha \|\partial_x \tilde{u}\|^\alpha_{L^2(-1,1)} \|\tilde{u}\|^{1-\alpha}_{L^2(-1,1)},$$

$$\|\tilde{u}\|^2_{L^2(-1,1)} = \int_0^1 u^2(x)\,\mathrm{d}x + \int_{-1}^0 |(x+1)u(-x)|^2\,\mathrm{d}x \le 2\|u\|^2_{L^2(I)},$$

and

$$\begin{aligned}
\|\partial_x \tilde{u}\|^2_{L^2(-1,1)} &= \int_0^1 (\partial_x u)^2(x)\,\mathrm{d}x + \int_{-1}^0 |u(-x) - (x+1)\partial_x u(-x)|^2\,\mathrm{d}x \\
&= \int_0^1 (\partial_x u)^2(x)\,\mathrm{d}x + \int_0^1 |u(x) - (1-x)\partial_x u(x)|^2\,\mathrm{d}x \\
&\le \|\partial_x u\|^2_{L^2(I)} + 2\left(\|u\|^2_{L^2(I)} + \|\partial_x u\|^2_{L^2(I)}\right) \\
&= 2\|u\|^2_{L^2(I)} + 3\|\partial_x u\|^2_{L^2(I)}.
\end{aligned}$$

Using the previous inequalities, we obtain

$$\begin{aligned}
\|u\|^2_{L^q(I)} &\le \|\tilde{u}\|^2_{L^q(-1,1)} \\
&\le 2^{2\alpha} \|\partial_x \tilde{u}\|^{2\alpha}_{L^2(-1,1)} \|\tilde{u}\|^{2-2\alpha}_{L^2(-1,1)} \\
&\le 2^{2\alpha}\left(2\|u\|^2_{L^2(I)} + 3\|\partial_x u\|^2_{L^2(I)}\right)^\alpha 2^{1-\alpha}\|u\|^{2-2\alpha}_{L^2(I)} \\
&\le 2^{2\alpha}\left(2^\alpha\|u\|^{2\alpha}_{L^2(I)} + 3^\alpha\|\partial_x u\|^\alpha_{L^2(I)}\right) 2^{1-\alpha}\|u\|^{2-2\alpha}_{L^2(I)} \\
&= 2^{1+2\alpha}\|u\|^2_{L^2(I)} + 2^{1+\alpha}3^\alpha \|\partial_x u\|^{2\alpha}_{L^2(I)} \|u\|^{2-2\alpha}_{L^2(I)}.
\end{aligned}$$

∎

## 3. Regularity of the solution

We know (see [4]) that Problem (2) admits a unique solution $u \in H^{1,2}(R)$. Then to prove Theorem 1.1 we have to obtain the regularity $u \in H^{2,4}(R)$.

We construct approximate solutions to (2) in the form

$$u_n(t,x) = \sum_{j=1}^{n} c_j(t) e_j(x), \quad (t,x) \in R,$$

where $c_j = (u_n, e_j)_{L^2(I)}$ and $(e_j)_{j\geq 1}$ is solution of Dirichlet problem

$$\begin{cases} -e_j'' = \lambda_j e_j, \quad j \geq 1, \\ e_j(0) = e_j(1) = 0. \end{cases}$$

$(e_j)_{j\geq 1}$ is an orthonormal basis in $L^2(I)$.
Consider the approximate problem

$$\begin{cases} \int_0^1 \partial_t u_n e_j \,\mathrm{d}x + \int_0^1 u_n \partial_x u_n e_j \,\mathrm{d}x + \nu \int_0^1 \partial_x u_n \partial_x e_j \,\mathrm{d}x = \int_0^1 f e_j \,\mathrm{d}x, \\ u_n(0,x) = u_{0n}(x), \quad x \in (0,1), \end{cases} \tag{7}$$

the sequence $u_{0n}$ will be chosen to converge in $H^3(\Gamma_0) \cap H_0^1(\Gamma_0)$ to $\psi$.

We have

$$\int_0^1 \partial_t u_n e_j \,\mathrm{d}x = \sum_{i=1}^{n} c_i'(t) \int_0^1 e_i e_j \,\mathrm{d}x = c_j'(t),$$

and $-e_j'' = \lambda_j e_j$, then $\partial_x^2 u_n(t) = -\sum_{i=1}^{n} c_i(t)\lambda_i e_i$. Therefore, for all $t \in [0,T]$

$$-\int_0^1 \partial_x^2 u_n e_j \,\mathrm{d}x = \sum_{i=1}^{n} c_i(t)\lambda_i \int_0^1 e_i e_j \,\mathrm{d}x = \lambda_j c_j(t).$$

Putting

$$f_j(t) = \int_0^1 f e_j \,\mathrm{d}x, \quad k_j(t) = -\int_0^1 u_n \partial_x u_n e_j \,\mathrm{d}x,$$

$$h_j(t) = -\int_0^1 \partial_x u_n e_j \,\mathrm{d}x,$$

for $j \in \{1, \ldots, n\}$, then (7) is equivalent to the following system of $n$ uncoupled linear ordinary differential equations

$$c_j'(t) = -\lambda_j c_j(t) + k_j(t) + h_j(t) + f_j(t), \quad j = 1, \ldots, n, \tag{8}$$

$k_j(t), h_j(t)$ are well defined because $e_j$ are regular and as $f$ is in $L^2(R)$, $f_j$ is integrable. Taking into account the initial condition $c_j(0)$, for each fixed $j \in \{1, \ldots, n\}$, (8) has a unique regular solution $c_j$ in some interval $(0, T')$ with $T' \leq T$. In fact, we can prove here that $T' = T$.

