Proper colorings of plane quadrangulations without rainbow faces

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This paper is dedicated to Professor Hikoe Enomoto on the occasion of his 75th birthday

Abstract

We consider a proper coloring of a plane graph such that no face is rainbow, where a face is rainbow if any two vertices on its boundary have distinct colors. Such a coloring is said to be proper anti-rainbow. A plane quadrangulation G is a plane graph in which all faces are bounded by a cycle of length 4. In this paper, we show that the number of colors in a proper anti-rainbow coloring of a plane quadrangulation G does not exceed $3\alpha(G)/2$, where $\alpha(G)$ is the independence number of G. Moreover, if the minimum degree of G is 3 or if G is 3-connected, then this bound can be improved to $5\alpha(G)/4$ or $7\alpha(G)/6 + 1/3$, respectively. All of these bounds are tight.

Keywords: quadrangulation, anti-rainbow coloring, valid coloring, looseness, dividing system,
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1 Introduction

In this paper, we consider finite undirected simple graphs. Colorings of plane graphs or of graphs embedded on surfaces with facial constraints have attracted many topological graph theorists. In particular, facially-constrained colorings of plane graphs were overviewed by Czap and Jendrol' [3].

Let G be a plane graph (or a graph embedded on a surface). An *anti-rainbow coloring*, which is also called a *valid coloring* or a *non-rainbow coloring*, is a (not necessarily proper) coloring of G such that each face is not *rainbow*, where a face is *rainbow* if any two vertices on its boundary have distinct colors. The *anti-rainbowness* of G is the maximum integer k such that G has a surjective anti-rainbow k-coloring, denoted by $\chi_f(G)$. This type of coloring was introduced by Ramamurthi and West [10] and Negami [8] independently. (See also [1, 2, 9] for some acts of taking the initiative.) By requiring the properness for anti-rainbow colorings, we additionally define a *proper anti-rainbow coloring* and the *proper anti-rainbowness*, denoted by $\chi_f^p(G)$. Note that $\chi_f^p(G)$ cannot defined for a plane graph G with a triangular face, while it can be defined for any triangle-free plane graph, which admits a proper 3-coloring by Grötzsch's theorem [5]. By the definition, $\chi_f^p(G) \leq \chi_f(G)$ for any triangle-free plane

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graph G. We should notice that the anti-rainbowness (resp. the proper anti-rainbowness) may depend on the embedding, that is, for another embedding G' of a graph G embedded on a surface, $\chi_f(G')$ (resp. $\chi_f^p(G')$) may not be equal to $\chi_f(G)$ (resp. $\chi_f^p(G)$).

Ramamurthi and West [10] noticed that for every plane graph G of order n with an edge, it holds that $\chi_f(G) \geq \alpha(G) + 1 \geq \lceil n/\chi(G) \rceil + 1$, where $\alpha(G)$ is the independence number of G and $\chi(G)$ is the chromatic number of G. Then, Grötzsch's theorem [5, 11] implies that $\chi_f(G) \geq \lceil n/3 \rceil + 1$ for every triangle-free plane graph G of order n. However, Ramamurthi and West [10] conjectured that this bound can be improved to $\lceil n/2 \rceil + 1$. Jungić, Král' and Škrekovki [6] proved this conjecture. Moreover, in the same paper, they gave a lower bound on $\chi_f(G)$ if a plane graph G has larger girth.

On the other hand, Dvořák, Král' and Škrekovki [4] considered upper bounds on the anti-rainbowness of plane graphs, and proved that every 3-connected plane graph G of order n satisfies $\chi_f(G) \leq \lfloor \frac{7n-8}{9} \rfloor$. They also gave upper bounds for the 4-connected or 5-connected cases, and show that the obtained bounds are tight for the 3-connected and 4-connected cases.

A *triangulation* on a surface is an embedding on the surface so that each face is bounded by a cycle of length 3. Negami [8] introduced the notion of the "looseness" of triangulations on surfaces, which was inspired by Arocha, Bracho and Neumann-Lara's works [1, 2]. Actually, the value of looseness of a triangulation G on a surface is equal to $\chi_f(G) - 2$. Nakamoto, Negami, Ohba and Suzuki [7] proved that for a triangulation G on a surface F^2, $\chi_f(G) \leq 2\alpha(G) + \lfloor \epsilon(F^2)/2 \rfloor$, where $\epsilon(F^2)$ is the Euler genus of F^2. Moreover, they proved that this upper bound can be improved to $\frac{11\alpha(G)+2}{6}$ for a plane triangulation G, and the bound is tight. Thus, the anti-rainbowness $\chi_f(G)$ of a plane triangulation G behaves the same as $\alpha(G)$ up to the constant factor.

As with a triangulation, a plane *quadrangulation* is defined as a plane graph with each face bounded by a cycle of length 4. Note that any plane quadrangulation is bipartite. Compared with the results on triangulations, it is natural to think the anti-rainbowness of quadrangulations. This does not seem an easy question, but instead, we give a non-trivial upper bound for the proper anti-rainbowness of a plane quadrangulation in terms of the independence number $\alpha(G)$:

Theorem 1 *Let G be a plane quadrangulation. Then all of the following statements hold.*

(I) $\chi_f^p(G) \leq \frac{3}{2}\alpha(G)$.

(II) *If the minimum degree of G is at least 3, then $\chi_f^p(G) \leq \frac{5}{4}\alpha(G)$.*

(III) *If G is 3-connected, then $\chi_f^p(G) \leq \frac{7}{6}\alpha(G) + \frac{1}{3}$.*

These results show that similarly to the anti-rainbowness of triangulations, the proper anti-rainbowness of a plane quadrangulation G behaves the same as $\alpha(G)$ up to the constant factor. All upper bounds on $\chi_f^p(G)$ in Theorem 1(I)–(III) are tight, which are shown in Section 4.

Recall that Dvořák, Král' and Škrekovki [4] proved that every 3-connected plane graph G of order n satisfies $\chi_f^p(G) \leq \chi_f(G) \leq \lfloor \frac{7n-8}{9} \rfloor$. For a 3-connected plane quadrangulation G, the upper bound of $\chi_f^p(G)$ in Theorem 1(III) is better than this bound when $\alpha(G) < \frac{2}{3}n - \frac{22}{21}$. Note that it follows from Euler's formula that for every plane quadrangulation G with n vertices, if the minimum degree is three, then we have $3\alpha(G) \leq |E(G)| \leq 2n - 4$, or $\alpha(G) \leq \frac{2}{3}n - \frac{4}{3}$. Their result on 4-connected or 5-connected plane graphs cannot be compared with Theorem 1, since no plane quadrangulation is 4-connected by Euler's formula.

