Colorings and dominating sets of graphs on surfaces

Toshiki Abe
Graduate School of Environment and Information Science
Yokohama National University
79-7 Tokiwadai, Hodogaya-ku, Yokohama 240-8501 JAPAN

Contents

Preface

This thesis is written on the subject 'Colorings and dominating sets of graphs on surfaces' and is to be submitted to the degree of Doctor at Yokohama National University.

When I was a junior high school student, I was recommended to take Japan Junior Mathematical Olympic by my homeroom teacher. In those days, I could not solve those problems at all. However, I saw another view of mathematics, which I could not see from the regular educations. These problems taught me to have a wide and various views in order to solve problems. Then I would like to know mathematics more and more.

After my entering in Yokohama National University, I met discrete mathematics and graph theory at Professor Negami and Professor Nakamoto. In these subjects, I got a lot of lessens, not only abundant ideas or techniques, but also mathematical attitudes, backgrounds of the problems as well. Especially, I was very impressed when difficult problems are solved without hard calculations by considering the ideas of discrete mathematics. These incidents made me a decision to study graph theory.

From 4th year grade, I study the domination of graphs. The problems of domination in Graph Theory, there are several open problems still now. Meanwhile, Professor Nakamoto gave me very attractive problems about this subject. In those days, I was so immature that I got many failure in an effort to solve the problem. However, Professor Nakamoto, Professor Matsumoto, who was a student in the same laboratory at that time, and Professor Ozeki gave me very helpful comments both solving problems and writing papers. Moreover, by discussing with Professor Tokunaga, who works in Tokyo Medical and Dental University, we develop the coloring methods and finally complete the problems.

From the last half of my master's course, I studied Combinatorial Nullstellensatz and its applications under the instruction of Professor Ozeki. This theorem seems to belong to the Algebra rather than Graph Theory, but I learned that this has a lot of applications in the graph theory as well, such as finding specific subgraphs, graph coloring, orientations and so on. Since I was interested in the graph coloring, I have studied list-coloring up to now.

At first, we roughly introduce our research and results on graph colorings and its application. In Chapter 1, we prepare some basic terminologies and notations on Graph theory. In Chapter 2, we introduce the research and results about some vertex colorings for planar graphs. In Chapter 3, we focus on some known results and proofs of our results about the Alon-Tarsi number. In Chapter 4, we give some known and our results about the applications of the Alon-Tarsi number. In Chapter 5, we introduce some known results and prove our results about dominating set, which are obtained by using coloring method in planar graphs.

Finally, I am grateful to Professor Ozeki for enormous supports and insightful comments. He gave me a lot of opportunities to glow up in the various things. I would also like to appreciate to Professor Nakamoto to give me a lot of chances in my developments. I gratefully acknowledge the work of past and present members of my laboratory and researchers. I would like to appreciate my family for their supports and encouragements.

Papers underlying the thesis

- T. Abe, The differences between between the list-coloring and DP-coloring for planar graphs, submitted to *Discrete Mathematics*
- T. Abe S. Kim and K. Ozeki, The Alon-Tarsi number of K_5-minor-free graphs, submitted to *Discrete Mathematics*
- T. Abe and K. Ozeki, Signature of edge-colorings on the projective plane,
- T. Abe J. Higa and S. Tokunaga, Domination number of annulus triangulations, *Theory and Applications of Graphs*, **7** (2020) Iss. 1 , Article 6.

Introduction

A *graph* consists of points, called *vertices* and arcs, called *edges*, each of which joins a pair of vertices. For a graph G, we denote a vertex set by $V(G)$ and edge set by $E(G)$ respectively. A graph can be regarded as a mathematical model which expresses such structures of finite sets with some relation.

An *embedding* of a graph G on a surface $\mathbb{F}$ is a drawing of G on $\mathbb{F}$ without crossing edges. We sometimes call an embedded graph on a surface an *embedding*. For embedded graph G on $\mathbb{F}$, each region which is bounded by vertices and edges is called a *face*. We denote the set of faces of G by $F(G)$. An embedded graph G is said to be a 2-*cell embedding* if each face of G is isomorphic to a 2-cell, that is, $\{(x, y) \in \mathbb{R}^2 : x^2 + y^2 \leq 1\}$. A graph G is said to be *planar* if G can be embedded on the plane. An embedded graph on the plane is said to be a *plane graph*. A *triangulation* G on a surface is a graph embedded on $\mathbb{F}$ such that each face of G is bounded by a cycle of length 3 and any two faces of G share at most one edge.

A vertex k-coloring of a graph G is a mapping $c : V(G) \to \{1, ..., k\}$. A coloring c is called *proper* if $c(x) \neq c(y)$ for each $xy \in E(G)$. We often refer to a k-coloring instead of a proper vertex k-coloring. A graph G is *k-colorable* if G admits a k-coloring.

At first, we introduce the well known Four Color Theorem.

Theorem 0.0.1 (Appel and Haken [5]) *Every planar graph is* 4*-colorable.*

After this theorem, a lot of variations of graph colorings are considered and studied by many researchers. In this thesis, we study several graph colorings and dominating set as the application of graph colorings.

First, we focus on the list coloring, which was introduced by P. Erdős, A.L.Rubin and H. Taylor [12] and Vizing [29] independently. We associate a *list assignment, L*, with a graph G such that each vertex v is assigned a list $L(v)$ of colors. The graph G is L-colorable if G has a proper coloring c such that $c(v) \in L(v)$ for each vertex v. Let $f : V(G) \to \mathbb{N}$ be a function and let k be a positive integer. We say that G is *f-choosable* if G is L-colorable for every list assignment L such that $|L(v)| \geq f(v)$ for every vertex v. Especially, we say that G is *k-choosable* if G is f-choosable where f is the constant function taking the value k. The *list-chromatic number* of G, denoted by $\chi_\ell(G)$, is the minimum integer k such that G is k-choosable. For the chromatic number and list-chromatic number, the following holds in general graphs.

Proposition 0.0.2 *Let G be a graph. Then $\chi(G) \leq \chi_\ell(G)$.*

Proof. Let G be a graph with $\chi_\ell(G) = k$. Then we give the list L with $L(v) = \{1, ..., k\}$ for each $v \in V(G)$. Since G is k-choosable, we have an L-coloring c. This implies that G is k-colorable. ■

The converse of the inequality of Proposition 0.0.2 does not hold since the complete bipartite graph $K_{2,4}$ satisfies $\chi_\ell(K_{2,4}) = 3$. Moreover, it is also shown that the gap between $\chi(G)$ and $\chi_\ell(G)$ can be arbitrary large [12].

In the following, let us focus on planar graphs. By a celebrated result of Appel and Haken, the chromatic number of a planar graph is at most 4 [5] and one may think that list-chromatic number of every planar graph is also bounded by 4. However, the latter one is not true and Voigt [30] constructed a planar graph G with $\chi(G) = 4$ and $\chi_\ell(G) = 5$ in 1993. Moreover, Thomassen proved the following.

Theorem 0.0.3 (Thomassen [27]) *Let G be a planar graph. Then $\chi_\ell(G) \leq 5$.*

In the following, we focus on two extensions of the list coloring, DP-coloring and the Alon-Tarsi number. First, we define the DP-coloring. This notion was introduced by Dvořák and Postle [9] as the *correspondence coloring* in [9]. However, this notion is also called *DP-coloring* by taking their initials. In this thesis, we also use the notation DP-coloring instead of correspondence coloring.

Suppose that G is a graph and L is a list assignment of G. For each edge uv in G, let M_{uv} be a matching between the sets $\{u\} \times L(u)$ and $\{v\} \times L(v)$. Moreover, let $\mathscr{M}_L = \{M_{uv} : uv \in E(G)\}$, which we call a *matching assignment*. Let $H = H(G, L, \mathscr{M}_L)$ be the graph that satisfies all of the following conditions:

(i) $V(H) = \{(w, a) : w \in V(G), a \in L(w)\}$,

(ii) for each $w \in V(G)$ and distinct $a, b \in L(w)$, $(w, a)(w, b) \in E(H)$,

(iii) if $uv \in E(G)$, then the set of edges between $\{u\} \times L(u)$ and $\{v\}L(v)$ form M_{uv}, and

(iv) if $uv \notin E(G)$, then there are no edges between $L(u)$ and $L(v)$.

If H contains an independent set of size $|V(G)|$, then G is said to be *$\mathscr{M}_L$-colorable.* A graph G is *DP-k-colorable* if G is $\mathscr{M}_L$-colorable for any list assignment L with $|L(v)| \geq k$ for every $v \in V(G)$ and for any matching assignment $\mathscr{M}_L$. The minimum integer k such that G is DP-k-colorable is the *DP-chromatic number* of G, denoted by $\chi_{DP}(G)$.
Let G be a graph and let L be a list assignment. If we take the matching assignment as $M_{uv} = \{(u, a)(v, b) : a \in L(u), b \in L(v), a = b\}$ for every $uv \in E(G)$, then the $\mathscr{M}_L$-coloring coincides with the L-coloring. Therefore, we have the following.

Proposition 0.0.4 *Let G be a graph. Then $\chi_\ell(G) \leq \chi_{DP}(G)$.*

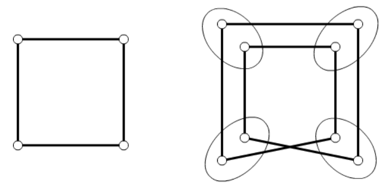

Figure 1: The left is C_4 and the right denotes a graph $H(C_4, L, \mathscr{M}_L)$, where the ellipse in the right shows the set of the pair of vertex and given colors contained in the list. In this case, we see that $H(C_4, L, \mathscr{M}_L)$ does not have an independent set of size $|C_4|$ and this implies that C_4 is not DP-2-colorable..

Since $\chi_\ell(C_{2n}) = 2$ and $\chi_{DP}(C_{2n}) = 3$, the converse of inequality does not hold. (See Figure 2.1.)

For the DP-coloring, Dvořák and Postle [9] observed that the following holds.

Theorem 0.0.5 ([9]) *Let G be a planar graph. Then $\chi_{DP}(G) \leq 5$.*

Next, we define the Alon-Tarsi number, which is another extension of list coloring. Let G be a graph and let '$<$' be an arbitrary fixed ordering of the vertices of G. The *graph polynomial* of G is defined as

$$P_G(\boldsymbol{x}) = \prod_{u \sim v, u < v} (x_u - x_v),$$

where $u \sim v$ means that u and v are adjacent, and $\boldsymbol{x} = (x_v)_{v \in V(G)}$ is a vector of $|V(G)|$ variables indexed by the vertices of G. It is easy to see that a mapping $c : V(G) \to \mathbb{N}$ is a proper coloring of G if and only if $P_G(\boldsymbol{c}) \neq 0$, where $\boldsymbol{c} = \big(c(v)\big)_{v \in V(G)}$. Therefore, to find a proper coloring of G is equivalent to find an assignment of $\boldsymbol{x}$ so that $P_G(\boldsymbol{x}) \neq 0$. The following theorem, which was proved by Alon and Tarsi, gives sufficient conditions for the existence of such assignments as above.

Theorem 0.0.6 (Alon and Tarsi [1]) *(Combinatorial Nullstellensatz) Let $\mathbb{F}$ be an arbitrary field and let $f = f(x_1, x_2, \ldots, x_n)$ be a polynomial in $\mathbb{F}[x_1, x_2, \ldots, x_n]$. Suppose that the degree $\deg(f)$ of f is $\sum_{i=1}^n t_i$ where each t_i is a nonnegative integer, and suppose that the coefficient of $\prod_{i=1}^n x_i^{t_i}$ of f is nonzero. Then if $S_1, S_2, \ldots, S_n$ are subsets of $\mathbb{F}$ with $|S_i| \geq t_i + 1$, then there are $s_1 \in S_1$,$s_2 \in S_2$,...,$s_n \in S_n$ so that $f(s_1, s_2, \ldots, s_n) \neq 0$.*

In particular, a graph polynomial $P_G(\boldsymbol{x})$ is a homogeneous polynomial and $\deg(P_G)$ is equal to $|E(G)|$. Therefore, if there exists a monomial $c \prod_{v \in V(G)} {x_v}^{t_v}$ in the expansion of the graph polynomial P_G so that $c \neq 0$ and $t_v < k$ for each $v \in V(G)$, then G is k-choosable. With this in mind, Jensen and Toft [19] defined the *Alon-Tarsi number* of a graph as follows.

Definition 0.0.7 The *Alon-Tarsi number* of a graph G, denoted by $AT(G)$, is the minimum k for which there exists a monomial $c\prod_{v\in V(G)} x_v^{t_v}$ in the expansion of $P_G(\boldsymbol{x})$ such that $c \neq 0$ and $t_v < k$ for all $v \in V(G)$.

By the definition of the Alon-Tarsi number, we have the following.

Proposition 0.0.8 *Let G be a graph. Then $\chi_\ell(G) \leq AT(G)$.*

It is also known that the gap between $\chi_\ell(G)$ and $AT(G)$ can be arbitrary large. Nevertheless, it is also known that the upper bounds of $\chi_\ell(G)$ and $AT(G)$ are the same for several graph classes. For example, Zhu proved the following.