In this section, all constants $(K_i)_{1\leq i\leq 5}$, $(C_i)_{1\leq i\leq 8}$ and $C$ are independent of $n$.

**LEMMA 3.1.** *There exists a positive constant $K_1$ such that*

$$\|u_n\|^2_{L^2(I)} + \nu \int_0^T \|\partial_x u_n(s)\|^2_{L^2(I)} \,\mathrm{d}s \leq K_1. \tag{9}$$

*Proof.* We have

$$|u_n|^2 = \Big| \int_0^x \partial_x u_n \,\mathrm{d}s \Big|^2 \leq x \int_0^x |\partial_s u_n|^2 \,\mathrm{d}s, \tag{10}$$

integrating from 0 to 1, we obtain

$$\int_0^1 |u_n|^2 \,\mathrm{d}x \leq \int_0^1 x \int_0^x |\partial_s u_n|^2 \,\mathrm{d}s \,\mathrm{d}x,$$

hence

$$\int_0^1 |u_n|^2 \,\mathrm{d}x \leq \int_0^1 |\partial_x u_n|^2 \,\mathrm{d}x.$$

Then,

$$\|u_n\|^2_{L^2(I)} \leq \|\partial_x u_n\|^2_{L^2(I)}. \tag{11}$$

We also deduce from (10) that

$$\|u_n\|^2_{L^\infty(I)} \leq \ |\partial_x u_n\|^2_{L^2(I)}. \tag{12}$$

Multiplying both sides of (7) by $c_j(t)$ and summing for $j = 1, \ldots, n$, we obtain

$$\frac{1}{2}\frac{d}{dt} \int_0^1 u_n^2 \,\mathrm{d}x + \int_0^1 \partial_x u_n u_n^2 \,\mathrm{d}x + \nu \int_0^1 (\partial_x u_n)^2 \,\mathrm{d}x = \int_0^1 f u_n \,\mathrm{d}x.$$

As

$$\int_0^1 \partial_x u_n u_n^2 \, \mathrm{d}x = \frac{1}{3}\int_0^1 \partial_x (u_n)^3 \, \mathrm{d}x = 0,$$

then

$$\frac{1}{2}\frac{d}{dt}\int_0^1 u_n^2 \, \mathrm{d}x + \nu \int_0^1 (\partial_x u_n)^2 \, \mathrm{d}x = \int_0^1 f u_n \, \mathrm{d}x. \tag{13}$$

Integrating (13) from 0 to $t$ and using Cauchy-Schwartz inequality, we find

$$\frac{1}{2}\|u_n\|^2_{L^2(I)} + \nu \int_0^t \|\partial_x u_n(s)\|^2_{L^2(I)} \, \mathrm{d}s$$
$$\leq \frac{1}{2}\|u_{0n}\|^2_{L^2(I)} + \int_0^t \|f(s)\|_{L^2(I)} \|u_n(s)\|_{L^2(I)} \, \mathrm{d}s.$$

By the inequality

$$|rs| \leq \frac{\varepsilon}{2} r^2 + \frac{s^2}{2\varepsilon}, \quad \forall r, s \in R, \ \forall \varepsilon > 0, \tag{14}$$

with $\varepsilon = \nu$, we obtain

$$\frac{1}{2}\|u_n\|^2_{L^2(I)} + \nu \int_0^t \|\partial_x u_n(s)\|^2_{L^2(I)} \, \mathrm{d}s$$
$$\leq \frac{1}{2}\|u_{0n}\|^2_{L^2(I)} + \frac{1}{2\nu}\int_0^t \|f(s)\|^2_{L^2(I)} \, \mathrm{d}s + \frac{\nu}{2}\int_0^t \|u_n(s)\|^2_{L^2(I)} \, \mathrm{d}s.$$

Thanks to (11)

$$\|u_n\|^2_{L^2(I)} + \nu \int_0^t \|\partial_x u_n(s)\|^2_{L^2(I)} \, \mathrm{d}s \leq \|u_{0n}\|^2_{L^2(I)} + \frac{1}{\nu}\int_0^t \|f(s)\|^2_{L^2(I)} \, \mathrm{d}s,$$

as $f \in L^2(R)$ and $\|u_{0n}\|^2_{L^2(I)}$ is bounded. Then, there exists a positive constant $K_1$ such that

$$\|u_n\|^2_{L^2(I)} + \nu \int_0^T \|\partial_x u_n(s)\|^2_{L^2(I)} \, \mathrm{d}s \leq K_1. \tag{15}$$

$\square$

**Lemma 3.2.**

*There exists a positive constant $K_2$ such that*

$$\|\partial_x u_n\|_{L^2(I)}^2 + \nu \int_0^T \|\partial_x^2 u_n(s)\|_{L^2(I)}^2 \,\mathrm{d}s \leq K_2.$$

*Proof.* $\lambda_j e_j = -e_j^{''}$, then

$$\sum_{j=1}^n c_j(t)\lambda_j e_j = -\sum_{j=1}^n c_j(t)e_j^{''} = -\partial_x^2 u_n(t).$$