Compared with the (non-proper) anti-rainbowness, it is natural to ask whether there exists a plane quadrangulation G such that $\chi_f^p(G) < \chi_f(G)$. We give a positive answer to this question using Theorem 1. Let G be a *pseud double wheel*, which is a plane quadrangulation obtained from an even cycle, say $C_{2\ell} = v_1w_1v_2w_2 \dots v_\ell w_\ell$, by adding one vertex x inside and another vertex y outside so that x is adjacent to v_i for $1 \le i \le \ell$ and y is adjacent to w_i for $1 \le i \le \ell$; see Figure 1 for the case $\ell = 6$. Note that G is 3-connected and $\alpha(G) = \frac{n}{2}$, where $n = |V(G)| = 2\ell + 2$. By Theorem 1(III), we have $\chi_f^p(G) \le \frac{7}{6}\alpha(G) + \frac{1}{3} = \frac{7}{12}n + \frac{1}{3}$. On the other hand, suppose that ℓ is a multiple of three, and let S be the set of every third vertex in $C_{2\ell}$, which are indicated by a circle in Figure 1. Note that each face contains exactly two vertices in $S \cup \{x, y\}$. Then by assigning the same color to all vertices $S \cup \{x, y\}$, and a distinct color to every other vertex, we obtain an anti-rainbow coloring with $\frac{4}{3}\ell + 1 = \frac{2}{3}n - \frac{1}{3}$ colors. Thus, we have $\chi_f(G) \ge \frac{2}{3}n - \frac{1}{3}$, and hence $\chi_f^p(G) < \chi_f(G)$ if $n \ge 14$ and $n - 2 = 2\ell$ is multiple of six.

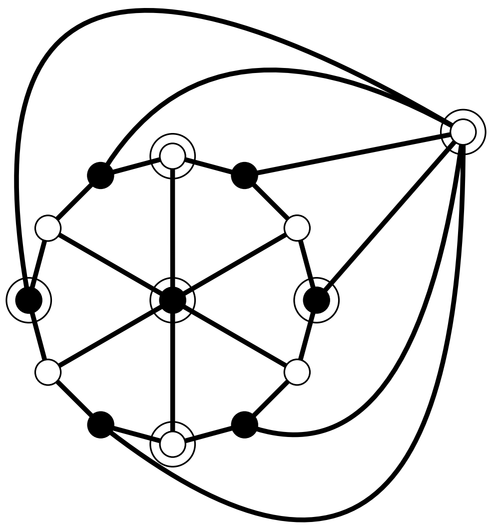

Figure 1: A pseud double wheel of order 14, where the vertices indicated by a circle form $S \cup \{x, y\}$.

The main idea for our proofs is a certain subgraph of the medial graph of a plane quadrangulation, called the dividing system and introduced in Section 2. We prove Theorem 1 in Section 3. In Section 4, we show the best possibility of the upper bounds on $\chi_f^p(G)$ in Theorem 1(I)–(III).

2 Dividing system

Let G be a plane graph. The *medial graph* of G, denoted by $M(G)$, is defined as follows: For each edge e of G, we put a new vertex $[e]$ in the middle of e, connect two vertices $[e]$ and $[e']$ if e and e' consecutively appear in the boundary of a face of G, and delete all vertices and edges in G. In other words, $M(G)$ is the graph with vertex set $\big\{[e] : e \in E(G)\big\}$ such that $[e][e'] \in E\big(M(G)\big)$ if and only if the edges e and e' in G share an end vertex and e appears just before or after e' on the rotation of the end vertex. Note that each face f of G contains k edges on $M(G)$ in its inside, where k is the length of the boundary of f.

For a plane quadrangulation G and a proper anti-rainbow coloring c, every face of G receives exactly two or three colors on its boundary by c. By the following lemma, we can focus on a proper anti-rainbow coloring such that no face receives exactly two colors.

Lemma 2 *Let G be a plane quadrangulation. Then, in each proper anti-rainbow $\chi_f^p(G)$-coloring, no face receives exactly two colors.*

Proof. Let c be a proper anti-rainbow $\chi_f^p(G)$-coloring of a plane quadrangulation G. For each edge e of G, we denote by C_e the set of colors appearing on an end vertex of e. Since c is a proper coloring, $|C_e| = 2$ for each edge e. Then, consider the spanning subgraph Λ_c of the medial graph $M(G)$ of G induced by all edges $[e][e']$ with $C_e = C_{e'}$. Note that each face f of G receives exactly two or three colors on its boundary, and the situation of each case is presented in Figures 2 and 3, respectively. Thus, Λ_c separates the plane into some regions, and all vertices of G in the same region receive the same color by c, where there might exist some regions containing no vertices of G, as in Figure 2.

Suppose that there is a face f of G that receives exactly two colors on its boundary by c. Let $v_1v_2v_3v_4$ be the boundary of f. Since c is proper, we have $c(v_1) = c(v_3)$ and $c(v_2) = c(v_4)$. Let σ be the region of Λ_c containing v_1. Then we consider the following two cases.

- Suppose that σ does not contain v_3. In this case, we change the color of all vertices contained in σ with a new color. Since $c(v_2) = c(v_4)$, f now receives exactly three colors on its boundary. Similarly, each face receiving exactly two colors by c does not become a rainbow face. For a face receiving exactly three colors by c, where $w_1w_2w_3w_4$ is its boundary with $c(w_1) = c(w_3)$, if w_1 is contained in σ, then so is w_3. Thus, the change of colors does not create a new rainbow face, and hence this contradicts the maximality of $\chi_f^p(G)$.

- Suppose that σ contains v_3. By Jordan's curve theorem, σ together with the quadrangular region inside f as in Figure 2, separates the region of Λ_c that contains v_2 from the region that contains v_4. Then, we can change the color of all vertices contained in the region that contains v_2 with a new color as in the previous case, and obtain a contradiction to the maximality of $\chi_f^p(G)$.

In either case, we have a contradiction, and thus complete the proof of Lemma 2. □

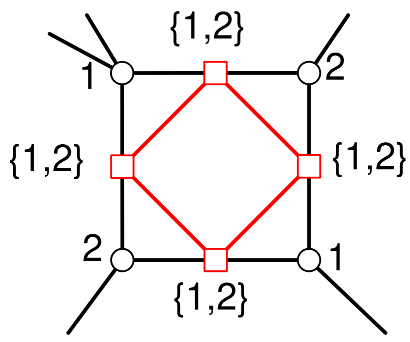

Figure 2: A face with exactly two colors on the boundary

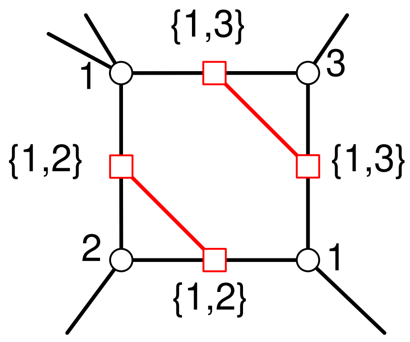

Figure 3: A face with exactly three colors on the boundary

Let G be a plane quadrangulation. A *dividing system* of G is a spanning subgraph of the medial graph $M(G)$ of G such that inside each face of G, exactly two edges that form a matching are chosen. See Figure 4 for example, where the dotted lines represent a plane quadrangulation and the red bold lines represent its dividing system. This concept was first defined by Negami and Midorikawa [9]. Since each edge of G belongs to the boundary of exactly two faces, a dividing system of G is a 2-factor (i.e. a spanning subgraph in which every vertex has degree exactly two) of $M(G)$. Thus, it consists of pairwise vertex-disjoint cycles. We first show a property of a dividing system.