Theorem 0.0.9 ([34]) *Let G be a planar graph. Then $AT(G) \leq 5$.*

In Chapter 2,, we focus on the list-chromatic number and DP-chromatic number of planar graphs.

As mentioned above, the gap between the chromatic number and the list-chromatic number can be arbitrary large in general graphs. On the other hand, we see that the one in a planar graph is bounded by the constant by Theorem 0.0.3. Therefore, it seems natural to ask whether there exists a planar graph G such that $\big(\chi(G), \chi_\ell(G)\big) = (i, j)$ for a given pair $(i, j) \in \mathbb{N}^2$. We study this problem and obtain the following.

Proposition 0.0.10 *For $(i, j) \in \mathbb{N}^2$, there exists a plane graph G with $(\chi(G), \chi_\ell(G)) = (i, j)$ if and only if $(i, j) \in \{(1, 1), (2, 2), (2, 3), (3, 3), (3, 4), (3, 5), (4, 4), (4, 5)\}$.*

Moreover, similarly to the case between chromatic number and list-chromatic number, we considered which types of planar graphs exist when we are given a triple $(\chi, \chi_\ell, \chi_{DP})$. We answer this problem as follows.

Theorem 0.0.11 *For $(i, j, k) \in \mathbb{N}^3 - \{(3, 3, 5)\}$, there exists a plane graph G with $(\chi(G), \chi_\ell(G), \chi_{DP}(G)) = (i, j, k)$ if and only if $(i, j, k) \in \{(1, 1, 1), (2, 2, 2), (2, 2, 3),$ $(2, 3, 3), (2, 3, 4), (3, 3, 3), (3, 3, 4), (3, 4, 4), (3, 4, 5), (4, 4, 4), (4, 4, 5), (4, 5, 5)\}$.*

As mentioned above, every planar graph is 4-colorable. Grytczuk and Zhu have asked how many numbers of edges in planar graphs G are need to be removed in order to bound the list-chromatic number of resulting graph is at most 4. They solved this question as follows.

Theorem 0.0.12 ([15]) *Let G be a planar graph. Then there exists a matching M of G such that $AT(G - M) \leq 4$.*

In this context, it seems natural to ask whether there exists a subgraph H of G such that $AT(G - E(H)) \leq 3$. For this problem, Kim, Kim and Zhu proved the following in [18].

Theorem 0.0.13 ([18]) *Let G be a planar graph. Then there exists a forest F in G such that $AT(G - E(F)) \leq 3$.*

A graph H is a *minor* of a connected graph G if we obtain H from G by deleting or contracting some edges recursively. If multiple edges appear by a contraction, we replace them with simple edge. A graph G is *H-minor-free* if H is not a minor of G.

As another extension of Thomassen's result, it was shown in [17] and [25] that every K_5-minor-free graph is 5-choosable. We extend these results from the list-chromatic number to the Alon-Tarsi number.

Theorem 0.0.14 *Let G be a K_5-minor-free graph. Then all of the following hold.*

(i) $AT(G) \leq 5$.

(ii) *There exists a matching M of G such that $AT(G - M) \leq 4$.*

(iii) *There exsits a forest F in G such that $AT(G - E(F)) \leq 3$.*

In Chapter 4, we consider the edge-colorings. A *k-edge-coloring* of a graph G is a map $\varphi : E(G) \to \{1, 2, ..., k\}$ such that for any pair of edges $uv, vw \in E(G)$ sharing an end vertex v, we have $\varphi(uv) \neq \varphi(vw)$. A graph G is *k-edge-colorable* if there exists a k-edge-coloring of G. Moreover, $EC_k(G)$ denotes the set of k-edge-colorings of G. Similarly to the vertex coloring, list-edge-coloring is defined as follows.

A map $L : E(G) \to 2^{\mathbb{N}}$ is called an *edge-list-assignment* (or simply *list*) of G. If G has an edge-coloring φ such that $\varphi(e) \in L(e)$ for any $e \in E(G)$, we say that G is *L-list-edge-colorable* and such an edge-coloring φ is called an L-edge-coloring. If G is L-list-edge-colorable for any list L that satisfies $|L(e)| \geq k$ for any $e \in E(G)$, we say that G is k-list-edge-colorable. Similarly to Proposition 0.0.2, it is easy to see the following.

Proposition 0.0.15 *Let G be a k-list-edge-colorable graph. Then G is k-edge-colorable.*

It seems natural to ask whether the converse of Proposition 0.0.15 holds or not. This question is known as *List Coloring Conjecture* and still open.

Conjecture 0.0.16 ([4]) *(List Coloring Conjecture) Let k be a positive integer. If G is a graph with k-edge-colorable, then G is k-list-edge-colorable.*

It is considered to be extremely difficult to solve Conjecture 0.0.16 completely. On the other hand, Alon and Tarsi invented the tool to solve this conjecture for regular graphs by using Theorem 0.0.6. In order to state this theorem, first we introduce the signature of the k-edge-coloring.

Let G be a k-regular k-edge-colorable graph and let φ be a k-edge-coloring of G. For $i \in \{1, 2, \ldots, k\}$, the set of edges of color i by φ is denoted by $\varphi^{-1}(i)$. For a vertex $v \in V(G)$, we denote by $E(v)$ the set of edges that are incident with v. Let ρ_v be a bijective map from $E(v)$ to $\{1, 2, ..., k\}$. We call $\rho = \{\rho_v : v \in V(G)\}$ a *basis* of G. For a basis ρ, a k-edge-coloring φ, and a vertex v of the graph G, we have a permutation $\pi_v = \varphi \circ \rho_v^{-1}$ of degree k. The signature $\mathrm{sign}(\pi_v)$ is defined as follows.

$$\mathrm{sign}(\pi_v) = \begin{cases} +1 & \text{if } \pi_v \text{ is an even permutation,} \\ -1 & \text{otherwise.} \end{cases}$$

Especially, we might denote $\mathrm{sign}_\rho(\pi_v)$ instead of $\mathrm{sign}(\pi_v)$ when we emphasize the basis ρ. We call $\prod_{v \in V(G)} \mathrm{sign}(\pi_v)$ the *signature of an edge-coloring φ (with respect to ρ)* and denoted by $\mathrm{sign}_\rho(\varphi)$. Alon and Tarsi gave the following interpretation in the edge-coloring of regular graphs.

Theorem 0.0.17 ([2]) *Let G be a k-regular k-edge-colorable graph and ρ be a basis of G. If G satisfies $\sum_{\varphi \in EC_k(G)} \mathrm{sign}_\rho(\varphi) \neq 0$, then G is k-list-edge-colorable.*

From this result, we have the following corollary.

Corollary 0.0.18 ([2]) *Let G be a k-regular k-edge-colorable graph and let ρ be a basis of G. If a k-edge-colorings φ_1 and φ_2 of G satisfy $\mathrm{sign}_\rho(\varphi_1) = \mathrm{sign}_\rho(\varphi_2)$, then G is k-list-edge-colorable.*

Now, let us focus on planar graphs. For the signatures of edge-colorings, the following are known.

Theorem 0.0.19 ([10]) *Let G be a planar k-regular k-edge-colorable graph. Then G satisfies $\mathrm{sign}_\rho(\varphi_1) = \mathrm{sign}_\rho(\varphi_2)$ for a basis ρ and k-edge-colorings φ_1, φ_2.*

By Theorem 0.0.18 and 0.0.19, List Coloring Conjecture holds for planar regular graphs. In this context, we consider the graphs embedded on the projective plane. The big difference between plane and projective plane is that the plane is orientable but the projective plane is not orientable. Moreover, it is easy to see that projective planar graphs might have two edge-colorings with different signatures under a common basis ρ. Thus we introduce the notion called a *type* in edge-colorings of the projective planar graphs, where the types are determined by the topological conditions of G. The definition of the type is defined in Chapter 4. We characterize the signature of edge-coloring of regular graphs on the projective plane by the type of the edge-coloring.

Theorem 0.0.20 *For $k \geq 3$, let G be a k-regular k-edge-colorable graph, let M be a perfect matching of G, and let D be a dual boundary. Moreover, let $\rho[M, D]$ be a basis of G. Then for a k-edge-coloring φ of type-s for M, where $s \in \{0, ..., k\}$, both of following holds.*

(i) If $k \equiv 0, 3 \pmod 4$,

then $$\mathrm{sign}_{\rho[M,D]}(\varphi) = \begin{cases} (-1)^{\frac{|G|}{2} + |M \cap D|} & \text{if } k - s \equiv 0, 1 \pmod 4, \\ (-1)^{\frac{|G|}{2} + |M \cap D| + 1} & \text{otherwise.} \end{cases}$$

(ii) If $k \equiv 1, 2 \pmod 4$*,*

$$\textit{then } \operatorname{sign}_{\rho[M,D]}(\varphi) = \begin{cases} +1 & \text{if } k - s \equiv 0, 1 \pmod 4, \\ -1 & \text{otherwise.} \end{cases}$$

By using Theorem 0.0.20, we find the new graph class which List Coloring Conjecture holds. The details are mentioned in Chapter 4.

In the following, we focus on the applications of the graph colorings. For $S \subset V(G)$, the set of vertices which are adjacent to a vertex of S are denoted by $N(S)$. For $S, T \subset V(G)$, we say that S *dominates* T if $T \subset S \cup N(S)$. If $D \subset V(G)$ dominates $V(G)$, then D is called a *dominating set* of G. The *domination number* of G is the minimum cardinality over all dominating sets of G and denoted by $\gamma(G)$. In Chapter 5, we study the domination number of planar graphs by using graph colorings.

A *disk triangulation* is a 2-connected plane graph such that every face except for the infinite face is triangular. Matheson and Tarjan proved the following theorem by an elegant coloring method.

Theorem 0.0.21 (Matheson and Tarjan [21]) *Let G be a disk triangulation with n vertices. Then $\gamma(G) \leq \lfloor \frac{n}{3} \rfloor$.*

They constructed a disk triangulation with n vertices in which any dominating sets have cardinality at least $\lfloor \frac{n}{3} \rfloor$, and hence the upper bound in Theorem 0.0.21 is best possible. The examples they constructed are *maximal outerplanar graphs*, (i.e., a 2-connected plane graph such that there is a single face f containing all vertices on the boundary cycle, and that every face other than f is triangular). After that, Campos and Wakabayashi [7] pointed out that maximal outerplanar graphs with a large domination number have many vertices of degree 2, and they (and Tokunaga independently) proved the following theorem.

Theorem 0.0.22 (Campos and Wakabayasi [7] and Tokunaga [28]) *Let G be a maximal outerplanar graph with n vertices and t vertices of degree 2. Then $\gamma(G) \leq \lfloor \frac{n+t}{4} \rfloor$, where the bound is sharp.*

We introduce an "annulus triangulation" and consider its domination number. An *annulus triangulation* is a 2-connected plane graph with two disjoint special faces f_1 and f_2 such that every face of G except for f_1 and f_2 are triangular, and that every vertex of G is contained in the boundary cycle of f_1 or f_2. This seems to be a natural extension of maximal outerplanar graphs.

A big difference between maximal outerplanar graphs and annulus triangulations is that an annulus triangulation G is not necessarily 3-colorable, and that G might not have vertices of degree 2. We elaborate a coloring method in [21, 28] and prove the following Theorem.

Theorem 0.0.23 *Let G be an annulus triangulation with n vertices and t vertices of degree 2. If $n \geq 7$, then $\gamma(G) \leq \lfloor \frac{n+t+1}{4} \rfloor$, where this bound is sharp.*

Chapter 1

Foundation

1.1 Graphs

In this section, we will give several terminologies for graphs. A *graph* consists of points, called *vertices* and arcs, called *edges*, each of which joins a pair of vertices. For a graph G, we denote a vertex set $V(G)$ and edge set $E(G)$ respectively. A graph can be regarded as one of mathematical models which expresses such structures of finite sets with some relations. We represent an edge of G as $e = uv$ if e is joined by the pair of vertices u and v. For an edge e, we say the vertex u is *endpoint* of e if $uv = e \in E(G)$. If uv is an edge of G, then u and v are *adjacent*. Two adjacent vertices are *neighborhood* of each other. The set of neighborhood of v, denoted by $N_G(v)$ or simply $N(v)$, is called the *open neighborhood* of v (or simply neighborhood of v). In particular, we call the set $N[v] = \{v\} \cup N(v)$ the *closed neighborhood* of v. Moreover, for $S \subset V(G)$, let $N(S)$ denote the *neighborhood* of S, i.e., the set of vertices adjacent to a vertex of S in G. The vertex u and an edge $e = uv$ are *incident* to each other. If an edge e joins a vertex itself, that is, for $e = uv$ and $u = v$, then it is called a *loop*. A graph G is called *loopless* if G dose not have any loop edges. Edges which join the same pair of vertices are called *multiple edges*. A graph is said to be *simple* if G contains no loop nor multiple edges. The *degree* of a vertex v is the number of edges incident with v, denoted by $\deg_G(v)$ or simply $\deg(v)$ and we call v a vertex of degree k or k-vertex when $\deg_G(v) = k$. Moreover, a graph G is *Eulerian* if each vertex is even degree. The *maximum degree* (*minimum degree* respectively) of a graph G, denoted by $\Delta(G)$ ($\delta(G)$ respectively) is defined as follows.

$$\Delta(G) := \max\{\deg_G(v) : v \in V(G)\}$$

$$\delta(G) := \min\{\deg_G(v) : v \in V(G)\}$$

A graph is k-regular if each vertex has degree k.