Multiplying both sides of (7) by $c_j\lambda_j$ and summing for $j = 1, \ldots, n$, we get

$$\begin{aligned} &\frac{1}{2}\frac{d}{dt}\int_0^1 (\partial_x u_n)^2 \mathrm{d}x + \nu \int_0^1 (\partial_x^2 u_n)^2 \,\mathrm{d}x \\ &= -\int_0^1 f\partial_x^2 u_n \,\mathrm{d}x + \int_0^1 u_n \partial_x u_n \partial_x^2 u_n \,\mathrm{d}x. \end{aligned} \tag{16}$$

Using Cauchy-Schwartz inequality and (14) with $\varepsilon = \frac{\nu}{2}$, we deduce that

$$\begin{aligned} \left|\int_0^1 f\partial_x^2 u_n \,\mathrm{d}x\right| &\leq \Big(\int_0^1 |\partial_x^2 u_n|^2 \,\mathrm{d}x\Big)^{\frac{1}{2}} \Big(\int_0^1 |f|^2 \,\mathrm{d}x\Big)^{\frac{1}{2}} \\ &\leq \frac{\nu}{4}\int_0^1 |\partial_x^2 u_n|^2 \,\mathrm{d}x + \frac{1}{\nu}\int_0^1 |f|^2 \,\mathrm{d}x. \end{aligned} \tag{17}$$

For the last term of (16). An integration by parts gives

$$\int_0^1 u_n \partial_x u_n \partial_x^2 u_n \,\mathrm{d}x = \frac{1}{2}\int_0^1 u_n \partial_x (\partial_x u_n)^2 \,\mathrm{d}x = -\frac{1}{2}\int_0^1 (\partial_x u_n)^3 \,\mathrm{d}x.$$

Since $\partial_x u_n$ satisfies $\int_0^1 \partial_x u_n \,\mathrm{d}x = 0$, we deduce that the continuous function $\partial_x u_n$ is zero at some point $y_{0n} \in (0,1)$, and by integrating $2\partial_x u_n \partial_x^2 u_n$ between $y_{0n}$ and $x$, we obtain

$$2\int_{y_{0n}}^x \partial_s u_n \partial_s^2 u_n \,\mathrm{d}s = \int_{y_{0n}}^x \partial_s (\partial_s u_n)^2 \,\mathrm{d}s = (\partial_x u_n)^2.$$

Then

$$|\partial_x u_n|^2 \leq 2\left|\int_{y_{0n}}^{x} \partial_s u_n \partial_s^2 u_n \,\mathrm{d}s\right| \leq 2\int_0^1 |\partial_s u_n||\partial_s^2 u_n| \,\mathrm{d}s,$$

Cauchy-Schwartz inequality gives

$$\|\partial_x u_n\|^2_{L^\infty(I)} \leq 2\|\partial_x u_n\|_{L^2(I)}\|\partial_x^2 u_n\|_{L^2(I)},$$

on the other hand

$$\|\partial_x u_n\|^3_{L^3(I)} \leq \|\partial_x u_n\|^2_{L^2(I)}\|\partial_x u_n\|_{L^\infty(I)},$$

so,

$$\left|\int_0^1 u_n \partial_x u_n \partial_x^2 u_n \,\mathrm{d}x\right| \leq \Big(\int_0^1 |\partial_x^2 u_n|^2 \,\mathrm{d}x\Big)^{\frac{1}{4}}\Big(\int_0^1 |\partial_x u_n|^2 \,\mathrm{d}x\Big)^{5/4}.$$

Recall Young's inequality $|AB| \leq \frac{|A|^p}{p} + \frac{|B|^{p'}}{p'}$, where $1 < p < \infty$ and $p' = \frac{p}{p-1}$. Choosing $p = 4$ (then $p' = \frac{4}{3}$)

$$A = \Big(\nu\int_0^1 |\partial_x^2 u_n|^2 \,\mathrm{d}x\Big)^{\frac{1}{4}}, \quad B = \Big(\frac{1}{\nu}\Big)^{\frac{1}{4}}\Big(\int_0^1 |\partial_x u_n|^2 \,\mathrm{d}x\Big)^{5/4},$$

we get

$$\left|\int_0^1 u_n \partial_x u_n \partial_x^2 u_n \,\mathrm{d}x\right| \leq \frac{\nu}{4}\int_0^1 |\partial_x^2 u_n|^2 \,\mathrm{d}x + \frac{3}{4}\Big(\frac{1}{\nu}\Big)^{\frac{1}{3}}\Big(\int_0^1 |\partial_x u_n|^2 \,\mathrm{d}x\Big)^{\frac{5}{3}}. \tag{18}$$

Let us return to (16): By integrating between 0 and $t$, from the estimates (17) and (18), we obtain

$$\begin{aligned}
&\|\partial_x u_n\|^2_{L^2(I)} + \nu\int_0^t \|\partial_x^2 u_n(s)\|^2_{L^2(I)} \,\mathrm{d}s \\
&\leq \|\partial_x u_{0n}\|^2_{L^2(I)} + \frac{2}{\nu}\int_0^t \|f(s)\|^2_{L^2(I)} \,\mathrm{d}s + \frac{3}{2}\Big(\frac{1}{\nu}\Big)^{\frac{1}{3}}\int_0^t \Big(\|\partial_x u_n(s)\|^2_{L^2(I)}\Big)^{\frac{5}{3}} \,\mathrm{d}s,
\end{aligned}$$

$f \in L^2(R)$ and $\|\partial_x u_{0n}\|^2_{L^2(I)}$ is bounded. Then, there exists a constant $C_1$ such that

$$\begin{aligned}
&\|\partial_x u_n\|^2_{L^2(I)} + \nu\int_0^t \|\partial_x^2 u_n(s)\|^2_{L^2(I)} \,\mathrm{d}s \\
&\leq C_1 + C_2\int_0^t \Big(\|\partial_x u_n(s)\|^2_{L^2(I)}\Big)^{2/3}\|\partial_x u_n(s)\|^2_{L^2(I)} \,\mathrm{d}s.
\end{aligned}$$

where $C_2 = \frac{3}{2}\left(\frac{1}{\nu}\right)^{\frac{1}{3}}$.