Lemma 3 *Let G be a plane quadrangulation, and let Λ be a dividing system of G. Then all vertices in the same region separated by Λ belong to the same partite set of G.*

Proof. Let σ be a region separated by Λ. For two vertices v_1 and v_2 in σ that are contained in the same face f of G, it follows from the definition of a dividing system that the boundary of f can be written as v_1wv_2w' for some $w, w' \in V(G)$. Thus, v_1 and v_2 belong to the same partite set of G. Since σ is arcwise-connected, this completes the proof of the lemma. $\square$

In the next lemma, we show the relation of a dividing system to a proper anti-rainbow coloring.

Lemma 4 *Let G be a plane quadrangulation. Then the anti-rainbowness $\chi_f^p(G)$ is equal to the maximum number of regions separated by a dividing system Λ of G, where Λ is taken over all dividing systems of G.*

Proof. Let λ be the maximum number of regions separated by a dividing system Λ of G, where Λ is taken over all dividing systems of G.

Let c be a proper anti-rainbow $\chi_f^p(G)$-coloring of a plane quadrangulation G. By Lemma 2, no face receives exactly two colors. Thus, if we take the spanning subgraph Λ of the medial graph $M(G)$ of G induced by all edges $[e][e']$ with $C_e = C_{e'}$ as in the proof of Lemma 2, then each face of G has the situation of Figure 3, and hence Λ is actually a dividing system of G. Since all vertices in the same region of Λ have the same color by c, the number of colors of c is at most the number of regions separated by Λ. This implies $\chi_f^p(G) \leq \lambda$.

On the other hand, suppose that there is a dividing system Λ of a plane quadrangulation G such that Λ separates the plane into λ regions. By assigning distinct colors to each region, and coloring all vertices of G contained in the same region by the color being assigning to the region, we have the coloring of G with λ colors. By Lemma 3, the coloring is proper and by the property of a dividing system (see Figure 3), it is anti-rainbow. Thus, we have $\chi_f^p(G) \geq \lambda$. $\square$

By Lemma 4, to evaluate the anti-rainbowness of a plane quadrangulation G, it suffices to consider a dividing system of G with maximum number of regions. This is the main idea to prove our main theorems. We say that a dividing system of G attaining the maximum number of regions is *optimal*.

For a dividing system Λ of G, the *division graph* T_Λ is defined as follows: We put a vertex on each region separated by Λ, and connect two vertices if the boundary of the corresponding two regions share a cycle of Λ. See Figure 5 for an example. Since each simple curve on the plane is separating, we have the following lemma.

Lemma 5 *Let G be a plane quadrangulation, and let Λ be a dividing system of G. Then, the division graph T_Λ is a tree with λ vertices, where λ is the number of regions separated by Λ.*

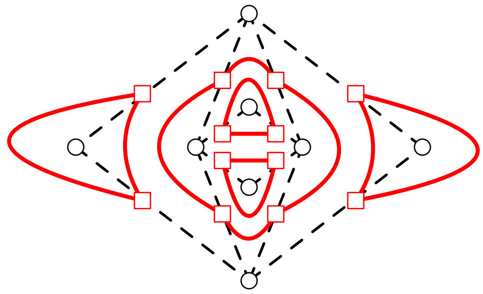

Figure 4: A dividing system of a plane quadrangulation G

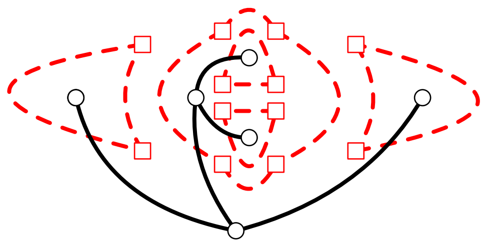

Figure 5: The division tree of the dividing system in Figure 4.

By Lemma 5, we call the division graph the *division tree* from now on. For a vertex x of the division tree T_Λ of a dividing system Λ of a plane quadrangulation G, we denote by $R(x)$ the set of vertices in G contained in the region corresponding to x. By Lemma 3, $R(x)$ is an independent set. The set of leaves of T_Λ that are adjacent with x is denoted by $\mathcal{L}_x$. Let $R(\mathcal{L}_x) = \bigcup_{y\in\mathcal{L}_x} R(y)$. We have the following lemma.

Lemma 6 *Let x be a vertex of the division tree T_Λ of a dividing system Λ of a plane quadrangulation G. If the degree of x in T_Λ is at least 2, then $|R(x)| \geq 2$.*

Proof. Suppose that $|R(x)| = 1$ and let v be the unique vertex contained in $R(x)$. For a face whose boundary contains v, say $vv_1v_2v_3$, either $[vv_1][v_1v_2], [vv_3][v_3v_2] \in E(\Lambda)$ or $[vv_1][vv_3], [v_1v_2][v_3v_2] \in E(\Lambda)$. Since $|R(x)| = 1$, the latter holds for all such faces. Thus, a cycle in Λ contains only v in its inside or outside, which contradicts that the degree of x in T_Λ is at least two. $\square$

The following lemma is trivial from the definition but useful.

Lemma 7 *Let T_Λ be the division tree of a dividing system Λ of a plane quadrangulation G. If two vertices x and y in T_Λ are not adjacent, then no vertices in $R(x)$ and no vertices in $R(y)$ are adjacent in G.*

3 Proof of Theorem 1

In this section, we prove Theorem 1(I)–(III) in each subsection, respectively.

3.1 Proof of Theorem 1(I)

Let Λ be an optimal dividing system of G and T_Λ be its division tree. Note that $|V(T_\Lambda)| = \chi_f^p(G)$. For $i \geq 1$, let $V_i = \{x \in V(T_\Lambda) : \deg_{T_\Lambda}(x) = i\}$. Then $\sum_{i\geq 1} |V_i| = |V(T_\Lambda)| = \chi_f^p(G)$. We divide the proof into two cases, depending on the value of $|V_1|$.

Case (i): $|V_1| \geq \frac{2}{3}\chi_f^p(G)$.

By Lemma 2, $|V(T_\Lambda)| \geq 3$, and hence no two leaves of T_Λ are adjacent. By Lemma 7, $\bigcup_{x\in V_1} R(x)$ is an independent set in G. Since $|R(x)| \geq 1$ for each $x \in V_1$, we have $\alpha(G) \geq \sum_{x\in V_1} |R(x)| \geq |V_1| \geq \frac{2}{3}\chi_f^p(G)$, or $\chi_f^p(G) \leq \frac{3}{2}\alpha(G)$.

Case (ii): $|V_1| < \frac{2}{3}\chi_f^p(G)$.