Two simple graphs G and H are *isomorphic*, denoted by $G \simeq H$, if there exists a bijective mapping $\sigma : V(G) \to V(H)$ such that for $uv \in E(G)$ if and only if $\sigma(u)\sigma(v) \in E(H)$.

For two vertices u and v, a *walk* W is a sequence of vertices beginning with u and ending at v such that consecutive vertices in W are adjacent in G. Such a walk can be expressed as

$$W = v_1 v_2 ... v_k$$

where $v_1 = u$, $v_k = v$ and $v_i v_{i+1} \in E(G)$ for each $1 \leq i \leq k-1$. A walk W is called *path* when no vertices appear twice in W. A walk W is said to be *closed* if $u = v$ and $k \geq 4$. A closed walk $W = v_1 ... v_k$ for $k \geq 2$ is called a *cycle* if all v_i for $1 \leq i \leq k-1$ are distinct. The *length* of a walk W is the number of edges contained in W. A cycle of length k is called a *k-cycle*. For a given graph G, a graph H is a *subgraph* of G if $V(H) \subset V(G)$ and $E(H) \subset E(G)$. In particular, a subgraph H is said to be induced if $V(H) = S$ and $E(H)$ consists of the edges of G whose ends are both in S. For a given graph G and subset S of $V(G)$, $G - S$ denotes a subgraph of G induced by $V(G) - S$. A subgraph H of G is said to be *spanning* if $V(H) = V(G)$. A graph G is connected if for each pair of distinct vertices u, v, there exists a path between u and v. On the other hand, if there is a pair of distinct vertices u and v such that there are no paths between u and v, we say a graph is *disconnected*. A *vertex-cut* of a graph G is a set of vertices of G such that $G - S$ is disconnected. In particular, a vertex-cut S with $|S| = k$ is said to be a *k-cut*. If a graph G is not complete graph, then G has a vertex cut. The *vertex connectivity*, denoted by $\kappa(G)$ is the minimum cardinality of the vertex cut of G if G is not complete and it is $n-1$ if G is a complete graph with n vertices. A graph G with $|V(G)| \geq k$ is k-connected if $\kappa(G) \leq k$. An *edge-cut* of a graph G is a set X of edges of G such that $G - X$ is disconnected. An edge-cut of minimum cardinality in G is called *minimum edge-cut* of G and this cardinality is called the *edge-connectivity* of G and it is denoted by $\kappa_e(G)$. A graph G with $|E(G)| \geq k$ is k-edge-connected if $\kappa_e(G) \geq k$.

A vertex k-coloring of a graph G is a mapping $c : V(G) \to \{1, ..., k\}$. A coloring c is called *proper* if $c(x) \neq c(y)$ for each $xy \in E(G)$. We often refer to a k-coloring instead of a proper vertex k-coloring. A graph G is *k-colorable* if G admits a k-coloring. It is not hard to see that a graph G has a $|V(G)|$-coloring. The *chromatic number* of G, denoted by $\chi(G)$ is the minimum number k such that G is k-colorable. In particular, G is said to be *k-chromatic* if $\chi(G) = k$.

1.2 Embedding a graph on a surface

In this section, we will give some terminologies for graphs on surfaces. Throughout this thesis, we call a connected 2-dimensional manifold without boundaries a *closed surface*. Let D be a disk, that is, $\{(x, y) \in \mathbb{R}^2 : x^2 + y^2 \leq 1\}$. If we identify the antipodals of the boundary of D, then we obtain a closed surface $\mathbb{P}^2$. We call this surface $\mathbb{P}^2$ *projective plane*. If we can distinguish clockwise and counter clockwise orientations around all the points on it, then we say $\mathbb{F}$ is *orientable*. Otherwise $\mathbb{F}$ is *non-orientable*. For example, the plane is orientable and the projective plane is not orientable.

A *closed curve* on a surface $\mathbb{F}$ is a continuous function $l : S^1 \to \mathbb{F}$ or its image, where S^1 is the 1-dimensional sphere, that is, $\{(x, y) \in \mathbb{R}^2 : x^2 + y^2 = 1\}$. A closed curve is said to be *simple* if the function l is an injection. A simple closed curve is called *separating* (resp. *non-separating*) if $\mathbb{F} - l$ is disconnected (resp. connected). A simple closed curve is called *contractible* if l is bounds a 2-cell on $\mathbb{F}$. Two closed curve l_1 and l_2 is said to be *homotopic* to each other on $\mathbb{F}$ if there exists a continuous function $\Phi : [0, 1] \times S^1 \to \mathbb{F}$ such that $\Phi(0, x) = l_1(x)$ and $\Phi(1, x) = l_2(x)$ for each $x \in S^1$.

For a curve on the plane, we introduce *Jordan Curve Theorem* and *Shönflies Theorem.*

Theorem 1.2.1 (Jordan Curve Theorem) *Any simple closed curve C on the plane divides into exactly two connected components, the interior and the exterior. Both of these region have the common boundary C.*

Theorem 1.2.2 (Shönflies Theorem) *The interior of any two simple closed curve on the plane is homotopic to an open 2-cell.*

When we discuss embedding of graphs into surfaces mathematically, we regard graphs as 1-dimensional topological spaces, not only as combinatorial objects. Let $\mathbb{F}$ be a closed surface, let G be a graph and let $f : G \to \mathbb{F}$ is an injective and continuous mapping. We say f is an *embedding* of G. We often deal with G and $f(G)$ as the same objects and denote embedded graph by G. On the other hand, in order to distinguish G from the embedded one $f(G)$, we call G an *abstract graph* while we call $f(G)$ an *embedding* or a *map*. For an embedded graph G on $\mathbb{F}$, each region which is bounded by vertices and edges is called a *face*. We denote the set of faces of G by $F(G)$.

A graph G is said to be *planar* if G is embeddable on the plane. Similarly, A graph G is said to be *projective planar* if G is embeddable on the projective plane.

For a given graph G embedded on a surface $\mathbb{F}$, the *dual graph* (or simply *dual*) of G is defined as follows: A vertex is placed on each face of G and two distinct vertices are joined by an edge for each common edge on the boundaries of the two corresponding faces of G. Lastly, by deleting G, we obtain the dual graph of G.

1.3 Polynomials

Let X be a set and let $+$ and $\cdot$ be binary operations of X. We say $(X, +, \cdot)$ or simply X is a *field* if the triple $(X, +, \cdot)$ satisfies all of the following.

(i) For arbitrary $a, b, c \in X$, $(a + b) + c = a + (b + c)$.

(ii) There exists an element $0 \in X$ such that $a + 0 = a$ for any $a \in X$.

(iii) For arbitrary $a \in X$, there exists $(-a) \in X$ such that $a + (-a) = 0$.

(iv) For arbitrary $x, y \in X$, $x + y = y + x$.

(v) For arbitrary $a, b, c \in X$,$(a \cdot b) \cdot c = a \cdot (b \cdot c)$.

(vi) There exists an element $1 \in X$ such that $a \cdot 1 = a$ for any $a \in X$.

(vii) For arbitrary $a \in X$ except for 0, there exists $(a^{-1}) \in X$ such that $a \cdot (a^{-1}) = 1$.

(viii) For arbitrary $x, y \in X$, $x \cdot y = y \cdot x$.

(ix) For arbitrary $x, y, z \in X$, $x \cdot (y + z) = x \cdot y + x \cdot z$ and $(x + y) \cdot z = x \cdot z + y \cdot z$.

$\mathbb{R}$ denotes the set of real numbers. The following fact is well-known.

Proposition 1.3.1 $\mathbb{R}$ *is a field.*

For a given field F, a *polynomial* f as a variable $x = (x_1, ..., x_m)$ on F is defined as follows,

$$f(x_1, ..., x_m) := \sum_{0 \le i_1, ..., i_m} a_{i_1,...,i_m} x_1^{i_1} x_2^{i_2} ... x_m^{i_m}$$

where $a_{i_1,...,i_m} \in F$ for all nonnegative integers $i_1, ..., i_m$. For a given polynomial, each $x_1^{i_1} x_2^{i_2} ... x_m^{i_m}$ is said to be a *term* and $a_{i_1,...,i_m}$ is called the *coefficient* of $x_1^{i_1} x_2^{i_2} ... x_m^{i_m}$. The *degree* of a term $x_1^{i_1} x_2^{i_2} ... x_m^{i_m}$ is defined by $i_1 + ... + i_m$. The *degree* of f is defined as follows.

$$\deg(f) := \max \left\{ i_1 + ... + i_m : x_1^{i_1} x_2^{i_2} ... x_m^{i_m} \text{ is a term of } f \text{ and } a_{i_1,...,i_m} \neq 0 \right\}.$$

We note that $\deg(0) = -\infty$. The set of polynomials on F with m variables is denoted by $F[x_1, ..., x_m]$. We say a polynomial f is *homogeneous* if the degree of each term $x_1^{i_1} x_2^{i_2} ... x_m^{i_m}$ with its coefficient $a_{i_1,...,i_m} \neq 0$ is exactly $\deg(f)$.

Chapter 2

Colorings for planar graphs.

In this chapter, we introduce list-coloring and DP-coloring. Then we focus on the differences between chromatic number, list-chromatic number and DP-chromatic number for planar graphs. In Section 2.2, we examine which types of plane graphs exist when we are given a triple $(\chi, \chi_\ell, \chi_{DP})$. In Section 2.3, we prove Theorem 2.2.9.

2.1 Introduction

One of the interesting variations of the graph coloring problems is the list coloring, which was introduced independently by Vizing [29] and Erdős, Rubin and Taylor [12].

We associate a *list assignment,* L, with a graph G such that each vertex v is assigned a list of colors $L(v)$. The graph G is L-colorable if G has a proper coloring c such that $c(v) \in L(v)$ for each vertex v. Let $f : V(G) \to \mathbb{N}$ be a function and let k be a positive integer. We say that G is f*-choosable* if G is L-colorable for every list assignment L such that $|L(v)| \geq f(v)$ for every vertex v. Especially, we say that G is k*-choosable* if G is f-choosable where f is the constant function taking the value k. The *list-chromatic number* of G, denoted by $\chi_\ell(G)$, is the minimum integer k such that G is k-choosable. It is easy to see that the following holds.

Proposition 2.1.1 *Let G be a graph. Then $\chi(G) \leq \chi_\ell(G)$.*

The converse of the inequality of Proposition 2.1.1 does not hold since the complete bipartite graph $K_{2,4}$ satisfies $\chi_\ell(K_{2,4}) = 3$. Moreover, it is also shown that the gap between χ and χ_ℓ can be arbitrary large [12].

Next, we define the *DP-coloring*. This notion was introduced by Dvořák and Postle [9]. Suppose that G is a graph and L is a list assignment of G. For each edge uv in G, let M_{uv} be a matching between the sets $\{u\} \times L(u)$ and $\{v\} \times L(v)$. Moreover, let $\mathscr{M}_L = \{M_{uv} : uv \in E(G)\}$, which we call the *matching assignment*. Let $H = H(G, L, \mathscr{M}_L)$ be the graph that satisfies all of the following conditions:

(i) $V(H) = \{(w, a) : w \in V(G), a \in L(w)\}$,

(ii) For each $w \in V(G)$ and distinct $a, b \in L(w)$, $(w,a)(w,b) \in E(H)$,

(iii) if $uv \in E(G)$, then the set of edges between $\{u\} \times L(u)$ and $\{v\} \times L(v)$ are M_{uv}, and

(iv) if $uv \notin E(G)$, then there are no edges between $\{u\} \times L(u)$ and $\{v\} \times L(v)$.

If H contains an independent set of size $|V(G)|$, then G is *$\mathscr{M}_L$-colorable*. The graph G is *DP-k-colorable* if G is $\mathscr{M}_L$-colorable for any list assignment L with $|L(v)| \geq k$ for every $v \in V(G)$ and for any matching assignment $\mathscr{M}_L$. The minimum integer k such that G is DP-k-colorable is the *DP-chromatic number* of G, denoted by $\chi_{DP}(G)$.
Let G be a graph and let L be a list assignment. If we take the matching assignment as $M_{uv} = \{(u,a)(v,b) : a \in L(u), b \in L(v), a = b\}$ for every $uv \in E(G)$, then the $\mathscr{M}_L$-coloring coincides with the L-coloring. Therefore, we have the following.

Proposition 2.1.2 *Let G be a graph. Then $\chi_\ell(G) \leq \chi_{DP}(G)$.*

Since $\chi_\ell(C_{2n}) = 2$ and $\chi_{DP}(C_{2n}) = 3$, the converse of inequality does not hold. (See Figure 2.1.)

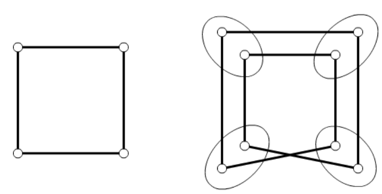

Figure 2.1: C_4 is not DP-2-colorable. The left is C_4 and the right denotes a graph $H(C_4, L, \mathscr{M}_L)$, which does not have an independent set of size $|C_4|$.