Consequently, the function

$$\varphi(t) = \|\partial_x u_n\|^2_{L^2(I)} + \nu \int_0^t \|\partial_x^2 u_n(s)\|^2_{L^2(I)}\,\mathrm{d}s$$

satisfies the inequality

$$\varphi(t) \le C_1 + C_2 \int_0^t \|\partial_x u_n(s)\|^{\frac{4}{3}}_{L^2(I)}\varphi(s)\mathrm{d}s,$$

Gronwall's inequality shows that

$$\varphi(t) \le C_1 \exp\Big(C_2 \int_0^t \|\partial_x u_n(s)\|^{\frac{4}{3}}_{L^2(I)}\mathrm{d}s\Big).$$

According to (9) the integral $\int_0^t \|\partial_x u_n\|^{\frac{4}{3}}_{L^2(I)}\mathrm{d}s$ is bounded by a constant independent of $n$. So there exists a positive constant $K_2$ such that

$$\|\partial_x u_n\|^2_{L^2(I)} + \nu \int_0^T \|\partial_x^2 u_n(s)\|^2_{L^2(I)}\,\mathrm{d}s \le K_2. \tag{19}$$

$\square$

**Lemma 3.3.** *There exists a positive constant $K_3$ such that*

$$\|\partial_t u_n\|^2_{L^2(I)} + \nu \int_0^T \|\partial_s\partial_x u_n(s)\|^2_{L^2(I)}\,\mathrm{d}s \le K_3.$$

*Proof.* Differentiating (7) with respect to $t$, multiplying both sides by $e^x c_j'$ and summing for $j = 1, \cdots, n$, we get

$$\begin{aligned}&\int_0^1 e^x \partial_t^2 u_n \partial_t u_n\,\mathrm{d}x + \int_0^1 e^x (\partial_t u_n)^2 \partial_x u_n\,\mathrm{d}x\\ &+ \int_0^1 e^x u_n \partial_t\partial_x u_n \partial_t u_n\,\mathrm{d}x + \nu \int_0^1 e^x\left(\partial_t\partial_x u_n\right)^2\,\mathrm{d}x\\ &= \int_0^1 e^x \partial_t f \partial_t u_n\,\mathrm{d}x.\end{aligned}$$

An integration by parts gives

$$\int_0^1 e^x \partial_t u_n \partial_t^2 u_n \,\mathrm{d}x = \frac{1}{2}\frac{d}{dt}\int_0^1 e^x (\partial_t u_n)^2 \,\mathrm{d}x,$$

$$\int_0^1 e^x (\partial_t u_n)^2 \partial_x u_n \,\mathrm{d}x = -\int_0^1 e^x (\partial_t u_n)^2 u_n \,\mathrm{d}x - 2\int_0^1 e^x u_n \partial_t \partial_x u_n \partial_t u_n \,\mathrm{d}x$$

then

$$\begin{aligned} &\frac{1}{2}\frac{d}{dt}\int_0^1 e^x (\partial_t u_n)^2 \,\mathrm{d}x + \nu \int_0^1 e^x (\partial_t \partial_x u_n)^2 \,\mathrm{d}x \\ &= \int_0^1 e^x \partial_t f \partial_t u_n \,\mathrm{d}x + \int_0^1 e^x (\partial_t u_n)^2 u_n \,\mathrm{d}x \\ &+ 2\int_0^1 e^x u_n \partial_t \partial_x u_n \partial_t u_n \,\mathrm{d}x - \int_0^1 e^x u_n \partial_t u_n \partial_t \partial_x u_n \,\mathrm{d}x. \end{aligned} \tag{20}$$

Let's find an estimates for all the terms of the right hand side of (20). By Cauchy-Schwartz inequality and (14) with $\varepsilon = 1$, we find

$$\left| \int_0^1 e^x \partial_t f \partial_t u_n \,\mathrm{d}x \right| \le \frac{e^2}{2} \|\partial_t f\|_{L^2(I)}^2 + \frac{1}{2} \|\partial_t u_n\|_{L^2(I)}^2$$