By Lemma 6, we have

$$\begin{aligned}|V(G)| &\geq |V_1| + \sum_{i\geq 2} 2|V_i| = |V_1| + 2\left(\chi_f^p(G) - |V_1|\right) \\ &= 2\chi_f^p(G) - |V_1| > 2\chi_f^p(G) - \frac{2}{3}\chi_f^p(G) = \frac{4}{3}\chi_f^p(G).\end{aligned}$$

Since G is bipartite, we have $\alpha(G) \geq \frac{1}{2}|V(G)| > \frac{2}{3}\chi_f^p(G)$. Thus, $\chi_f^p(G) < \frac{3}{2}\alpha(G)$. $\square$

3.2 Proof of Theorem 1(II)

Let Λ be an optimal dividing system of G and T_Λ be its division tree. Let

$$\begin{aligned}A &= \{x \in V(T_\Lambda) : \mathcal{L}_x = \emptyset,\ \deg_{T_\Lambda}(x) \geq 2\}, \\ B &= \{x \in V(T_\Lambda) - A : |R(x)| = 3\},\ \text{and} \\ V_i &= \{x \in V(T_\Lambda) - (A \cup B) : \deg_{T_\Lambda}(x) = i\} \text{ for } i \geq 1.\end{aligned}$$

By Lemma 6, each vertex $x \in A$ satisfies $|R(x)| \geq 2$. We have the following claims.

Claim 1 *Each vertex $x \in B$ satisfies $|R(\mathcal{L}_x)| \leq 2$.*

Proof of Claim 1. By Lemma 7, any vertex in $R(\mathcal{L}_x)$ can be adjacent in G with only vertices of $R(x)$. Since $|R(x)| = 3$ and the minimum degree of G is 3, each vertex in $R(\mathcal{L}_x)$ is adjacent to all vertices in $R(x)$. Thus, if $|R(\mathcal{L}_x)| \geq 3$, then $R(x) \cup R(\mathcal{L}_x)$ induces a graph containing $K_{3,3}$, which contradicts the planarity of G. Therefore, $|R(\mathcal{L}_x)| \leq 2$. $\blacksquare$

Claim 2 *For $i \geq 2$, each vertex $x \in V_i$ satisfies $|R(x)| \geq 4$.*

Proof of Claim 2. Let $x \in V_i$ for $i \geq 2$. Since $x \notin A$, we have $\mathcal{L}_x \neq \emptyset$, say $v \in \mathcal{L}_x$. Since the degree of v in G is at least 3 and all neighbors of v are contained in $R(x)$, we have $|R(x)| \geq 3$. Since $x \notin B$, we have $|R(x)| \neq 3$, and the claim holds. $\blacksquare$

We divide the proof into two cases, depending on the value of $|V_1|$.

Case (i): $|V_1| \geq \frac{4}{5}\chi_f^p(G) - \frac{4}{5}|A| - \frac{2}{5}|B|$.

By Lemma 2, $|V(T_\Lambda)| \geq 3$, and hence two leaves of T_Λ are not adjacent. Thus, by Lemma 7, $S = \bigcup_{x\in V_1} R(x)$ is an independent set in G, and $|S| \geq |V_1|$. We can observe the following:

- Let $a \in A$. Since $\mathcal{L}_a = \emptyset$, if we add $R(a)$ to S, then the set is still an independent set in G. It follows from Lemma 6 that this addition increases the size of S by $|R(a)| \geq 2$.
- Let $b \in B$. By Claim 1, we have $|R(b)| - |R(\mathcal{L}_b)| \geq 1$. Thus, if we replace $R(\mathcal{L}_b)$ in S with $R(b)$, then the set is still an independent set in G, which is larger than S.

Let $f : A \cup B \to \{1, 2\}$ be the mapping such that $f(a) = 2$ for $a \in A$ and $f(b) = 1$ for $b \in B$. Note that $|R(x)| - |R(\mathcal{L}_x)| \geq f(x)$ for $x \in A \cup B$. Since $A \cup B$ induces a subgraph of the division tree T_Λ, which is a bipartite graph, one of its partite sets, say X, satisfies $\sum_{x\in X} f(x) \geq \frac{1}{2}\sum_{x\in A\cup B} f(x) = \frac{1}{2}\big(2|A| + |B|\big)$. Since X is an independent set in T_Λ, it follows from Lemma 7 that $S\cup\left(\bigcup_{x\in X}R(x)\right)-\bigcup_{x\in X}R(\mathcal{L}_x)$ is an independent set in G, and hence

$$\begin{aligned}
\alpha(G) &\geq \left|S\cup\left(\bigcup_{x\in X}R(x)\right)-\bigcup_{x\in X}R(\mathcal{L}_x)\right| = |S|+\sum_{x\in X}|R(x)|-\sum_{x\in X}|R(\mathcal{L}_x)| \\
&\geq |V_1|+\sum_{x\in X}f(x) \geq |V_1|+\frac{1}{2}\big(2|A|+|B|\big) \\
&\geq \frac{4}{5}\chi_f^p(G)+\frac{1}{5}|A|+\frac{1}{10}|B| \geq \frac{4}{5}\chi_f^p(G).
\end{aligned}$$

Therefore, $\chi_f^p(G)\leq\frac{5}{4}\alpha(G)$.

Case (ii): $|V_1|<\frac{4}{5}\chi_f^p(G)-\frac{4}{5}|A|-\frac{2}{5}|B|$.

Note that $\sum_{i\geq 1}|V_i|+|A|+|B|=|V(T_\Lambda)|=\chi_f^p(G)$. By Lemma 6 and Claim 2, we have

$$\begin{aligned}
|V(G)| &\geq |V_1|+2|A|+3|B|+\sum_{i\geq 2}4|V_i| \\
&= |V_1|+2|A|+3|B|+4\left(\chi_f^p(G)-|A|-|B|-|V_1|\right) \\
&= 4\chi_f^p(G)-2|A|-|B|-3|V_1| \\
&> 4\chi_f^p(G)-2|A|-|B|-3\left(\frac{4}{5}\chi_f^p(G)-\frac{4}{5}|A|-\frac{2}{5}|B|\right) \\
&= \frac{8}{5}\chi_f^p(G)+\frac{2}{5}|A|+\frac{1}{5}|B| \geq \frac{8}{5}\chi_f^p(G).
\end{aligned}$$

Since G is a bipartite graph, we obtain $\alpha(G)\geq\frac{1}{2}|V(G)|>\frac{4}{5}\chi_f^p(G)$, or $\chi_f^p(G)<\frac{5}{4}\alpha(G)$, which completes the proof. $\square$

3.3 Proof of Theorem 1(III)

Let Λ be an optimal dividing system of G, and T_Λ be its division tree. Let $V_1=\{x\in V(T_\Lambda):\deg_{T_\Lambda}(x)=1\}$, and $T'_\Lambda=T_\Lambda-V_1$. Note that T'_Λ is a tree. For $i\geq 0$ and $k\geq 1$, let

$$\begin{aligned}
V'_{i,k} &= \{x\in V(T'_\Lambda):\deg_{T'_\Lambda}(x)=i,|R(x)|-|R(\mathcal{L}_x)|=k\}, \text{ and} \\
V'_{i,0} &= \{x\in V(T'_\Lambda):\deg_{T'_\Lambda}(x)=i,|R(x)|-|R(\mathcal{L}_x)|\leq 0\}.
\end{aligned}$$

Note that $V'_{0,k}\neq\emptyset$ only when T'_Λ consists of only one vertex.