In what follows, we focus on planar graphs as a restricted graph class. By a celebrated result of Appel and Haken, the chromatic number of a planar graph is at most 4 [5] and one may think that list-chromatic number of every planar graph is also bounded by 4. However, the latter one is not true and Voigt [30] constructed a planar graph G with $\chi(G) = 4$ and $\chi_\ell(G) = 5$ in 1993. As mentioned above, the gap between chromatic number and list-chromatic number can be arbitrary large in general graphs. On the other hand, the one in a planar graph is bounded by the constant. Therefore, it seems natural to ask whether there exists a planar graph G such that $\big(\chi(G), \chi_\ell(G)\big) = (i, j)$ for a given pair $(i, j) \in \mathbb{N}^2$.

This problem can be solved as follows. For bipartite plane graphs, Alon and Tarsi [3] proved that their list-chromatic numbers are at most 3 and this upper bound is tight since $K_{2,4}$ satisfies $\chi_\ell(K_{2,4}) = 3$. Moreover, for non-bipartite planar graphs, Gutner [16] constructed a planar graph G with $(\chi(G), \chi_\ell(G)) = (3, 5)$ and we can construct the planar graph G with $(\chi(G), \chi_\ell(G)) = (4, 5)$ from this result. Thomassen [27] proved that every planar graph has list-chromatic number at most 5. Thus the left cases are only $(\chi, \chi_\ell) = (3,3), (3,4), (4,4)$. It is easy to see that $(\chi(K_3), \chi_\ell(K_3)) = (3,3)$ and $(\chi(K_4), \chi_\ell(K_4)) = (4,4)$. Finally, we constructed the graph G with $(\chi_G, \chi_\ell) = (3,4)$ in Proposition 2.2.8. Thus we have the following.

Proposition 2.1.3 *For $(i,j) \in \mathbb{N}^2$, there exists a plane graph G with $(\chi(G), \chi_\ell(G)) = (i,j)$ if and only if $(i,j) \in \{(1,1), (2,2), (2,3), (3,3), (3,4), (3,5), (4,4), (4,5)\}$.*

It is shown in [12] that the gap between χ and χ_ℓ can be arbitrary large in general graphs. However, we have that the one in planar graphs is at most 2 from the above discussions.

In this paper, we consider which types of planar graphs exist when we are given a triple $(\chi, \chi_\ell, \chi_{DP})$ and answer this problem except for the case $(\chi, \chi_\ell, \chi_{DP}) = (3,3,5)$.

Theorem 2.1.4 *For $(i,j,k) \in \mathbb{N}^3 - \{(3,3,5)\}$, there exists a plane graph G with $(\chi(G), \chi_\ell(G), \chi_{DP}(G)) = (i,j,k)$ if and only if $(i,j,k) \in \{(1,1,1), (2,2,2), (2,2,3), (2,3,3), (2,3,4), (3,3,3), (3,3,4), (3,4,4), (3,4,5), (4,4,4), (4,4,5), (4,5,5)\}$.*

2.2 A study of the existence of planar graphs

A graph G is *k-degenerate* if each nonempty subgraph of G has a vertex of degree at most k. For the upper bound of the DP-chromatic number, it is easy to see the following.

Proposition 2.2.1 *Let G be a k-degenerate graph. Then $\chi_{DP}(G) \leq k+1$.*

Moreover, Dvořák and Postle observed the following.

Theorem 2.2.2 ([9]) *Let G be a planar graph. Then $\chi_{DP}(G) \leq 5$.*

In the following, we divide the study of the existence of graphs by the chromatic number.

2.2.1 Bipartite planar graphs

At first, we focus on bipartite plane graphs.

By Euler's formula, every bipartite plane graph is 3-degenerate. Thus $\chi_{DP}(G) \leq 4$ for a bipartite plane graph G by Proposition 2.2.1. Since $\chi_{DP}(C_n) = 3$ for any $n \geq 3$ and every tree is 1-degenerate, it is easy to see the following.

Proposition 2.2.3 *Let G be a graph. Then $\chi_{DP}(G) \leq 2$ if and only if G is a forest.*

For graphs with list-chromatic number 2, Erdős et al. [12] gave the following characterization.

Theorem 2.2.4 ([12]) *Let G be a connected graph. Then $\chi_\ell(G) \le 2$ if and only if the core of G is isomorphic to K_1, C_{2n+2} or $\theta_{2,2,2n}$, where n is a positive integer.*

The *core* of a graph G is the resulting graph from G by deleting the vertices of degree 1 recursively. It is a direct consequence of Theorem 2.2.4 that all the graphs with list-chromatic number 2 are 2-degenerate and hence we have the following corollary.

Corollary 2.2.5 *There are no graphs G with $\chi_\ell(G) = 2$ and $\chi_{DP}(G) \ge 4$.*

Thus 2-list-chromatic graphs have DP-chromatic number at most 3. For example, the even cycle C_{2n} satisfies $(\chi, \chi_\ell, \chi_{DP}) = (2, 2, 3)$.
Next, we consider the graphs with list-chromatic number 3. Since every bipartite plane graph is 3-degenerate, its DP-chromatic number is at most 4 and hence we may only consider the cases $(\chi, \chi_\ell, \chi_{DP}) = (2, 3, 3), (2, 3, 4)$. For instance, $K_{2,4}$ satisfies the former one. Moreover, the latter one was constructed by Bernshteyn and Kostochka [6].
Finally, we note that there are no planar graphs with $(\chi, \chi_\ell) = (2, 4)$ since Alon and Tarsi [3] showed that each bipartite plane graph has list-chromatic number at most 3. Therefore we have the following table for bipartite plane graphs.

	$\chi_\ell = 2$	$\chi_\ell = 3$
$\chi_{DP} = 2$	$\circ$	$\emptyset$
$\chi_{DP} = 3$	$\circ$	$\circ$
$\chi_{DP} = 4$	$\emptyset$	$\circ$

Table 2.1: The existences of bipartite plane graphs. The symbol "$\circ$" shows that there is such a graph with corresponding χ_ℓ and χ_{DP} and "$\emptyset$" represents that no such graphs exist.

2.2.2 3-chromatic planar graphs

In this subsection, we focus on 3-chromatic planar graphs.
At first, we consider the graphs with list-chromatic number 3. It is easy to show that K_3 satisfies $\big(\chi(K_3), \chi_\ell(K_3), \chi_{DP}(K_3)\big) = (3, 3, 3)$. Moreover, it is not hard to show the following.

Proposition 2.2.6 *The octahedron satisfies $(\chi, \chi_\ell, \chi_{DP}) = (3, 3, 4)$.*

Proof. Let G be the octahedron as shown in Figure 2.2. First we show that G is 3-choosable. If the list assignments $L(v_1)$ and $L(v_4)$ have a common color, say a, then we color v_1 and v_4 by a. On the other hand, if $L(v_1)$ and $L(v_4)$ are disjoint, without loss of generality, let $L(v_1) = \{1, 2, 3\}$ and let $L(v_4) = \{4, 5, 6\}$. For each v_i for $i \in \{2, 3, 5, 6\}$, we have $\min\{|L(v_i) \cap L(v_1)|, |L(v_i)| \cap L(v_4)|\} \le 1$. Thus we obtain at most two pairs of colors

(a, b) such that $a \in L(v_1) \cap L(v_i)$ and $b \in L(v_4) \cap L(v_i)$ for each $i \in \{2, 3, 5, 6\}$. Since $|L(v_1) \times L(v_4)| = 9$, we have a pair $(s, t) \in L(v_1) \times L(v_4)$ such that $|L(v_i) - \{s, t\}| \geq 2$ for each $i \in \{2, 3, 5, 6\}$. In both cases, the vertices v_i have at least two available colors in their lists and the graph induced by them form the cycle of length 4. Thus we conclude that G is 3-choosable.

Next, we will show $\chi_{DP}(G) = 4$. At first, in order to show that $\chi_{DP}(G) \leq 4$, we color the vertex v_1 arbitrarily. In this situation, the vertex v_i has at least three available colors for each $i \in \{2, 3, 5, 6\}$ and the vertex v_4 has 4 available colors. Thus there exists a color $c(v_4)$ in $L(v_4)$ which is not covered by a matching assignment $M_{v_2v_4}$. In this situation, we can color the vertices greedy v_4, v_6, v_5, v_3 and v_2 as in this order. Next, we prove that there exists a list assignment L and a matching-assignment $\mathscr{M}$ which implies that G is not DP-3-colorable. We give the list $L(v) = \{0, 1, 2\}$ for each $v \in V(G)$. Moreover, we assign $+1$ to the edges v_iv_j if $(i, j) = (4, 5), (4, 6)$ and assign $+0$ to any other edges. Then we give the matching assignment $\mathscr{M}$ to each edge depending on the assigned number as shown in Figure 2.2. It is easy to check that the vertices v_1, v_2 and v_3 receive three distinct colors and hence the vertices v_4, v_5 and v_6 must be also colored by different colors. If $c(v_4) = 0$, then we have $c(v_5) = 1$ or $c(v_6) = 1$ and this contradicts the matching assignment in the edges v_4v_5 and v_4v_6 respectively. Similarly, we get the contradictions in either cases with $c(v_4) = 1$ or $c(v_4) = 2$. $\square$

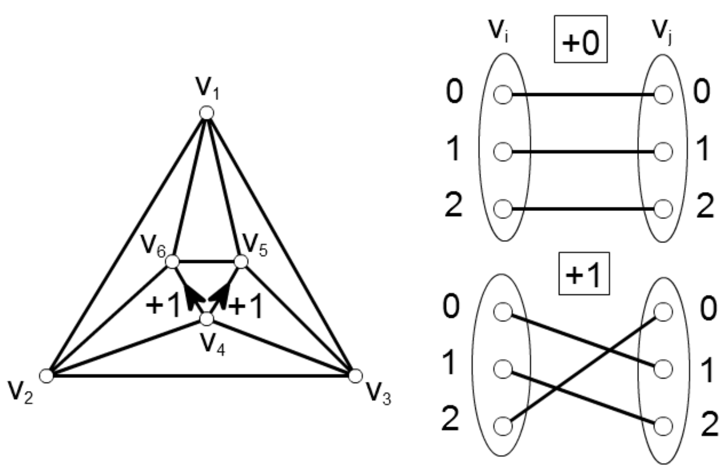

Figure 2.2: The edges v_iv_j are assigned $+1$ if $(i, j) = (4, 5), (4, 6)$ and any other edges are assigned $+0$.

Planar graphs with $(\chi, \chi_\ell, \chi_{DP}) = (3, 3, 3), (3, 3, 4)$ have been already constructed. Therefore, in this context, it is natural to ask whether there exists a graph with $(\chi, \chi_\ell, \chi_{DP}) = (3, 3, 5)$. We conjecture as follows.

Conjecture 2.2.7 *There are no planar graphs with* $(\chi, \chi_\ell, \chi_{DP}) = (3, 3, 5)$.

Since every planar graph has DP-chromatic number at most 5 by Theorem 2.2.2, there are no graphs with $(\chi, \chi_\ell, \chi_{DP}) = (3, 3, k)$, where $k > 5$.

Next, we focus on the graphs with chromatic number 3 and list-chromatic number 4. A graph with $(\chi, \chi_\ell, \chi_{DP}) = (3, 4, 4)$ is easily obtained as follows.

Proposition 2.2.8 *Let G_i be the graph as shown in Figure 2.3, where $i \in \{1, 2, 3\}$. Moreover, let G be the graph so that $G = \bigcup_i G_i$ and $V(G_j) \cap V(G_k) = \{v\}$ for distinct $j, k \in \{1, 2, 3\}$. Then G satisfies $(\chi, \chi_\ell, \chi_{DP}) = (3, 4, 4)$.*

Proof. Similarly to the proof of Proposition 2.2.6, we have $\chi_{DP}(G) \leq 4$. So it suffices to show that $\chi_\ell(G) \geq 4$. let L_i be a list assignment of G_i as shown in Figure 2.3, where $i \in \{1, 2, 3\}$. Without loss of generality, we color the vertex v by 1. Then we focus on the subgraph G_1. Since the cycle $s_{11}s_{12}s_{13}s_{14}$ has 4 colors for any L_i-colorings, we cannot color the vertex s_{15}. $\square$

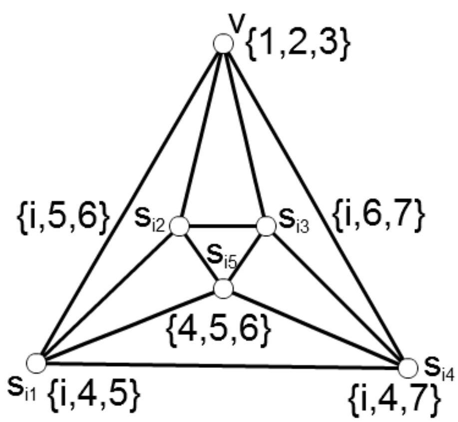

Figure 2.3: The graph G_i and the list assignment L_i.

Thus, it is natural to ask whether there exists a planar graph with $(\chi, \chi_\ell, \chi_{DP}) = (3, 4, 5)$. We answer this problem affirmatively as follows and the proof is written in Section 3.