Using (12) and (19), then (14) with $\varepsilon = \frac{\nu}{2}$, we get

$$\begin{aligned} \left| \int_0^1 e^x (\partial_t u_n)^2 u_n \,\mathrm{d}x \right| &\le e \|u_n\|_{L^\infty(I)} \|\partial_t u_n\|_{L^2(I)}^2 \\ &\le e \|\partial_x u_n\|_{L^2(I)} \|\partial_t u_n\|_{L^2(I)}^2 \\ &\le e K_2 \|\partial_t u_n\|_{L^2(I)}^2, \end{aligned}$$

$$\begin{aligned} \left| 2\int_0^1 e^x u_n \partial_t \partial_x u_n \partial_t u_n \,\mathrm{d}x \right| &\le 2e \|u_n\|_{L^\infty(I)} \|\partial_t u_n\|_{L^2(I)} \|\partial_t \partial_x u_n\|_{L^2(I)} \\ &\le 2e K_2 \|\partial_t u_n\|_{L^2(I)} \|\partial_t \partial_x u_n\|_{L^2(I)} \\ &\le \frac{\nu}{4} \|\partial_t \partial_x u_n\|_{L^2(I)}^2 + \frac{4e^2 K_2^2}{\nu} \|\partial_t u_n\|_{L^2(I)}^2, \end{aligned}$$

and

$$\left|\int_0^1 e^x u_n \partial_t u_n \partial_t \partial_x u_n \,\mathrm{d}x\right| \le e\|u_n\|_{L^\infty(I)}\|\partial_t u_n\|_{L^2(I)}\|\partial_t\partial_x u_n\|_{L^2(I)}$$
$$\le eK_2\|\partial_t u_n\|_{L^2(I)}\|\partial_t\partial_x u_n\|_{L^2(I)}$$
$$\le \frac{e^2K_2^2}{\nu}\|\partial_t u_n\|^2_{L^2(I)} + \frac{\nu}{4}\|\partial_t\partial_x u_n\|^2_{L^2(I)}.$$

Submitting the previous inequalities into (20), we deduce that

$$\frac{1}{2}\frac{d}{dt}\int_0^1 e^x(\partial_t u_n)^2\,\mathrm{d}x + \frac{\nu}{2}\int_0^1(\partial_t\partial_x u_n)^2\,\mathrm{d}x \le C_3\|\partial_t u_n\|^2_{L^2(I)} + \frac{e^2}{2}\|\partial_t f\|^2_{L^2(I)}.$$

where $C_3 = eK_2 + \frac{5e^2K_2}{\nu} + \frac{1}{2}$.

Integrating the last inequality with respect to $t$ ($t \in (0,T)$), we find

$$\|\partial_t u_n\|^2_{L^2(I)} + \nu\int_0^t \|\partial_s\partial_x u_n(s)\|^2_{L^2(I)}\,\mathrm{d}s$$
$$\le e\|\partial_t u_n(0)\|^2_{L^2(I)}\,\mathrm{d}s + e^2\int_0^t \|\partial_s f\|^2_{L^2(I)}\,\mathrm{d}s + 2C_3\int_0^t \|\partial_s u_n\|^2_{L^2(I)}\,\mathrm{d}s.$$

Observe that $f \in H^{1,2}(R)$, then from (8) there exist a positive constant $C_3$ such that

$$e\|\partial_t u_n(0)\|^2_{L^2(I)} + e^2\int_0^t \|\partial_s f\|^2_{L^2(I)}\,\mathrm{d}s \le C_4. \tag{21}$$

Hence,

$$\|\partial_t u_n\|^2_{L^2(I)} + \nu\int_0^t \|\partial_s\partial_x u_n(s)\|^2_{L^2(I)}\,\mathrm{d}s$$
$$\le C_4 + 2C_3\int_0^t\left(\|\partial_s u_n(s)\|^2_{L^2(I)} + \nu\int_0^s \|\partial_\tau\partial_x u_n(\tau)\|^2_{L^2(I)}\,\mathrm{d}\tau\right)\mathrm{d}s,$$

by Gronwall's inequality

$$\|\partial_t u_n\|^2_{L^2(I)} + \nu\int_0^t \|\partial_s\partial_x u_n(s)\|^2_{L^2(I)}\,\mathrm{d}s \le C_5 e^T.$$

Taking $K_3 = C_5 e^T$, we get

$$\|\partial_t u_n\|^2_{L^2(I)} + \nu \int_0^t \|\partial_s \partial_x u_n(s)\|^2_{L^2(I)} \, \mathrm{d}s \leq K_3. \tag{22}$$

□

**Remark 3.4.** From (9), (19) and (22), we deduce that there exists a positive constant $K$ independent of $n$, such that for all $t \in [0, T]$

$$\|u_n\|^2_{L^2(I)} + \|\partial_t u_n\|^2_{L^2(I)} + \|\partial_x u_n\|^2_{L^2(I)} \leq K.$$

**Lemma 3.5.** *There exists a positive constant $K_4$ such that*

$$\|\partial_t \partial_x u_n\|^2_{L^2(I)} + \nu \int_0^T \|\partial_s \partial_x^2 u_n(s)\|^2_{L^2(I)} \, \mathrm{d}s \leq K_4.$$

*Proof.* We have

$$\partial_x^2 u_n(t) = -\sum_{j=1}^n c_j(t) \lambda_j e_j,$$

then

$$\partial_t \partial_x^2 u_n(t) = -\sum_{j=1}^n c_j'(t) \lambda_j e_j.$$

Differentiating (7) with respect to $t$, multiplying both sides by $c_j' \lambda_j$ and summing for $j = 1, \ldots, n$, we obtain

$$\begin{aligned} &\int_0^1 \partial_t^2 u_n \partial_t \partial_x^2 u_n \, \mathrm{d}x - \nu \int_0^1 (\partial_t \partial_x^2 u_n)^2 \, \mathrm{d}x \\ &= -\int_0^1 \partial_t u_n \partial_x u_n \partial_t \partial_x^2 u_n \, \mathrm{d}x - \int_0^1 u_n \partial_t \partial_x u_n \partial_t \partial_x^2 u_n \, \mathrm{d}x + \int_0^1 \partial_t f \partial_t \partial_x^2 u_n \, \mathrm{d}x. \end{aligned} \tag{23}$$

An integration by parts gives

$$\int_0^1 \partial_t^2 u_n \partial_t \partial_x^2 u_n \, \mathrm{d}x = -\int_0^1 \partial_t^2 \partial_x u_n \partial_t \partial_x u_n \, \mathrm{d}x = -\frac{1}{2} \frac{d}{dt} \int_0^1 (\partial_t \partial_x u_n)^2 \, \mathrm{d}x.$$

Firstly, we have to obtain an estimates for the terms of the right-hand side of (23).