Claim 3 *For $i\geq 0$ and $k\geq 0$, every vertex $x\in V'_{i,k}$ satisfies $|R(x)|\geq i-k+4$.*

Proof of Claim 3. Let $x\in V'_{i,k}$. For each component C of $T_\Lambda-x$ with $|C|\geq 2$, we contract all vertices of $\bigcup_{y\in V(C)}R(y)$ in G into one vertex, and let G' be the obtained graph. By Lemmas 3 and 7, G' is a planar bipartite graph such that one of the partite sets is $R(x)$, and the other consists of the vertices in $R(\mathcal{L}_x)$ and exactly i vertices obtained by the contraction. Thus, $|V(G')|=|R(x)|+|R(\mathcal{L}_x)|+i$. By Lemma 6, G' has at least three vertices. It follows from Euler's formula that $|E(G')|\leq 2|V(G')|-4$. On the other hand, since G is 3-connected and no two vertices in $R(\mathcal{L}_x)$ are adjacent, each vertex of $G'-R(x)$ has degree at least 3 in G'. Thus, $|E(G')|\geq 3\left(|R(\mathcal{L}_x)|+i\right)$. These inequalities imply

$$|R(x)|\geq i+|R(\mathcal{L}_x)|-|R(x)|+4.$$

Since $x \in V'_{i,k}$, we have $|R(\mathcal{L}_x)| - |R(x)| \geq -k$ regardless $k \geq 1$ or $k = 0$, and hence we obtain the desired inequality. ■

We divide the remaining proof into two cases depending on $|V_1|$.

Case (i): $|V_1| \geq \frac{6}{7}\chi_f^p(G) - \frac{2}{7} - \frac{2}{7}\sum_{i\geq 0}\sum_{k\geq 1} k|V'_{i,k}|$.

By Lemma 2, $|V(T_\Lambda)| \geq 3$, and hence two leaves of T_Λ are not adjacent. Thus, by Lemma 7, $S = \bigcup_{x\in V_1} R(x)$ is an independent set in G, and $|S| \geq |V_1|$. For $i \geq 0$, $k \geq 0$ and each vertex $x \in V'_{i,k}$, if we replace $R(\mathcal{L}_x)$ in S with $R(x)$, then the set is still an independent set in G. Now let f be a mapping on $\bigcup_{i\geq 0}\bigcup_{k\geq 1} V'_{i,k}$ such that $f(x) = k$ for each $x \in V'_{i,k}$. By the definition, we have $f(x) = |R(x)| - |R(\mathcal{L}_x)|$ for each $x \in V'_{i,k}$. Since $T'_\Lambda - \bigcup_{i\geq 0} V'_{i,0}$ is bipartite, one of its partite sets, say X, satisfies

$$\sum_{x\in X} f(x) \;\geq\; \frac{1}{2}\sum_{x\in\bigcup_{i\geq 0}\bigcup_{k\geq 1}V'_{i,k}} f(x) \;=\; \frac{1}{2}\sum_{i\geq 0}\sum_{k\geq 1} k|V'_{i,k}|.$$

Since X is an independent set in T_Λ, it follows from Lemma 7 that $S \cup \left(\bigcup_{x\in X} R(x)\right) - \bigcup_{x\in X} R(\mathcal{L}_x)$ is an independent set in G. Thus,

$$\begin{aligned}
\alpha(G) &\geq \left|S \cup \left(\bigcup_{x\in X} R(x)\right) - \bigcup_{x\in X} R(\mathcal{L}_x)\right| \;=\; |S| + \sum_{x\in X}\Big(|R(x)| - |R(\mathcal{L}_x)|\Big)\\
&= |V_1| + \sum_{x\in X} f(x) \;\geq\; |V_1| + \frac{1}{2}\sum_{i\geq 0}\sum_{k\geq 1} k|V'_{i,k}|\\
&\geq \frac{6}{7}\chi_f^p(G) - \frac{2}{7} + \frac{3}{14}\sum_{i\geq 0}\sum_{k\geq 1} k|V'_{i,k}| \;\geq\; \frac{6}{7}\chi_f^p(G) - \frac{2}{7}.
\end{aligned}$$

This implies that $\chi_f^p(G) \leq \frac{7}{6}\alpha(G) + \frac{1}{3}$.

Case (ii): $|V_1| < \frac{6}{7}\chi_f^p(G) - \frac{2}{7} - \frac{2}{7}\sum_{i\geq 0}\sum_{k\geq 1} k|V'_{i,k}|$.

Note that $|V_1| + \sum_{i\geq 0}\sum_{k\geq 0}|V'_{i,k}| = |V(T_\Lambda)| = \chi_f^p(G)$. Since T'_Λ is a tree with $\chi_f^p(G) - |V_1|$ vertices, it follows from the handshaking lemma that $\sum_{i\geq 0}\sum_{k\geq 1} i|V'_{i,k}| = 2\left(\chi_f^p(G) - |V_1| - 1\right)$. Thus, by Claim 3, we have

$$\begin{aligned}
|V(G)| &= \sum_{x\in V(T_\Lambda)} |R(x)| \;\geq\; |V_1| + \sum_{i\geq 0}\sum_{k\geq 0}(i - k + 4)|V'_{i,k}|\\
&= |V_1| + 4\sum_{i\geq 0}\sum_{k\geq 0}|V'_{i,k}| - \sum_{i\geq 0}\sum_{k\geq 0} k|V'_{i,k}| + \sum_{i\geq 0}\sum_{k\geq 0} i|V'_{i,k}|\\
&\geq |V_1| + 4\left(\chi_f^p(G) - |V_1|\right) - \sum_{i\geq 0}\sum_{k\geq 1} k|V'_{i,k}| + 2\left(\chi_f^p(G) - |V_1| - 1\right)\\
&= 6\chi_f^p(G) - 5|V_1| - 2 - \sum_{i\geq 0}\sum_{k\geq 1} k|V'_{i,k}|\\
&> 6\chi_f^p(G) - 5\left(\frac{6}{7}\chi_f^p(G) - \frac{2}{7} - \frac{2}{7}\sum_{i\geq 0}\sum_{k\geq 1} k|V'_{i,k}|\right) - 2 - \sum_{i\geq 0}\sum_{k\geq 1} k|V'_{i,k}|\\
&= \frac{12}{7}\chi_f^p(G) - \frac{4}{7} + \frac{3}{7}\sum_{i\geq 0}\sum_{k\geq 1} k|V'_{i,k}| \;\geq\; \frac{12}{7}\chi_f^p(G) - \frac{4}{7}.
\end{aligned}$$

Since G is bipartite, $\alpha(G) \geq \frac{1}{2}|V(G)| > \frac{6}{7}\chi_f^p(G) - \frac{2}{7}$, and hence $\chi_f^p(G) < \frac{7}{6}\alpha(G) + \frac{1}{3}$. This completes the proof of Theorem 1(III). □

4 Plane quadrangulations satisfying the equality

In this section, we show that the upper bounds of $\chi_f^p(G)$ in Theorem 1(I)–(III) are best possible.