Theorem 2.2.9 *There exists a plane graph with $(\chi, \chi_\ell, \chi_{DP}) = (3, 4, 5)$.*

Finally, we focus on the graphs with list-chromatic number 5. In this case, Gutner [16] constructed the graphs with $(\chi, \chi_\ell, \chi_{DP}) = (3, 5, 5)$. Therefore, we have the following table for the graphs with $\chi(G) = 3$.

2.2.3 4-chromatic planar graphs

Finally, we focus on the graphs with chromatic number 4. For example, K_4 satisfies $(\chi, \chi_\ell, \chi_{DP}) = (4, 4, 4)$. Moreover, a graph H with $(\chi(H), \chi_\ell(H), \chi_{DP}(H)) = (4, 4, 5)$ is obtained as follows. Let G be the graph in Theorem 2.2.9 and let v be the vertex shown in Figure 2.5. Let H be the graph with $V(H) = V(G) \cup V(K_4)$, $V(G) \cap V(K_4) =$

	$\chi_\ell = 3$	$\chi_\ell = 4$	$\chi_\ell = 5$
$\chi_{DP} = 3$	$\circ$	$\emptyset$	$\emptyset$
$\chi_{DP} = 4$	$\circ$	$\circ$	$\emptyset$
$\chi_{DP} = 5$	?	$\circ$	$\circ$

Table 2.2: The table for 3-chromatic graphs.

$\{v\}$ and $E(H) = E(G) \cup E(K_4)$. Since $(\chi(G), \chi_\ell(G), \chi_{DP}(G)) = (3, 4, 5)$, we have $(\chi(H), \chi_\ell(H), \chi_{DP}(H)) = (4, 4, 5)$. Furthermore, Voigt [30] constructs a graph with $(\chi, \chi_\ell, \chi_{DP}) = (4, 5, 5)$. Therefore, we obtain the following table.

	$\chi_\ell = 4$	$\chi_\ell = 5$
$\chi_{DP} = 4$	$\circ$	$\emptyset$
$\chi_{DP} = 5$	$\circ$	$\circ$

Table 2.3: The table for 4-chromatic graphs.

2.3 Proof of Theorem 2.2.9

2.3.1 Prepare for the proof.

First, in order to show the proof of Theorem 2.2.9, we have following lemmas.

Lemma 2.3.1 *Let G be the graph shown in Figure 2.4 and let L be the list assignment such that $L(v_1) = \{a, b, c, d\}, L(v_2) = \{a\}, L(v_3) = \{a, b, c\}, L(v_4) = \{b, c, d\}$ and $L(v_5) = \{a, b, d\}$. Then for any L-coloring of G, the vertices v_3 and v_5 receive the color b.*

Proof. If the cycle $C = v_2v_3v_4v_5$ has 4 colors, then we cannot color v_1. Thus, C has at most 3 colors and hence there exists two vertices which receive the same color. Since v_2 and v_4 does not have common colors, v_3 and v_5 receive the same color, which must be b. $\square$

Lemma 2.3.2 *Let G be a graph shown in the right of Figure 2.4 and let L be a list assignment such that $|L(v_1)| = 4, |L(v_2)| = 1$ and $|L(v_3)| = |L(v_4)| = |L(v_5)| = 3$. Then G is L-colorable.*

Proof. First, we color v_2 with its single available color. Then we may assume that the lists of available colors for v_1, v_3, v_4, and v_5 are of sizes 3, 2, 3, and 2 respectively. Since the number of available colors for v_1 exceeds that for v_3, we can assign a color to v_1 that does not correspond to any of the colors available to v_3. Now, the lists of v_3, v_4, and v_5 have sizes 2, 2, and 1 respectively. Thus we can color the vertices v_5, v_4, and v_3 greedily (in this order).$\square$

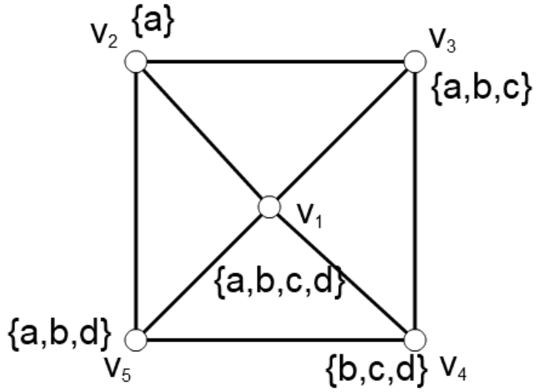

Figure 2.4: The graph and list assignment in Lemma 2.3.1 and Lemma 2.3.2.

2.3.2 Proof of Theorem 2.2.9

Proof of Theorem 2.2.9 At first, we construct the graph G_i for $i \in \{1, 2, 3, 4\}$ as shown in Figure 2.5. All semi-edges in G_i are incident with the vertex v.

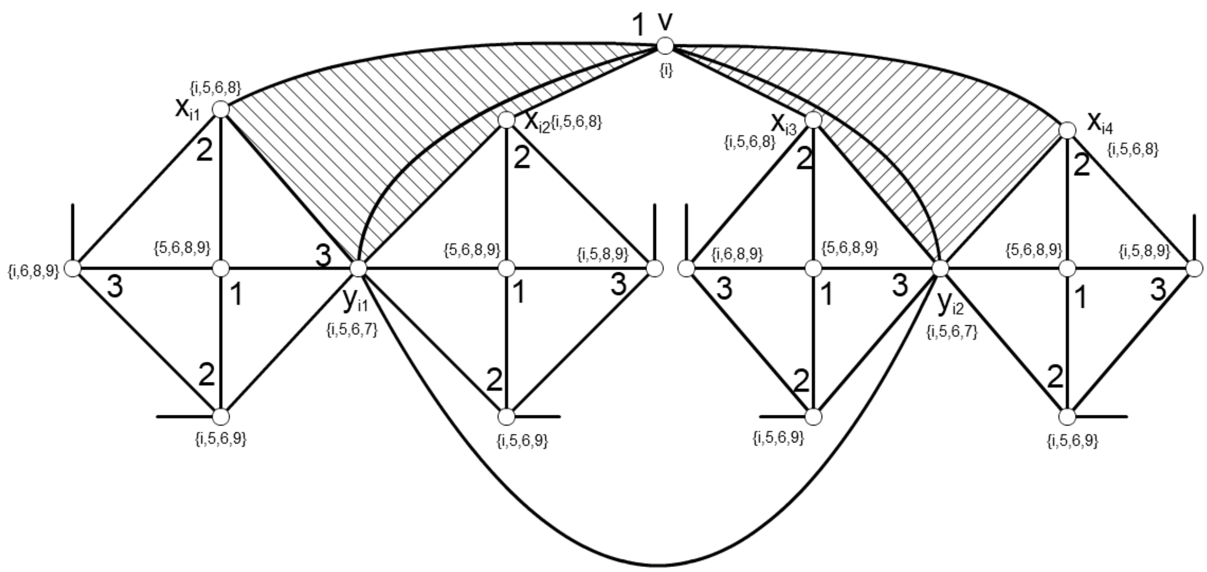

Figure 2.5: The graph G_i. All semi-edges are incident with the vertex v.

Then we have the following Claim.

Claim 1 *Let G_i be the graph and let L_i be the list assignment of G_i as shown in Figure 2.5. Moreover, let L be a list assignment of G_i such that $|L(v)| = 1$ and $|L(z)| = 4$ for $z \in V(G_i) - \{v\}$. Then all of the followings hold.*

(i) G_i is 3-colorable.

(ii) G_i is L-colorable.

(iii) For every L_i-coloring c satisfies $c(x_{i1}) = 6$, $c(x_{i2}) = 5$,$c(x_{i3}) = 6$ or $c(x_{i4}) = 5$.

Proof of Claim 1 For part (i), we give the 3-coloring as shown in Figure 2.5.
For part (ii), we assign to v the unique color from $L(v)$. After removing this color from the lists of its neighbors, we may shrink the lists as follows.

- $|L(y)| = 1$ for $y \in \{y_{i1}, y_{i2}\}$,
- $|L(u)| = 3$ for u, where $u \notin \{y_{i1}, y_{i2}\}$ is adjacent to v, and
- $|L(w)| = 4$ for w, where w is not adjacent to v.

Then we apply Lemma 2.3.1 and obtain an L-coloring.
For part (iii), for any L_i-coloring c, y_{i1} or y_{i2} is colored by 5 or 6. Now we assume $c(y_{i1}) \in \{5, 6\}$. If $c(y_{i1}) = 5$, then we have $c(x_{i1}) = 6$ by Lemma 2.3.1. On the other hand, if $c(y_{i1}) = 6$, then we have $c(x_{i2}) = 5$. For the case $c(y_{i2}) \in \{5, 6\}$, we have $c(x_{i3}) = 5$ or $c(x_{i4}) = 6$ by the symmetric argument as above. ■

Next, we insert the graph shown in Figure 2.6 into each triangular face $x_{ik}y_{i1}v$ for $k \in \{1, 2\}$ and $x_{ik}y_{i2}v$ for $k \in \{3, 4\}$ (the shaded areas in Figure 2.5). The resulting graph is denoted by G'_i. For the graph G'_i, we have the following.

Claim 2 *All of the followings hold.*

(i) G'_i has a 3-coloring c such that $c(s_{ik1}) = c(s_{ik4})$ and $c(s_{ik2}) = c(s_{ik3})$.

(ii) G'_i is L-colorable for a list assignment L such that $|L(v)| = 1$ and $|L(z)| = 4$ for $z \in V(G'_i) - \{v\}$.

(iii) There exists a list assignment L'_i of G'_i which satisfies all of the following.

- *$|L'_i(v)| = 1$ and $|L'_i(z)| = 4$ for $z \in V(G'_i) - \{v\}$.*
- *For any L_i-coloring c, there exists an integer k such that $c(s_{ik1}) = c(s_{ik4}) = 7$ and $c(s_{ik2}) = c(s_{ik3}) = 8$.*

Proof of Claim 2. First, we color the vertices of $V(G_i)$ by Claim 1. Without loss of generality, we may assume $c(v) = 1, c(x_{ik}) = 2$ and $c(y_{ij}) = 3$. Then we color the vertices of $V(G'_i)$ as in Figure 2.6.
For part (ii), at first we give an L-coloring to the vertices contained in $V(G_i)$ by Claim 1 (ii). Since each vertex of $V(G'_i) - V(G_i)$ which is adjacent to v has at least three available colors, we can apply Lemma 2.3.2 to the graphs which are induced by $W_{ik1} = \{x_{ik}, s_{ik1}, t_{ik1}, u_{ik1}, r_{ik1}\}$ or $W_{ik4} = \{x_{ik}, s_{ik4}, t_{ik4}, u_{ik4}, r_{ik4}\}$. After that, we apply Lemma 2.3.2 to the graphs which are induced by $W_{ik2} = \{s_{ik1}, s_{ik2}, t_{ik2}, u_{ik2}, r_{ik2}\}$ or $W_{ik3} = \{s_{ik4}, s_{ik3}, t_{ik3}, u_{ik3}, r_{ik3}\}$ and we obtain a desired L-coloring.
For part(iii), at first we give the list assignment L'_i so that $L'_i(w) = L_i(w)$ for each

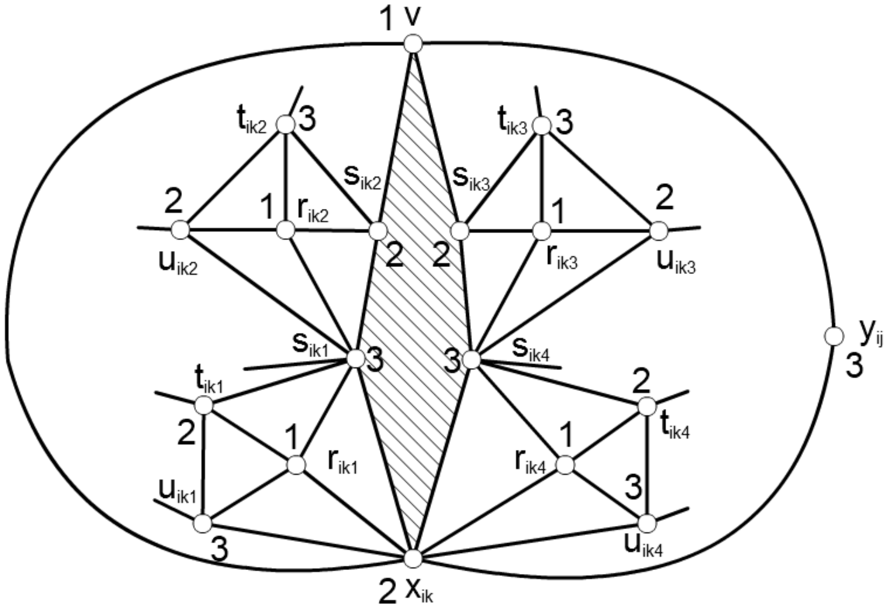

Figure 2.6: The fragment of the graph G'_i. All semi-edges are incident with the vertex v and the shaded area represents the hexagonal face.