Using Cauchy-Schwartz inequality twice , we get

$$\left|\int_0^1 \partial_t u_n \partial_x u_n \partial_t \partial_x^2 u_n \,\mathrm{d}x\right| \leq \|\partial_t u_n\|_{L^4(I)} \|\partial_x u_n\|_{L^4(I)} \|\partial_t \partial_x^2 u_n\|_{L^2(I)},$$

so, inequality (5) with $q = 4$(then $\alpha = \frac{1}{4}$), gives

$$\begin{aligned}\left|\int_0^1 \partial_t u_n \partial_x u_n \partial_t \partial_x^2 u_n \,\mathrm{d}x\right| &\leq C\|\partial_t \partial_x^2 u_n\|_{L^2(I)} \left(\|\partial_t u_n\|_{L^2(I)} + \|\partial_t \partial_x u_n\|^{\frac{1}{4}}_{L^2(I)} \|\partial_t u_n\|^{\frac{3}{4}}_{L^2(I)}\right) \\ &\times \left(\|\partial_x u_n\|_{L^2(I)} + \|\partial_x^2 u_n\|^{\frac{1}{4}}_{L^2(I)} \|\partial_x u_n\|^{\frac{3}{4}}_{L^2(I)}\right).\end{aligned}$$

By Young's inequality with $p = 4$, we find

$$\|\partial_t \partial_x u_n\|^{\frac{1}{4}}_{L^2(I)} \|\partial_t u_n\|^{\frac{3}{4}}_{L^2(I)} \leq \frac{1}{4}\|\partial_t \partial_x u_n\|_{L^2(I)} + \frac{3}{4}\|\partial_t u_n\|_{L^2(I)},$$

and

$$\|\partial_x^2 u_n\|^{\frac{1}{4}}_{L^2(I)} \|\partial_x u_n\|^{\frac{3}{4}}_{L^2(I)} \leq \frac{1}{4}\|\partial_x^2 u_n\|_{L^2(I)} + \frac{3}{4}\|\partial_x u_n\|_{L^2(I)}.$$

Using the previous inequalities and (14) with $\varepsilon = \frac{\nu}{2}$, we find

$$\begin{aligned}\left|\int_0^1 \partial_t u_n \partial_x u_n \partial_t \partial_x^2 u_n \,\mathrm{d}x\right| &\leq \frac{\nu}{4}\|\partial_t \partial_x^2 u_n\|^2_{L^2(I)} + C_6\left(\|\partial_t u_n\|_{L^2(I)} + \|\partial_t \partial_x u_n\|_{L^2(I)}\right)^2 \\ &\times \left(\|\partial_x u_n\|_{L^2(I)} + \|\partial_x^2 u_n\|_{L^2(I)}\right)^2 \\ &\leq \frac{\nu}{4}\|\partial_t \partial_x^2 u_n\|^2_{L^2(I)} + C_7.\end{aligned}$$

For the second term of the right-hand side in (23), inequality (12) and (14) with $\varepsilon = \frac{\nu}{2}$, yield

$$\begin{aligned}\left|\int_0^1 u_n \partial_t(\partial_x u_n) \partial_t \partial_x^2 u_n \,\mathrm{d}x\right| &\leq \|u_n\|_{L^\infty(I)} \|\partial_t \partial_x u_n\|_{L^2(I)} \|\partial_t \partial_x^2 u_n\|_{L^2(I)} \\ &\leq \|\partial_x u_n\|_{L^2(I)} \|\partial_t \partial_x u_n\|_{L^2(I)} \|\partial_t \partial_x^2 u_n\|_{L^2(I)} \\ &\leq K_2\|\partial_t \partial_x^2 u_n\|_{L^2(I)} \|\partial_t \partial_x u_n\|_{L^2(I)} \\ &\leq \frac{\nu}{4}\|\partial_t \partial_x^2 u_n\|^2_{L^2(I)} + \frac{K_2^2}{\nu}\|\partial_t \partial_x u_n\|^2_{L^2(I)}.\end{aligned}$$

Replacing the previous estimates into (23), we obtain

$$\begin{aligned}&\frac{1}{2}\frac{d}{dt}\int_0^1 (\partial_t \partial_x u_n)^2 \,\mathrm{d}x + \frac{\nu}{4}\|\partial_t \partial_x^2 u_n\|^2_{L^2(I)} \\ &\leq C_8 + \frac{1}{\nu}\|\partial_t f\|^2_{L^2(I)} + \frac{K_2^2}{2}\|\partial_t \partial_x u_n\|^2_{L^2(I)}.\end{aligned}$$

Then, integrating with respect to $t$ ($t \in (0,T)$), by Lemma 3.3, there exists a positive constant $K_4$ such that

$$\|\partial_t\partial_x u_n\|^2_{L^2(I)} + \nu \int_0^t \|\partial_s\partial_x^2 u_n(s)\|^2_{L^2(I)}\,\mathrm{d}s \le K_4.$$

$\square$

In [4], we have proved that the approximation $u_n$ converges to the unique solution $u \in H^{1,2}(R)$ of Problem (2).