4.1 Best possibility of Theorem 1(I)

In this section, we show the following proposition.

Proposition 8 *There are infinitely many plane quadrangulations G such that $\chi_f^p(G) = \frac{3}{2}\alpha(G)$.*

Proof. To construct quadrangulations with desired properties, we define a $K_{2,4}$-addition as follows. Let f be a face of a quadrangulation G, and let $v_1v_2v_3v_4$ be its boundary. We add two new vertices u_1 and u_3 inside f, and connect them to v_1 and v_3 so that it creates a new quadrangular face $v_1u_1v_3u_3$, say f'. Then we further add two more new vertices u_2 and u_4 inside f', and connect them to u_1 and u_3 so that it creates a new quadrangular face $u_1u_2u_3u_4$. We call this operation a $K_{2,4}$-*addition* to the face f at v_1 and v_3; see Figure 6. In the operation, we say that v_1 and v_3 are the *base* vertices and u_1 and u_3 are the *subbase* vertices.

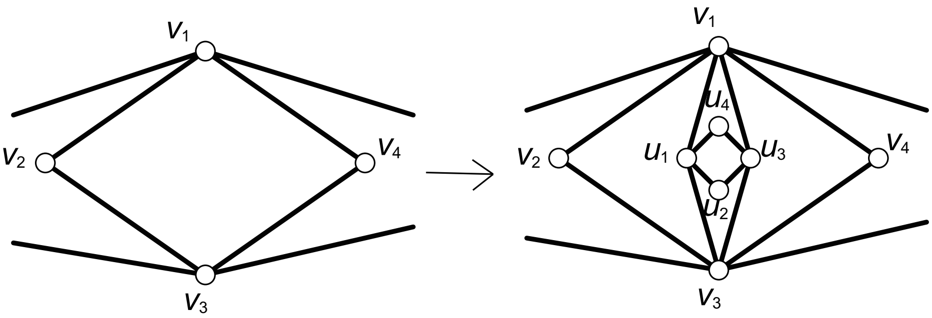

Figure 6: The quadrangulation obtained by a $K_{2,4}$-addition to the face f.

We define the set $\mathcal{G}_1$ of plane quadrangulations G with a family $\mathcal{P}_G$ of pairwise disjoint vertex subsets of G such that each $S \in \mathcal{P}_G$ is an independent set with $|S| = 2$ and

> $(*)$ for each face f of G, there exists $S \in \mathcal{P}_G$ such that the boundary of f contains the two vertices in S,

by recursively adding members as follows.

Let G_1 be the plane quadrangulation obtained from C_4 by a $K_{2,4}$-addition (see Figure 7), and let S_0 (resp. S_1) be the set of the base (resp. subbase) vertices of the $K_{2,4}$-addition, respectively. Note that G_1 satisfies the condition $(*)$ for $\mathcal{P}_{G_1} = \{S_0, S_1\}$. First we add G_1 to $\mathcal{G}_1$.

Let $G \in \mathcal{G}_1$, and let f be a face of G. By the condition $(*)$ for G, there exists $S \in \mathcal{P}_G$ such that the boundary of f contains the two vertices in S. We perform the $K_{2,4}$-addition to f at the two vertices in S. Let G' be the obtained plane quadrangulation, S' be the set of the subbase vertices, and $\mathcal{P}_{G'} = \mathcal{P}_G \cup \{S'\}$. Since there are five new faces in G' and each of their boundaries contains either the two vertices of S or the two vertices of S', G' satisfies the condition $(*)$ for $\mathcal{P}_{G'}$. We add G' to $\mathcal{G}_1$. Note that $|V(G')| = |V(G)|+4$, and $|\mathcal{P}_{G'}| = |\mathcal{P}_G|+1$.

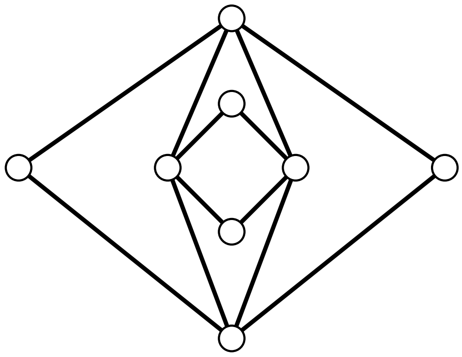

Figure 7: The plane quadrangulation G_1.

Since the vertices added to G form C_4, we have $\alpha(G') \leq \alpha(G)+2$. (To be exact, the equality holds, but the inequality suffices for our proof.)

Then we obtain the set $\mathcal{G}_1$. Let $G \in \mathcal{G}_1$. We color the two vertices in each $S \in \mathcal{P}_G$ by the same color which is different from the one used for any $S' \in \mathcal{P}_G - \{S\}$. Then we color the remaining vertices, that is, the vertices that do not belong to any $S \in \mathcal{P}_G$, by different colors. By the condition $(*)$, this coloring is a proper anti-rainbow coloring, and it uses $|V(G)|-|\mathcal{P}_G|$ colors. Since $|V(G_1)| = 8$, $|\mathcal{P}_{G_1}| = 2$, and $\alpha(G_1) = 4$, if G is obtained by $K_{2,4}$-addition k times, then

$$|V(G)| = 4k+8, \qquad |\mathcal{P}_G| = k+2, \qquad \text{and} \quad \alpha(G) \leq 2k+4.$$

Thus,

$$\chi_f^p(G) \ \geq \ |V(G)| - |\mathcal{P}_G| \ = \ 3k+6 \ \geq \ \frac{3}{2}\alpha(G).$$

Since $\chi_f^p(G) \leq \frac{3}{2}\alpha(G)$ by Theorem 1(I), every member G of $\mathcal{G}_1$ satisfices $\chi_f^p(G) = \frac{3}{2}\alpha(G)$. This completes the proof of Proposition 8. $\square$

Remark: If a plane quadrangulation G satisfies $\chi_f^p(G) = \frac{3}{2}\alpha(G)$, then all equalities hold in Case (i) of the proof of Theorem 1(I) in Section 3.1. In addition, if we follow the argument in Case (ii) replacing the assumption with $|V_1| = \frac{2}{3}\chi_f^p(G)$, then we also obtain the bound $\chi_f^p(G) \leq \frac{3}{2}\alpha(G)$. Therefore, all equalities in Case (ii) also hold. Those conditions imply that G is either a member of $\mathcal{G}_1$ or isomorphic to C_4. (We leave the detail of the proof to the readers.) This gives a complete characterization of plane quadrangulations G with $\chi_f^p(G) = \frac{3}{2}\alpha(G)$.

4.2 Best possibility of Theorem 1(II)

Similarly to the previous section, we prove the following proposition.

Proposition 9 *There are infinitely many plane quadrangulations G such that the minimum degree of G is 3 and $\chi_f^p(G) = \frac{5}{4}\alpha(G)$.*

Proof. We first define a cube-addition as follows, where the cube is the plane quadrangulation represented in Figure 8. Let f be a face of a quadrangulation G, and let $v_1v_2v_3v_4$ be its boundary. Let $u_1u_2u_3u_4$ be the boundary of the outer face of the cube. We embed the cube into the inside of f and connect v_i and u_j for all pairs $i, j \in \{1, 3\}$; see Figure 9 for example. We call this operation a *cube-addition to the face f at v_1 and v_3*. The partite set of the cube containing u_1 and u_3 is the *subbase* of the cube-addition.