$w \in V(G_i)$, $L'_i(s_{ikj}) = \{i, n(k), 7, 9\}$, $L'_i(u_{ikj}) = \{i, n(k), 7, 8\}$, $L_i(t_{ikj}) = \{i, 7, 8, 9\}$ and $L_i(r_{ikj}) = \{n(k), 7, 8, 9\}$ for $j \in \{1, 4\}$, $L'_i(s_{ikj}) = \{i, 7, 8, 9\}$, $L'_i(u_{ikj}) = \{i, 7, 8, 10\}$, $L'_i(t_{ikj}) = \{i, 8, 9, 10\}$ $L'_i(r_{ikj}) = \{7, 8, 9, 10\}$ for $j \in \{2, 3\}$, where $n(k) = 6$ if $k \in \{1, 3\}$ and $n(k) = 5$ if $k \in \{2, 4\}$. Then we fix an L'_i-coloring c of G_i. By Claim 1 (iii), c satisfies $c(x_{i1}) = 6$, $c(x_{i2}) = 5$,$c(x_{i3}) = 6$ or $c(x_{i4}) = 5$. In other words, there exists a positive integer k such that $c(x_{ik}) = n(k)$. Let k with $c(x_{ik}) = n(k)$ be fixed. In this situation, we regard the graph $G'_i[W_{ik1}]$ as the graph in Lemma 2.3.1 so that $v_1 = x_{ik}$, $v_2 = s_{ik1}$, $a = n(k), b = 7$. Thus we must have $c(s_{ik1}) = 7$ by Lemma 2.3.1. Similarly, we must have $c(s_{ik4}) = 7$. Moreover, we regarded the graph $G'_i[W_{ik2}]$ so that $v_1 = s_{ik1}$, $v_2 = s_{ik2}$, $a = 7$ and $b = 8$ as shown in Lemma 2.3.1. Thus we must have $c(s_{ik2}) = 8$ and similarly $c(s_{ik3}) = 8$. ■

Next, we insert the graph shown in Figure 2.7 into each hexagonal face $x_{ik}s_{ik1}s_{ik2}vs_{ik3}s_{ik4}$ for any $k \in \{1, 2, 3, 4\}$ in G'_i and the resulting graph is denoted by G''_i. For the graph G''_i, we have the following.

Claim 3 *All of the followings hold.*

(i) G''_i *is 3-colorable.*

(ii) G''_i *is L-colorable for a list assignment L of* G''_i *such that* $|L(v)| = 1$ *and* $|L(z)| = 4$ *for* $z \in V(G''_i) - \{v\}$.

(iii) There exists a list assignment L''_i *of* G'_i *which satisfies the following.*

- $|L''_i(v)| = 1$ *and* $|L''_i(z)| = 4$ *for* $z \in V(G'_i) - \{v\}$.

- *There exists a matching assignment $\mathscr{M}''_i$ in L''_i such that G''_i is not DP-$\mathscr{M}''_{L''_i}$-colorable.*

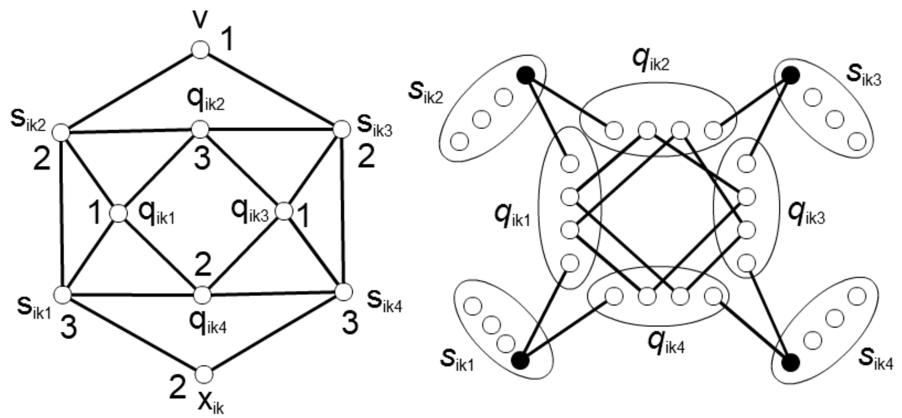

Figure 2.7: The left is a fragment of the graph G''_i. The right is the matching assignment $\mathscr{M}_{L''_i}$. The black vertices in s_{ik1} and s_{ik4} correspond to the color 7 and the ones in s_{ik2} and s_{ik3} correspond to the color 8.

Proof of Claim 3. For part (i), G'_i has a 3-coloring c such that $c(s_{ik1}) = c(s_{ik4})$ and $c(s_{ik2}) = c(s_{ik3})$ by Claim 2. Without loss of generality, we may assume $c(s_{ik1}) = c(s_{ik4}) = 2$ and $c(s_{ik2}) = c(s_{ik3}) = 3$. Then we extend the coloring c of G'_i to the 3-coloring of G''_i as shown in Figure 2.7.

For part (ii), first we give an L-coloring to the vertices of G'_i by Claim 2. In this situation, each vertex q_{ikj} has at least two available colors in L, where $j \in \{1, 2, 3, 4\}$. Since a cycle of even length is 2-choosable, G''_i is L-colorable.

For part (iii), at first, we give the list assignment L''_i such that $L''_i(u) = L'_i(u)$ for each $u \in V(G'_i)$ and $L''_i(q_{ikj})$=$\{7,8,9,10\}$ for $j \in \{1, 2, 3, 4\}$. By Claim 2 (iii), for any L'_i-coloring c, there exists a positive integer k such that $c(s_{ik1}) = c(s_{ik4}) = 7$ and $c(s_{ik2}) = c(s_{ik3}) = 8$. In other words, the vertices which correspond to the color 7 in s_{ik1} and s_{ik4} and which correspond to the color 8 in s_{ik2} and s_{ik3} must be chosen as an independent set of size $|V(G'_i)|$ in the graph $H(G'_i, L'_i, \mathscr{M}'_i)$, where $\mathscr{M}'_i$ is the matching assignment of G'_i so that each element of $\mathscr{M}'_i$ connects the same colors. Now, we take a matching assignment $\mathscr{M}''_i = \mathscr{M}'_i \cup \mathscr{M}_i$ of (G''_i, L''_i), where $\mathscr{M}_i$ is the matching assignment of the cycles $q_{k1}q_{k2}q_{k3}q_{k4}$ for $k \in \{1, 2, 3, 4\}$ as shown in Figure 2.7. It is easy to check that the graph $H(G''_i, L''_i, \mathscr{M}''_i)$ does not have an independent set of size $|V(G''_i)|$. ■

Finally, we construct the graph G so that $G = G''_1 \cup G''_2 \cup G''_3 \cup G''_4$ and $V(G''_i) \cap V(G''_j) = \{v\}$ for any distinct $i, j \in \{1, 2, 3, 4\}$. By Claim 3, we have $(\chi(G), \chi_\ell(G), \chi_{DP}(G)) = (3, 4, 5)$. □

Chapter 3

The Alon-Tarsi number of K_5-minor-free graphs.

In this Chapter, we introduce the Alon-Tarsi number.

3.1 Introductions

A *d-defective coloring* of G is a coloring $c : V(G) \to \mathbb{N}$ such that each color class induces a subgraph of maximum degree at most d. Especially, a 0-defective coloring is also called a *proper coloring* of G.

A *k-list assignment* of a graph G is a mapping L which assigns to each vertex v of G a set $L(v)$ of k permissible colors. Given a k-list assignment L of G, a d-defective L-coloring of G is a d-defective coloring c such that $c(v) \in L(v)$ for every vertex v. We say that G is *d-defective k-choosable* if G has a d-defective L-coloring for every k-list assignment L. Especially, we say that G is *k-choosable* if G is 0-defective k-choosable. The *list chromatic number* $\chi_\ell(G)$ is defined as the smallest integer k such that G is k-choosable.

Let G be a graph and let '$<$' be an arbitrary fixed ordering of the vertices of G. The *graph polynomial* of G is defined as

$$P_G(\boldsymbol{x}) = \prod_{u \sim v, u < v} (x_u - x_v),$$

where $u \sim v$ means that u and v are adjacent, and $\boldsymbol{x} = (x_v)_{v \in V(G)}$ is a vector of $|V(G)|$ variables indexed by the vertices of G. It is easy to see that a mapping $c : V(G) \to \mathbb{N}$ is a proper coloring of G if and only if $P_G(\boldsymbol{c}) \neq 0$, where $\boldsymbol{c} = \big(c(v)\big)_{v \in V(G)}$. Therefore, to find a proper coloring of G is equivalent to find an assignment of $\boldsymbol{x}$ so that $P_G(\boldsymbol{x}) \neq 0$. The following theorem, which was proved by Alon and Tarsi, gives sufficient conditions for the existence of such assignments as above.

Theorem 3.1.1 (Combinatorial Nullstellensatz [1]) *Let $\mathbb{F}$ be an arbitrary field and let $f = f(x_1, x_2, \ldots, x_n)$ be a polynomial in $\mathbb{F}[x_1, x_2, \ldots, x_n]$. Suppose that the degree*

$\deg(f)$ *of* f *is* $\sum_{i=1}^{n} t_i$ *where each* t_i *is a nonnegative integer, and suppose that the coefficient of* $\prod_{i=1}^{n} x_i^{t_i}$ *of* f *is nonzero. Then if* $S_1, S_2, \ldots, S_n$ *are subsets of* $\mathbb{F}$ *with* $|S_i| \geq t_i + 1$*, then there are* $s_1 \in S_1$*,*$s_2 \in S_2$*,...,*$s_n \in S_n$ *so that* $f(s_1, s_2, \ldots, s_n) \neq 0$*.*

In particular, a graph polynomial $P_G(\boldsymbol{x})$ is a homogeneous polynomial and $\deg(P_G)$ is equal to $|E(G)|$. Therefore, if there exists a monomial $c\prod_{v\in V(G)} {x_v}^{t_v}$ in the expansion of P_G so that $c \neq 0$ and $t_v < k$ for each $v \in V(G)$, then G is k-choosable. Jensen and Toft [19] defined the *Alon-Tarsi number* of a graph as follows.

Definition 3.1.2 The *Alon-Tarsi number* of a graph G, denoted by $AT(G)$, is the minimum k for which there exists a monomial $c\prod_{v\in V(G)} x_v^{t_v}$ in the expansion of $P_G(\boldsymbol{x})$ such that $c \neq 0$ and $t_v < k$ for all $v \in V(G)$.

As explained above, $\chi_\ell(G) \leq AT(G)$ for every graph G. Moreover, it is known that the gap between $\chi_\ell(G)$ and $AT(G)$ can be arbitrary large. Nevertheless, it is also known that the upper bounds of $\chi_\ell(G)$ and $AT(G)$ are the same for several graph classes. For example, Thomassen [27] proved that every planar graph is 5-choosable. Later, Zhu proved the following.

Theorem 3.1.3 ([34]) *Let* G *be a plane graph. Then* $AT(G) \leq 5$*.*

Moreover, it was shown in [8] that every planar graph is 1-defective 4-choosable. Recently, Grytczuk and Zhu have proved the following theorem.

Theorem 3.1.4 ([15]) *Let* G *be a plane graph. Then there exists a matching* M *of* G *such that* $AT(G - M) \leq 4$*.*

This result implies that every planar graph is 1-defective 4-choosable. Furthermore, it was shown independently in [11] and [24] that every planar graph is 2-defective 3-choosable. In this context, it seems natural to ask whether there exists a subgraph H of G such that $AT(G - E(H)) \leq 3$ and $d_H(v) \leq 2$ for every $v \in V(G)$. Since if it was true, this implies that every planar graph is 2-defective 3-choosable. However,this is not true and it was shown in [18] that there exists a planar graph G such that for any subgraph of H of G with maximum degree at most 3, $G - E(H)$ is not 3-choosable. On the other hand, the following was also proved in the same paper.

Theorem 3.1.5 ([18]) *Let* G *be a plane graph. Then there exists a forest* F *in* G *such that* $AT(G - E(F)) \leq 3$*.*

A graph H is a *minor* of a connected graph G if we obtain H from G by deleting or contracting some edges recursively. A graph G is H*-minor-free* if H is not a minor of G. If multiple edges appear by a contraction, we replace them with simple edge.

As another extension of Thomassen's result, it was shown in [17] and [25] that every K_5-minor-free graph is 5-choosable. Moreover, it is also shown in [33] that every

K_5-minor-free graph is 1-defective 4-choosable. In this paper, we extend these results from list chromatic number to Alon-Tarsi number.

Theorem 3.1.6 *Let G be a K_5-minor-free graph. Then all of the following hold.*

(i) $AT(G) \leq 5$.

(ii) There exists a matching M of G such that $AT(G - M) \leq 4$.

Theorem 3.1.7 *For every K_5-minor-free graph G, there exists a forest F such that $G - E(F)$ is 2-degenerate.*

Thus we have the following corollary.

Corollary 3.1.8 *For every K_5-minor-free graph G, there exists a forest F such that $AT(G - E(F)) \leq 3$.*

This paper is organized as follows. In Section 2, we prepare some lemmas in order to show the main theorems. And in Section 3, we prove Theorem 3.1.6 and Theorem 3.1.7. In Section 4, we have some remarks that Theorem 3.1.6 and Corollary 3.1.8 can be extended to singed graphs.