**Proposition 3.6.** *Under the hypotheses of Theorem 1.1, the solution of Problem* (2) *is in* $H^2(R)$ *and* $\partial_t\partial_x^2 u \in L^2(R)$.

*Proof.* Observe that Lemma 3.3 and 3.5 imply that the solution of Problem (2) satisfies $\partial_t\partial_x u \in L^2(R)$ and $\partial_t\partial_x^2 u \in L^2(R)$. So, it is enough to prove that $\partial_t^2 u \in L^2(R)$. Differentiating (2), taking $L^2$-norms and integrating the obtained equation with respect to $t$, we get

$$\begin{aligned}\int_0^t \|\partial_t^2 u(s)\|^2_{L^2(I)}\,\mathrm{d}s \le{}& \nu\int_0^t \|\partial_s\partial_x^2 u(s)\|^2_{L^2(I)}\,\mathrm{d}s + \int_0^t \|\partial_s u(s)\partial_x u(s)\|^2_{L^2(I)}\,\mathrm{d}s \\ &+ \int_0^t \|u(s)\partial_s\partial_x u(s)\|^2_{L^2(I)}\,\mathrm{d}s + \int_0^t \|\partial_s f(s)\|^2_{L^2(I)}\,\mathrm{d}s.\end{aligned} \tag{24}$$

We need to estimate the terms of the right-hand side in (24). Using (4) with $q = 4$, we obtain

$$\begin{aligned}\|\partial_t u\partial_x u\|^2_{L^2(I)} &\le \|\partial_t u\|^2_{L^4(I)}\|\partial_x u\|^2_{L^4(I)} \\ &\le C\left(\|\partial_t u\|^2_{L^2(I)} + \|\partial_t\partial_x u\|^{\frac{1}{2}}_{L^2(I)}\|\partial_t u\|^{\frac{3}{2}}_{L^2(I)}\right) \\ &\quad\times\left(\|\partial_x u\|^2_{L^2(I)} + \|\partial_x^2 u\|^{\frac{1}{2}}_{L^2(I)}\|\partial_x u\|^{\frac{3}{2}}_{L^2(I)}\right),\end{aligned} \tag{25}$$

and

$$\begin{aligned}\|u\partial_t\partial_x u\|^2_{L^2(I)} &\le \|u\|^2_{L^\infty(I)}\|\partial_t\partial_x u\|^2_{L^2(I)} \\ &\le \|\partial_x u\|^2_{L^2(I)}\|\partial_t\partial_x u\|^2_{L^2(I)}.\end{aligned} \tag{26}$$

Thanks to estimates (25), (26), Lemma 3.3 and 3.5, inequality (24) shows that

$$\int_0^t \|\partial_s^2 u(s)\|^2_{L^2(I)}\,\mathrm{d}s \le K_5, \tag{27}$$

where $K_5$ is a constant independent of $n$ and $t$. $\square$

*Proof of Theorem 1.1.* Recall that $f$ is given in $H^{1,2}(R)$. So, $\partial_t f \in L^2(R)$.

Let $v = \partial_t u$ and $g = v\partial_x v - v\partial_x u - u\partial_x v + \partial_t f$. From Lemma 3.5, we deduce $v \in L^\infty(0, T, H_0^1(I))$. Then $v \in L^\infty(R)$, consequently, $v\partial_x v \in L^2(R)$. On the other hand, $u \in H^2(R)$ implies that $v\partial_x u \in L^2(R)$, and choosing $s_1 = 2$, $s_2 = 0$ in Lemma 2.1, we obtain $u\partial_x v \in L^2(R)$. Finally, we get $g \in L^2(R)$.

Differentiating (2) with respect to $t$ we deduce

$$\partial_t v + v\partial_x u + u\partial_x v - \nu\partial_x^2 v = \partial_t f,$$

then

$$\partial_t v + v\partial_x v - \nu\partial_x^2 v = g.$$

Observe that $v$ is a solution of the problem

$$\begin{cases} \partial_t v + v\partial_x v - \nu\partial_x^2 v = g & (t, x) \in R, \\ v(0, x) = f_{|\Gamma_0} + \psi'' & x \in \Gamma_0, \\ v(t, 0) = v(t, 1) = 0 & t \in (0, T), \end{cases} \tag{28}$$

where (according to the hypothesis of Theorem 1.1) $f_{|\Gamma_0} + \psi'' \in H_0^1(\Gamma_0)$. Consequently, by the main result of [4] $v$ is in $H^{1,2}(R)$.

On the other hand, from (2) we have

$$\nu\partial_x^4 u = \partial_t\partial_x^2 u + 3\partial_x u\partial_x^2 u + u\partial_x^3 u - \partial_x^2 f, \tag{29}$$

as all the terms of the right-hand side in (29) are in $L^2(R)$, $\partial_x^4 u$ is in $L^2(R)$, we deduce that $u \in H^{2,4}(R)$ which is the maximal regularity of the solution $u$. □

**Remark 3.7.** As a simple example about Theorem 1.1, we can take a polynomial function $f$ which satisfies $f(0, 0) = f(0, 1) = 0$, that is

$$f(t, x) = x(1 - x) - 2t + t^2 x(1 - x)(1 - 2x).$$

So, we look for a solution $u$ as a polynomial $u(x, t) = v(t)w(x)$. The boundary conditions led to

$$u(t, x) = at(bt + c)x(1 - x).$$

Finally, we find that the unique solution of the equation $\partial_t u + u\partial_x u - \nu\partial_x^2 u = f$ is $u(x, t) = tx(1 - x)$ which satisfies the boundary conditions.