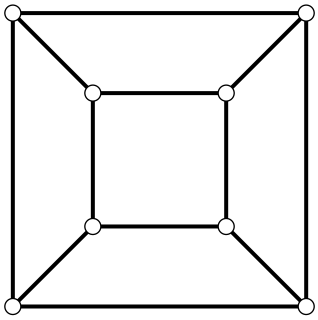

Figure 8: The cube

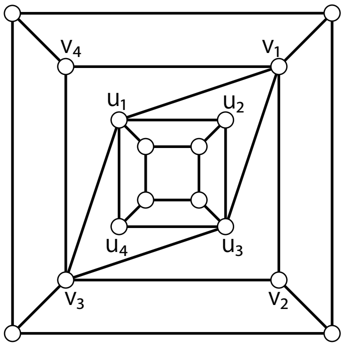

Figure 9: The plane quadrangulation obtained from the cube by a cube-addition at v_1 and v_3

We define the set $\mathcal{G}_2$ of plane quadrangulations G with a family $\mathcal{P}_G$ of pairwise disjoint vertex subsets of G such that each $S \in \mathcal{P}_G$ is an independent set with $|S| = 4$ and

> $(*2)$ for each face f of G, there exists $S \in \mathcal{P}_G$ such that the boundary of f contains exactly two vertices in S

by recursively adding members as follows.

Let G_2 be the cube and let S_2 be one of the partite sets. Note that G_2 satisfies the condition $(*2)$ for $\mathcal{P}_{G_2} = \{S_2\}$. First we add G_2 to $\mathcal{G}_2$.

Let $G \in \mathcal{G}_2$, and let f be a face of G. By the condition $(*2)$ for G, there exists $S \in \mathcal{P}_G$ such that the boundary of f contains exactly two vertices in S. We perform a cube-addition to f at the two vertices in S, and let G' be the obtained plane quadrangulation with $\mathcal{P}_{G'} = \mathcal{P}_G \cup \{S'\}$, where S' is the subbase of the cube-addition. Since there are nine new faces in G' and each of their boundaries contains either two vertices of S or two vertices of S', G' satisfies the condition $(*2)$ for $\mathcal{P}_{G'}$. We add G' to $\mathcal{G}_2$. Note that $|V(G')| = |V(G)| + 8$, and $|\mathcal{P}_{G'}| = |\mathcal{P}_G| + 1$. Since the vertices added to G form the cube, we have $\alpha(G') \leq \alpha(G) + 4$. (To be exact, the equality holds, but the inequality suffices for our proof.)

Then we obtain the set $\mathcal{G}_2$. Note that the minimum degree of every member $G \in \mathcal{G}_2$ is three. Let $G \in \mathcal{G}_2$. We color the four vertices in each $S \in \mathcal{P}_G$ by the same color which is different from the one used for any $S' \in \mathcal{P}_G - \{S\}$. Then we color the remaining vertices, that is, the vertices that do not belong to any $S \in \mathcal{P}_G$, by different colors. By the condition $(*2)$, this coloring is a proper anti-rainbow coloring, and it uses $|V(G)| - 3|\mathcal{P}_G|$ colors. Since $|V(G_2)| = 8$, $|\mathcal{P}_{G_2}| = 1$, and $\alpha(G_2) = 4$, if G is obtained by cube-addition k times, then

$$|V(G)| = 8k + 8, \qquad |\mathcal{P}_G| = k + 1 \qquad \text{and} \quad \alpha(G) \leq 4k + 4.$$

Thus,

$$\chi_f^p(G) \ \geq \ |V(G)| - 3|\mathcal{P}_G| \ = \ 5k + 5 \ \geq \ \frac{5}{4}\alpha(G).$$

Since $\chi_f^p(G) \leq \frac{5}{4}\alpha(G)$ by Theorem 1(II), every member G of $\mathcal{G}_2$ satisfices $\chi_f^p(G) = \frac{5}{4}\alpha(G)$. This completes the proof of Proposition 9. $\square$

4.3 Best possibility of Theorem 1(III)

Similarly to the previous sections, we prove the following proposition.

Proposition 10 *There are infinitely many* 3*-connected plane quadrangulations* G *such that* $\chi_f^p(G) = \frac{7}{6}\alpha(G) + \frac{1}{3}$.

Proof. We first define an H_t-addition for $t = 1, 2$ as follows, where H_1 and H_2 are the plane graphs in Figures 10 and 11, respectively.

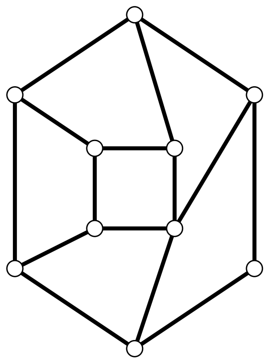

Figure 10: The graph H_1

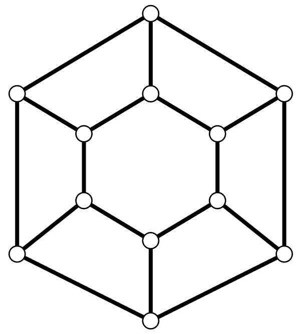

Figure 11: The graph H_2

Let f be a hexangular face of a plane graph G, and let $v_1v_2v_3v_4v_5v_6$ be its boundary. For $t = 1, 2$, let f' be the outer face of the graph H_t and $u_1u_2u_3u_4u_5u_6$ be its boundary. We embed the graph H_t into the inside of f and connect v_i and u_i for $i \in \{1, 3, 5\}$ and connect v_{i+2} and u_i for $i \in \{1, 3, 5\}$ where $v_7 = v_1$; see Figure 12 for example. We call this operation an *H_t-addition to the face f at v_1, v_3 and v_5*. The partite set of H_t that contains u_1, u_3 and u_5 is the *subbase* of the H_t-addition.

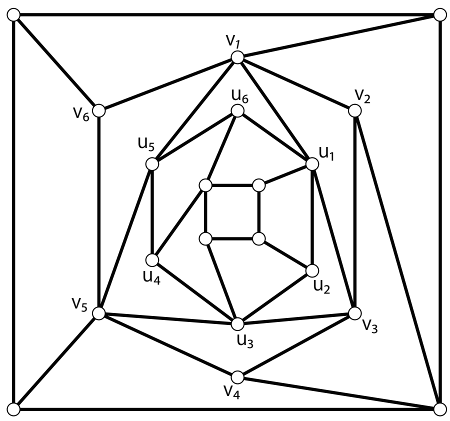

Figure 12: The 3-connected plane quadrangulation obtained from H_1 (with an embedding in which the hexangular face is an inner face) by an H_1-addition.