3.2 Orientations and Alon-Tarsi number

3.2.1 An alternative definition of the Alon-Tarsi number.

Indeed Alon-Tarsi number is already defined algebraically in Section 1, Alon and Tarsi [3] found a combinatorial interpretation of the coefficient for each monomial in the graph polynomials in terms of orientations and Eulerian subgraphs. For an orientation D of G, $d_D^+(v)$ (resp. $d_D^-(v)$) denotes out-degree (resp. in-degree) of a vertex v in D. A subgraph H of D is called *Eulerian* if $V(H) = V(G)$ and $d_H^-(v) = d_H^+(v)$ for every $v \in V(H)$ with respect to D. Note that H might not be connected. Let $EE(D)$ (resp. $OE(D)$) denote the set of all Eulerian subgraphs of D with even (resp. odd) number of edges. Especially, we say that an orientation D is *acyclic* if D does not contain any directed cycles.

Theorem 3.2.1 ([3]) *Let G be a graph, let P_G be the graph polynomial of G and let D be an orientation of G with out-degree sequence $\boldsymbol{d} = (d_v)_{v \in V(G)}$. Then the coefficient of $\prod_{v \in V(G)} x_v^{d_v}$ in the expansion of P_G is equal to $\pm(|EE(D)| - |OE(D)|)$*

We say that orientation D of G is an *AT-orientation* if D satisfies $|EE(D)| - |OE(D)| \neq 0$.

3.2.2 Orientations of planar graphs

Now let us focus on planar graphs. We say a plane graph G is a *near triangulation* if each internal face in G is triangular. In the papers [15], [18] and [34], the following are shown respectively.

Lemma 3.2.2 *Let G be a plane graph with simple boundary cycle $C = v_1v_2...v_m$. Then all of the following hold.*

(i) ([34]) *G has an AT-orientation D such that $d_D^+(v_1) = 0$, $d_D^+(v_2) = 1$, $d_D^+(v_i) \le 2$ for each $i \in \{3, ..., m\}$ and $d_D^+(u) \le 4$ for each interior vertex u.*

(ii) ([15]) *There exists a matching M and an AT-orientation D of $G - M$ such that $d_D^+(v_1) = d_D^+(v_2) = 0$, $d_D^+(v_i) \le 2 - d_M(v_i)$ for each $i \in \{3, ..., m\}$ and $d_D^+(u) \le 3$ for each interior vertex u.*

(iii) ([18]) *There exists a forest F in G and an acylic orientation D of $G - E(F)$ such that $d_D^+(v_1) = d_D^+(v_2) = 0$, $d_D^+(v_i) = 1$ for each $i \in \{3, ..., m\}$ and $d_D^+(u) \le 2$ for each interior vertex u.*

In order to show the main theorem, we need an orientation which has some stronger properties.

Lemma 3.2.3 *Let G be a plane graph with a boundary cycle $v_1v_2v_3$. Then all of the following hold.*

(i) There exists a matching M of G and an AT-orientation D of $G - M$ such that M does not cover v_3, $d_D^+(v_1) = d_D^+(v_2) = 0$, $d_D^+(v_3) = 2$ and $d_D^+(y) \le 3$ for $y \in V(G) - \{v_1, v_2, v_3\}$.

(ii) There exists a forest F of G and an acyclic orientation D of $G - E(F)$ such that $v_1v_3 \notin E(F)$, $d_D^+(v_1) = d_D^+(v_2) = 0$, $d_D^+(v_3) = 1$ and $d_D^+(y) \le 2$ for $y \in V(G) - \{v_1, v_2, v_3\}$.

Proof. Let $G' = G - v_3$ and let $N(v_3) = \{v_1, u_1, ..., u_k, v_2\}$ be the neighborhood of v_3 as this rotation. Since G' is a plane graph, we have a matching M and an AT-orientation D' of $G' - M$ such that $d_{D'}^+(v_1) = d_{D'}^+(v_2) = 0$, $d_{D'}^+(u_i) \le 2$ for $i \in \{1, 2, ..., k\}$ and $d_{D'}^+(u) \le 3$ for each interior vertex u by Lemma 4.3.1 (See Figure 3.1). Let D be the orientation of $G - M$ obtained from D' by adding the vertex v_3 and $k + 2$ oriented edges (u_i, v_3) for $i \in \{1, 2, ..., k\}$, (v_3, v_1) and (v_3, v_2).

It is easy to see that D also satisfies the out-degree conditions and that M does not cover v_3. Moreover, since the vertices v_1 and v_2 have out-degree 0, D is also an AT-orientation. $\square$

Let G and H be a graph which contain a clique of the same size. The *clique-sum* of G and H is a operation that forms a new graph obtained from their disjoint union by

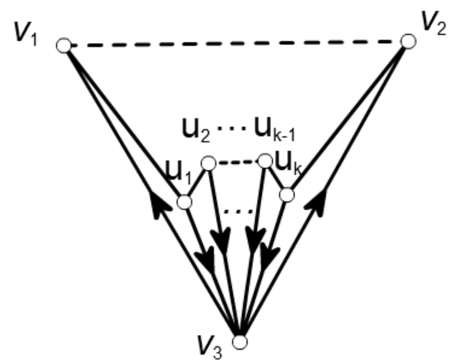

Figure 3.1: The orientation D of $G - M$

identifying a clique of G and one of H with the same size and possibly deleting some edges in the clique. A *k-clique-sum* is a clique-sum in which both cliques have at most k-vertices.

Lemma 3.2.4 *Let G be a graph which can be obtained by the 3-clique-sum of G_1 and G_2 and let $T = \{x_1, x_2, x_3\}$ be its clique. Moreover, let $G_i' = G \cap G_i$. Suppose that G_1' has an AT-orientation D_1' with maximum out-degree at most k and that G_2 has an AT-orientation D_2 such that $d^+_{D_2}(x_1) = 0$, $d^+_{D_2}(x_2) \leq 1$ and $d^+_{D_2}(x_3) \leq 2$ and x_i is directed only to $x_{i'}$, where $x_i, x_{i'} \in V(T)$. Then G has an AT-orientation D such that $d^+_D(v) = d^+_{D_1'}(v)$ for each $v \in V(G_1)$ and maximum out-degree of D is at most k.*

Proof. Let $D_2' \subset D_2$ be the orientation of G_2' and let $D = D_1' \cup \big(D_2' - E(T)\big)$. Then it is easy to see that $d^+_D(v) = d^+_{D_1'}(v)$ for each $v \in V(G_1')$, $d^+_D(v) = d^+_{D_2'}(v)$ for each $v \in V(G_2') - V(T)$ and hence maximum out-degree of D is at most k. For the orientation D_2, the vertices in T has a direction only to other vertices of T and no Eulerian subgraphs in D_2 contain the edge in T by the out-degree conditions of D_2. Thus D_2' is also an AT-orientation of G_2' and any spanning Eulerian sub-digraphs H of D has an edge-disjoint decomposition $H = H_1 \cup H_2$ where H_1 and H_2 are Eulerian sub-digraphs in D_1' and D_2', respectively. Therefore, we have the bijection τ so that

- $\tau\big(EE(D)\big) = \big(EE(D_1') \times EE(D_2')\big) \cup \big(OE(D_1') \times OE(D_2')\big)$ and

- $\tau\big(OE(D)\big) = \big(OE(D_1') \times EE(D_2')\big) \cup \big(EE(D_1') \times OE(D_2')\big)$.

Hence

$$\begin{aligned}
& |EE(D)| - |OE(D)| \\
= \ & (|EE(D'_1)| \times |EE(D'_2)| + |OE(D'_1)| \times |OE(D'_2)|) \\
& -(|EE(D'_1)| \times |OE(D'_2)| + |OE(D'_1)| \times |EE(D'_2)|) \\
= \ & (|EE(D'_1)| - |OE(D'_1)|) \cdot (|EE(D'_2)| - |OE(D'_2)|) \\
\neq \ & 0.
\end{aligned}$$

These imply that the orientation D is an AT-orientation of G with desired properties. □

3.2.3 Characterizations of K_5-minor-free graphs.

Now, let us focus on K_5-minor-free graphs. In order to show the main theorem, we use the following results.

Lemma 3.2.5 ([31]) *A graph G is K_5-minor-free if and only if G can be formed from some 3-clique-sums of graphs, each of which is either planar or the Wagner graph W as shown in Figure 3.2.*

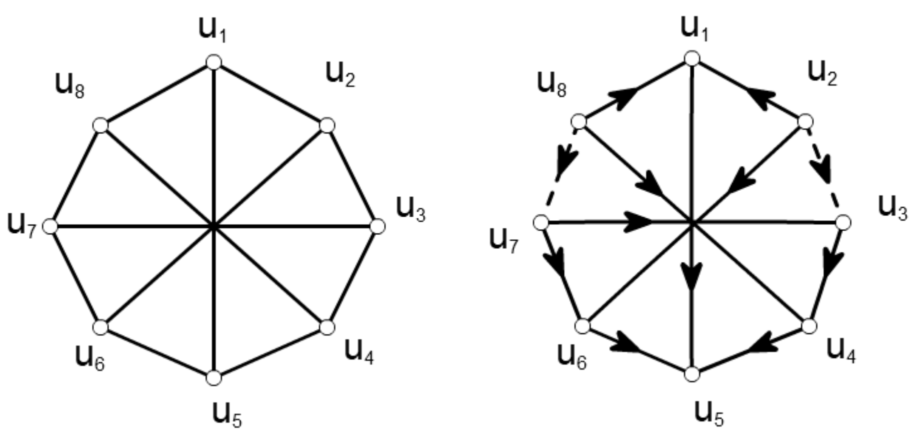

Figure 3.2: The left is the Wagner graph W and the right is an acyclic orientation with maximum out-degree 3. Doted edges denote elements of a matching or forest.

3.3 Proof of main Theorem.

Theorem 3.1.6 and Theorem 3.1.7 follow from the lemma below.

Lemma 3.3.1 *Let G be a K_5-minor-free graph and let H_i be a subgraph of G_i which is isomorphic to $uv \in E(G)$ or $\{uv, vw, wu\} \subset E(G)$ for each $i \in \{1, 2, 3\}$. Then all of the following hold.*

(i) There exists an AT-orientation D such that $d^+_D(u) = 0$, $d^+_D(v) = 1$, ($d^+_D(w) = 2$ if H_1 is isomorphic to K_3) and $d^+_D(y) \leq 4$ for $y \in V(G) - \{u, v, w\}$.

(ii) There exists a matching M of G and an AT-orientation D of $G - M$ such that $d^+_D(u) = d^+_D(v) = 0$, ($d^+_D(w) = 2$ and M does not cover w if H_2 is isomorphic to K_3) and $d^+_D(y) \leq 3$ for $y \in V(G) - \{u, v, w\}$.

(iii) There exists a forest F of G and an acyclic orientation D of $G - E(F)$ such that $d^+_D(u) = d^+_D(v) = 0$, ($d^+_D(w) = 1$ if H_3 is isomorphic to K_3) and $d^+_D(y) \leq 2$ for $y \in V(G) - \{u, v, w\}$.

Proof. Suppose that the Lemma is false and let G_i be a counterexample for each $i \in \{1, 2, 3\}$ respectively with $|V(G_i)|$ as small as possible. By the minimality of G_i, G_i is connected. Moreover, it is easy to check that G_i does not have a cut vertex by Lemma 3.2.4. Thus we may assume G_i is 2-connected.

First we suppose that G_i is a plane graph. If H_i is isomorphic to K_2 or K_3 which bounds a face, without loss of generality, H_i lies on the boundary of G. In this case, a desired AT-orientation exists by Lemma 4.3.1. Thus H_i consists a separating 3-cycle in G_i. We let $G_{i,1}$ and $G_{i,2}$ be subgraphs of G_i so that $G_{i,1} \cup G_{i,2} = G_i$ and $G_{i,1} \cap G_{i,2} = H_i$. By Lemma 4.3.1 and Lemma 3.2.3, for $j \in \{1, 2\}$ we have an AT-orientation $D_{1,j}$ of $G_{1,j}$ which satisfies the conditions. Similarly, we have that there exists a matching M_j and an AT-orientation of $G_{2,j} - M_j$ and that there exists a forest F_j and acyclic orientation of $G_{3,j} - E(F_j)$, which satisfy the conditions. It is easy to see that $M = M_1 \cup M_2$ is also a matching of G_2 and $F = F_1 \cup F_2$ is a forest of G_3. Therefore, we get a desired AT-orientation respectively by Lemma 3.2.4. Thus G_i is not planar.

Next, suppose that G_i is the Wagner graph W. Since W does not contain a triangle, H_i must be isomorphic to K_2. By the symmetry of W, we may assume that $H_i = u_1u_5$ or u_5u_6. In this case, the orientation in Figure 3.2 is a desired AT-orientation respectively.

Thus we assume that G_i is neither planar graph nor the graph W. By Lemma 3.2.5, there exists K_5-minor-free graphs $G_{i,1}$ and $G_{i,2}$ such that G_i can be obtained by a 3-clique-sum of $G_{i,1}$ and $G_{i,2}$. Let T be its clique and let $G'_{i,j} = G_{i,j} \cap G$ for $j \in \{1, 2\}$. It is easy to see that $H_i \subset G'_{i,1}$ or $G'_{i,2}$. Without loss of generality, we may assume that $H_i \subset G_{i,1}$. By the minimality of G_i, we have the following.