**Remark 3.8.** This work can be extended to the case where the rectangle $R$ is replaced by a polygonal domain and, more generally, by the domain $\Omega$ defined by (3). The study of this problem needs to consider two cases: $\varphi_1(t) < \varphi_2(t)$, $0 \le t \le T$ and $\varphi_1(0) = \varphi_2(0)$. In the second case, some singularities may appear and then, the solution is not necessarily in $H^{2,4}(\Omega)$. We are working on these cases.

## References

[ 1 ] L. Ait Mahiout, and Y. Benia, On the exact solutions and numerical simulation of the non-homogeneous Burgers equations, submitted.

[ 2 ] H. Bateman, Some recent researches on the motion of fluids, *Mon. Weather Rev.*, **43**, (1915), 163–170.

[ 3 ] A. Behzadan, Holst M., Multiplication in Sobolev spaces, revisited. Available as arXiv:1512.07379 [math.AP], (2015).

[ 4 ] Y. Benia, and B.-K. Sadallah, Existence of solutions to Burgers equations in domains that can be transformed into rectangles, *J. Diff. Eq.*, **157** (2016), 1–13.

[ 5 ] Y. Benia, and B. K. Sadallah, Existence of solutions to Burgers equations in a non-parabolic domain, *J. Diff. Eq.*, **20** (2018), 1–13.

[ 6 ] J. M. Burgers, *The nonlinear diffusion equation*, Reidel, Dordrecht, 1974.

[ 7 ] J. M. Burgers, A mathematical model illustrating the theory of turbulence, *Adv. Appl. Mech.*, **1** (1948), 171–199.

[ 8 ] C. I. Byrnes, D. S. Gilliam, and V. I. Shubov, Boundary control for a viscous Burgers' equation, in H. T. Banks, R. H. Fabiano, and K. Ito (Eds.), Identification and control for systems governed by partial differential equations, SIAM, (1993), 171–185.

[ 9 ] A. Chen, F.-F. Wang, and W. Zhang, Soliton solutions of Burgers equation and the modified Kadomtsev-Petviashvili equation, *Journal of Physics. A. Mathematical and Theoretical*, **43 (36)** (2010), Article ID 365202.

[ 10 ] J. D. Cole, On a quasi-linear parabolic equation occurring in aerodynamics, *Quart. Appl. Math.*, **9** (1951), 225–236.

[ 11 ] S. Eule, and R. Friedrich, A note on the forced Burgers equation, *Phys. Lett.*, **A 351** (2006), 238–241.

[ 12 ] E. Hopf, The partial differential equation $u_t + uu_x = \nu u_{xx}$, *Comm. Pure Appl. Math.*, **3** (1950), 201–230.

[ 13 ] J. Kevorkian, *Partial differential equations: Analytical solution techniquess*, Brooks-Cole Pub, Company, California, 1990.

[ 14 ] O. A. Ladyzhenskaya, and V. A. Solonnikov, Ural'ceva N. N., Linear and quasi linear equations of parabolic type, *Translations of AMS*, **23**, (1968).

[ 15 ] A. Orlowski, and K. Sobczyk, Solitons and shock waves under random external noise. *Rep. Math. Phys.*, (1989), 27–59.

[ 16 ] G. B. Whitham, *Lectures on wave propagation*, Narosa Pub. House, New Delhi, 1979.

[ 17 ] Ch. Srinivasa, R. Manoj, and K. Yadav, On the solution of a nonhomogeneous Burgers equation, *Stud. Appl. Math.*, **124** (2010), 411–422.

[ 18 ] A. Sirin, and K. Oktay Exact solutions of forced Burgers equations with time variable coefficients. *Commun Nonlinear Sci. Numer. Simulat.*, (2013), 1635–1651.

[ 19 ] C. S. Su, and C. S. Gardner, Korteweg-de Vries Equation and generalizations. III . derivation of the Korteweg-de Vries equation and Burgers equation, *J. Math. Phys.*, **10** (1969), 536.

[ 20 ] Vinogradova M. B., Rudenko O. V. , and Sukhorukov A. P., M. B. Vinogradova, O. V. Rudenko, and A. P. Sukhorukov, *Wave theory*, in Russian, Nauka, Moscow 1979.

[ 21 ] D. Xiaqi, J. Quansen, and H. Cheng, On a non-homogeneous Burgers' equation, *Sci. China Ser.*, **A 44** (2001), 984–993.

[ 22 ] J. A. Atwell, B. B. King, Stabilized finite element methods and feedback control for Burgers equation, in *Proceedings of the American Control Conference*, (2000), 2745–2749.

Yassine Benia
Department of Mathematics and Informatics,
Faculty of Sciences,
University of Algiers, 16000, Algiers, Algeria

Boubaker-Khaled Sadallah
Lab. PDE & Hist Maths;
Department of Mathematics, E.N.S.,
16050, Kouba, Algiers,
Algeria