We define the set $\mathcal{G}_3$ of plane graphs G with a family $\mathcal{P}_G$ of pairwise disjoint vertex subsets of G such that all faces of G are quadrangular except for one inner face that is hexangular, each $S \in \mathcal{P}_G$ is an independent set with $|S| = 6$ except for one independent set S_{H_1} with $|S_{H_1}| = 5$ and

(∗3) for each face f of G, there exists $S \in \mathcal{P}_G$ such that the boundary of f contains exactly two vertices in S if f is quadrangular and exactly three vertices in S if f

is hexangular,

by recursively adding members as follows.

First we re-embed the plane graph H_1 so that the hexangular face is an inner face. Note that H_1 satisfies the condition ($*3$) for $\mathcal{P}_{H_1} = \{S_{H_1}\}$, where S_{H_1} is one of the partite sets of H_1. (Note that there are two choices for S_{H_1}.) We add H_1 to $\mathcal{G}_3$.

Let $G \in \mathcal{G}_3$, and let f be the unique hexangular face of G. By the condition ($*3$) for G, there exists $S \in \mathcal{P}_G$ such that the boundary of f contains exactly three vertices in S. We perform an H_2-addition to f at the three vertices in S, and let G' be the obtained plane graph with $\mathcal{P}_{G'} = \mathcal{P}_G \cup \{S'\}$, where S' is the subbase of the H_2-addition. Note that there are 12 new quadrangular faces in G' and each of their boundaries contains either two vertices of S or two vertices of S'. In addition, the boundary of the unique hexangular face of G' contains three vertices of S'. Thus, G' satisfies the condition ($*3$) for $\mathcal{P}_{G'}$. We add G' to $\mathcal{G}_3$. Note that $|V(G')| = |V(G)| + 12$, $|\mathcal{P}_{G'}| = |\mathcal{P}_G| + 1$, and $\alpha(G') \leq \alpha(G) + 6$. (The equality also holds, but the inequality suffices for our proof.)

Then we obtain the set $\mathcal{G}_3$. Note that any member of $\mathcal{G}_3$ is 3-connected. Let $G \in \mathcal{G}_3$ with the unique haxangular face f. By the condition ($*3$) for G, there exists $S_0 \in \mathcal{P}_G$ such that the boundary of f contains exactly three vertices in S_0. We perform an H_1-addition to f at the three vertices in S_0, and let $\widetilde{G}$ be the obtained plane graph with $\mathcal{P}_{\widetilde{G}} = \mathcal{P}_G \cup \{S'_0\}$, where S'_0 is the subbase of the H_1-addition. Note that $\widetilde{G}$ is a 3-connected plane quadrangulation and $|S'_0| = 5$. We color the vertices in each $S \in \mathcal{P}_{\widetilde{G}}$ by the same color which is different from the one used for any $S' \in \mathcal{P}_{\widetilde{G}} - \{S\}$. Then we color the remaining vertices, that is, the vertices that do not belong to any $S \in \mathcal{P}_{\widetilde{G}}$, by different colors. By the condition ($*3$), this coloring is a proper anti-rainbow coloring, and it uses $|V(\widetilde{G})| - \sum_{S \in \mathcal{P}_{\widetilde{G}}} (|S| - 1)$ colors. Suppose that G is obtained from H_1 by H_2-addition k times. Since $|V(H_1)| = 10$, $|\mathcal{P}_{H_1}| = 1$, and $\alpha(H_1) = 5$, we have

$$|V(\widetilde{G})| = 12k + 20, \qquad |\mathcal{P}_{\widetilde{G}}| = k + 2 \qquad \text{and} \quad \alpha(\widetilde{G}) \leq 6k + 10.$$

Thus,

$$\begin{aligned} \chi_f^p(\widetilde{G}) \ &\geq \ |V(\widetilde{G})| - \sum_{S \in \mathcal{P}_{\widetilde{G}}} (|S| - 1) \ = \ 12k + 20 - (5k + 8) \\ &= \ 7k + 12 \ \geq \ \frac{7}{6}\alpha(\widetilde{G}) + \frac{1}{3}. \end{aligned}$$

Since $\chi_f^p(\widetilde{G}) \leq \frac{7}{6}\alpha(\widetilde{G}) + \frac{1}{3}$ by Theorem 1(III), the plane quadrangulation $\widetilde{G}$ for every member G of $\mathcal{G}_3$ satisfices $\chi_f^p(\widetilde{G}) = \frac{7}{6}\alpha(\widetilde{G}) + \frac{1}{3}$. This completes the proof of Proposition 10. $\square$

Remark: Note that all plane quadrangulations G we construct in this section satisfy $\alpha(G) = \frac{|V(G)|}{2}$. Therefore, the bounds in Theorem 1(I)–(III) might be improved for plane quadrangulations G such that $\alpha(G)$ is much larger than $\frac{|V(G)|}{2}$. We leave this for the readers as an open problem.

Acknowledgment

We thank the referees for reading the paper carefully and giving helpful comments and suggestions. In particular, one referee suggested the problem on the existence of a plane quadrangulation G with $\chi_f^p(G) < \chi_f(G)$, which we explain in Section 1. The other referee suggested the problem in the end of Section 4.

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References

[1] J. L. Arocha, J. Bracho and V. Neumann-Lara, On the minimum size of tight hypergraphs, *J. Graph Theory*, **16** (1992), 319–326.

[2] J. L. Arocha, J. Bracho and V. Neumann-Lara, Tight and untight triangulations of surfaces by complete graphs, *J. Combin. Theory Ser. B*, **63** (1995), 185–199.

[3] J. Czap and S. Jendrol', Facially-constrained colorings of plane graphs: a survey, *Discrete Math.*, **340** (2017), 2691–2703.

[4] Z. Dvořák, D. Král' and R. Škrekovski, Non-rainbow colorings of 3-, 4- and 5-connected plane graphs, *J. Graph Theory* **63** (2010), 129–145.

[5] H. Grötzsch, Ein Dreifarbensatz für dreikreisfreie Netze auf der Kugel, *Wiss. Z. Martin-Luther-Universit, Halle, Wittenberg, Math. Nat. Reihe* **8** (1959), 109–120.

[6] V. Jungić, D. Král' and R. Škrekovski, Colorings of plane graphs with no rainbow faces, *Combinatorica*, **26** (2006), 169–182.

[7] A. Nakamoto, S. Negami, K. Ohba and Y. Suzuki, Looseness and independence number of triangulations on closed surfaces, *Discuss. Math. Graph Theory*, **36** (2016), 545–554.

[8] S. Negami, Looseness ranges of triangulations on closed surface, *Discrete Math.*, **303** (2005), 167–174.

[9] S. Negami and T. Midorikawa, Loosely-tightness of triangulations of closed surface, *Sci. rep. Yokohama Nat. Univ. Sect. I Math Phys Chem*, **43** (1996), 25–41.

[10] R. Ramamurthi and D. B. West, Maximum face-constrained colorings of plane graphs, *Discrete Math.*, **274** (2004), 233–240.

[11] C. Thomassen, Grötzsch's 3-color theorem and its counterparts for the torus and the projective plane, *J. Combin. Theory Ser. B*, **62** (1994), 268–279.