(i) There exists an AT-orientation $D_{1,1}$ of $G'_{1,1}$ which satisfies the assumption (i) of Lemma 3.3.1.

(ii) There exists a matching M_1 and an AT-orientation $D_{2,1}$ of $G'_{2,1} - M$ which satisfies the assumption (ii) of Lemma 3.3.1.

(iii) There exists a forest F_1 and an acyclic orientation $D_{3,1}$ of $G'_{3,1} - E(F)$ which satisfies the assumption (iii) of Lemma 3.3.1.

First, we consider the case when $i = 1$. By the minimality of G_1, we get an AT-orientation $D_{1,2}$ of $G_{1,2}$ with $d^+_{D_{1,2}}(x_1) = 0$, $d^+_{D_{1,2}}(x_2) = 1$, $d^+_{D_{1,2}}(x_3) = 2$ and the maximum degree of $D_{1,2}$ is at most 4. By Lemma 3.2.4, we get a desired AT-orientation D in G_1.

Next, we consider the case when $i = 2$. By the minimality of G_2, we get a matching M_2 of $G_{2,2}$ and an AT-orientation $D_{2,2}$ of $G_{2,2} - M_2$ such that $d^+_{D_{2,2}}(x_1) = 0$, $d^+_{D_{2,2}}(x_2) = 0$, $d^+_{D_{2,2}}(x_3) = 2$, maximum out-degree of $D_{2,2}$ is at most 3 and M_2 does not cover x_3. Let $M = M_1 \cup \big(M_2 - \{x_1x_2\}\big)$. It is easy to see that M is a matching of G. By Lemma 3.2.4, we get a desired AT-orientation D in $G_2 - M$.

Finally, we consider the case when $i = 3$. By the minimality of G_3, we get a forest F_2 of $G_{3,2}$ and an acyclic orientation $D_{3,2}$ of $G_{3,2} - E(F_2)$ with $d^+_{D_{3,2}}(x_1) = d^+_{D_{3,2}}(x_2) = 0$, $d^+_{D_{3,2}}(x_3) = 1$ and maximum out-degree of $D_{3,2}$ is at most 2. Let $F = F_1 \cup (F_2 - E(T))$. Similarly, we can show that F is a forest and D_3 is an acyclic orientation of $G_3 - E(F)$ with desired properties. This is a contradiction and we completes the proof. $\square$

[Proof of Theorem 3.1.6 and Theorem 3.1.7]
Theorem 3.1.6 follows immediately from (i) and (ii) in Lemma 3.3.1. For Theorem 3.1.7, each K_5-minor-free graph G has a forest F and and an acyclic orientation D of $G - E(F)$ with maximum out-degree at most 2 by Lemma 3.3.1. Since G is finite and D is acyclic, there exists a vertex v with $d^-_D(v) = 0$. Therefore, the vertex v has degree at most 2 and hence $G - E(F)$ is 2-degenerate. $\square$

3.4 Some remarks

A *signed graph* is a pair (G, σ), where G is a graph and σ is a *signature* of G which assigns to each edge $e = uv$ of G a sign $\sigma_{uv} \in \{1, -1\}$. Let

$$N_k = \begin{cases} \{0, \pm 1, \ldots, \pm q\} & \text{if } k = 2q+1 \text{ is an odd integer,} \\ \{\pm 1, \ldots, \pm q\} & \text{if } k = 2q \text{ is an even integer.} \end{cases}$$

A *proper coloring* of (G, σ) is a mapping $c : V(G) \to N_k$ such that $c(x) \neq \sigma_{xy} c(y)$ for each edge xy. The *chromatic number* $\chi(G, \sigma)$ of (G, σ) is minimum integer t such that there exists a proper coloring $c : V(G) \to N_k$ with $|N_k| = t$. The list chromatic number $ch(G, \sigma)$ of (G, σ) is minimum integer k such that for every k-list assignment L, there exists a proper coloring c of (G, σ) so that $c(v) \in L(v)$ for every $v \in V(G)$.

Let (G, σ) be a signed graph and let '$<$' be an arbitrary fixed ordering of the vertices of (G, σ). The *singed graph polynomial* of (G, σ) is defined as

$$P_{G,\sigma}(\boldsymbol{x}) = \prod_{u \sim v, u < v} (x_u - \sigma_{uv} x_v),$$

where $u \sim v$ means that u and v are adjacent, and $\boldsymbol{x} = (x_v)_{v \in V(G)}$ is a vector of $|V(G)|$ variables indexed by the vertices of G. It is easy to see that a mapping $c : V(G) \to \mathbb{Z}$

is a proper coloring of (G,σ) if and only if $P_{G,\sigma}(\boldsymbol{c}) \neq 0$, where $\boldsymbol{c} = \big(c(v)\big)_{v\in V(G)}$. The Alon-Tarsi number of (G,σ) is defined similarly and we have $\chi(G,\sigma) \leq ch(G,\sigma) \leq AT(G,\sigma)$.

Let (G,σ) be a singed graph and let D be an orientation of (G,σ). Let $\sigma EE(D)$ (resp. $\sigma OE(D)$) denote the set of all spanning Eulerian sub-digraphs of D with even (resp. odd) number of positive edges on σ. It was shown in [32] that Theorem 3.2.1 can be extended to signed one as follows.

Theorem 3.4.1 ([32]) *Let (G,σ) be signed graph, let $P_{G,\sigma}$ be the signed graph polynomial of (G,σ) and let D be an orientation of (G,σ) with out-degree sequence $\boldsymbol{d} = (d_v)_{v\in V(G)}$. Then the coefficient of $\prod_{v\in V(G)} x_v^{d_v}$ in the expansion of $P_{G,\sigma}$ is equal to $\pm(|\sigma EE(D)| - |\sigma OE(D)|)$.*

We say orientation D of (G,σ) is a σAT-orientation if D satisfies $|\sigma EE(D)| - |\sigma OE(D)| \neq 0$.

Let us focus on planar graphs. In the papers [15] and [32], the following are shown respectively.

Lemma 3.4.2 *Let (G,σ) be a signed near triangulation and let $C = v_1v_2...v_m$ be the boundary cycle of G. Then all of the following hold.*

(i) ([32]) *G has a σAT-orientation D such that $d_D^+(v_1) = 0$, $d_D^+(v_2) = 1$, $d_D^+(v_i) \leq 2$ for each $i \in \{3,...,m\}$ and $d_D^+(u) \leq 4$ for each interior vertex u.*

(ii) ([15]) *There exists a matching M and a σAT-orientation D of $G - M$ such that $d_D^+(v_1) = d_D^+(v_2) = 0$, $d_D^+(v_i) \leq 2 - d_M(v_i)$ for each $i \in \{3,...,m\}$ and $d_D^+(u) \leq 3$ for each interior vertex u.*

Although the Lemma 3.4.2 only deal with near triangulations in the paper [32], it is not hard to extend the graph class from near triangulations to planar graphs. Moreover, since all the arguments of Lemmas in Section 2 and Lemma 3.3.1 work even if we replace AT-orientations into σAT-orientations, we have the following results.

Theorem 3.4.3 *Let (G,σ) be a signed K_5-minor-free graph. Then all of the following hold.*

(i) $AT(G,\sigma) \leq 5$.

(ii) *There exists a matching M of G such that $AT(G - M,\sigma) \leq 4$.*

(iii) *There exsits a forest F in G such that $AT(G - E(F),\sigma) \leq 3$.*

Chapter 4

Edge-colorings on the projective plane

In this Chapter, we focus on the edge-coloring of graphs.

4.1 Introduction

In this chapter, we only deal with graphs which are finite undirected and loopless but may have multiple edges. Let k be a positive integer. A *k-edge-coloring* of a graph G is a map $\varphi : E(G) \to \{1, 2, ..., k\}$ such that for any pair of edges $uv, vw \in E(G)$ sharing an end vertex v, we have $\varphi(uv) \neq \varphi(vw)$. A graph G is *k-edge-colorable* if there exists a k-edge-coloring of G. Moreover, $EC_k(G)$ denotes the set of k-edge-colorings of G
In this paper, we study the *signature of edge-colorings φ with respect to a base ρ*, which is denoted by $\mathrm{sign}_\rho(\varphi)$ and takes positive or negative. Before defining it formally in Section 3, we first introduce its application.

4.1.1 The List Coloring Conjecture

First, we focus on the List Coloring Conjecture. The definition of list coloring is mentioned in Section 4.4. The following problem is still open so far.

Conjecture 4.1.1 ([4]) *Let G be a graph. If G is k-edge-colorable, then G is k-list-edge-colorable.*

The converse of Conjecture 4.1.1 trivially holds from the definition, but it seems difficult to prove Conjecture 4.1.1. As a partial solution to Conjecture 4.1.1, Alon and Tarsi proved the following theorem.

Theorem 4.1.2 ([2]) *Let G be a k-regular k-edge-colorable graph and ρ be a basis of G. If G satisfies $\displaystyle\sum_{\varphi \in EC_k(G)} \mathrm{sign}_\rho(\varphi) \neq 0$, then G is k-list-edge-colorable.*

From Theorem 4.1.2, by considering the signatures, we could obtain list-edge-colorings. We explain this more details in Section 4.4.

4.1.2 The number of vertex colorings

Next, we consider the number of vertex colorings. For edge-colorings of plane graphs, the following fact is well known.

Theorem 4.1.3 ([26]) *Let G be a 2-edge-connected plane cubic graph. Then G is 3-edge-colorable if and only if the dual G^* has a 4-vertex-coloring.*

From the proof of Theorem 4.1.3, we have a one-to-one correspondence between 3-edge-colorings in cubic plane graphs and 4-vertex-colorings in the dual planar triangulations, up to the permutations of colors. Thus, the number of 4-colorings in planar triangulations can be counted by doing 3-edge-colorings in planar cubic graphs. On the other hand, it is known in [14] that the number of 3-edge-colorings in planar cubic graphs can be computed by using the signature.

4.1.3 Kempe switch

Finally, we will see the Kempe switch. Let φ be a k-edge-coloring of a graph G and let C be a cycle induced by edges with two distinct colors i, j of φ. A *Kempe switch* at C is an operation where we recolor the edges on C colored i to j and the ones colored j to i respectively. We say that φ_1 and φ_2 are *Kempe equivalent* if φ_2 is obtained from φ_1 by a sequence of Kempe switches. It is easy to see that Kempe equivalence forms an equivalence relations on the set of k-edge-colorings. For the Kempe switch, the following is obtained by the easy observations.

Proposition 4.1.4 *Let G be a k-regular k-edge-colorable graph with a base ρ and let φ_1 and φ_2 be k-edge-colorings of G. If φ_2 is obtained from φ_1 by the Kempe switch, then* $\mathrm{sign}_\rho(\varphi_1) = \mathrm{sign}_\rho(\varphi_2)$.

By Proposition 4.1.4, if two edge-colorings φ_1 and φ_2 satisfy that $\mathrm{sign}_\rho(\varphi_2) \neq \mathrm{sign}_\rho(\varphi_2)$ under a common basis ρ, then they are not Kempe equivalent. Thus the signature could be one of the tools to show that two edge-colorings are not Kempe equivalent.

Related to those works, it seems natural to ask which edge-colorings have the positive or negative signatures. For plane graphs, it is known that the signatures of any two edge-colorings are the same under a common basis ρ [10]. However, in general, a graph may have both edge-colorings with positive signatures and those with negative ones. In this paper, we study signatures of edge-colorings for graphs on the projective plane.

We organize this paper as follows; First we prepare terminologies and propositions. In Section 4.2, we give the definition and results of signatures of edge-colorings. and in

Section 4.3 we prove our main theorem. In Section 4.4, we introduce an application to the List Coloring Conjecture.

Before proceeding to the main part, we define some terminology. A *surface* is a connected compact 2-dimensional manifold without boundary. A *triangulation* (respectively a *quadrangulation*) of a surface $\mathbb{F}$ is a graph embedded in $\mathbb{F}$ with each face triangular (respectively a quadrangular).
A closed curve γ on $\mathbb{F}$ is *essential* if γ does not bound a 2-cell region on $\mathbb{F}$. Otherwise, γ is *contractible*. A non-separating curve is always essential but the converse does not generally hold. However, if $\mathbb{F}$ is the projective plane, a closed curve γ is non-separating if and only if it is essential. For the projective plane, the following is well-known.

Proposition 4.1.5 *Let γ and γ' be closed curves on the projective plane. Then γ and γ' are both essential if and only if they intersect transversally an odd number of times.*

4.2 Definition of signatures of edge-colorings

4.2.1 Abstract graphs

In this subsection, we give a formal definition of the signatures of edge-colorings. In the following, G is always a k-regular k-edge-colorable graph except as noted. Let φ be a k-edge-coloring of G. For $i \in \{1, 2, \ldots, k\}$, the set of edges of color i by φ is denoted by $\varphi^{-1}(i)$. For a vertex $v \in V(G)$, we denote by $E(v)$ the set of edges that are incident with v. Let ρ_v be a bijective map from $E(v)$ to $\{1, 2, ..., k\}$. We call $\rho = \{\rho_v : v \in V(G)\}$ a *basis* of G. For a basis ρ, a k-edge-coloring φ, and a vertex v of the graph G, we have a permutation $\pi_v = \varphi \circ \rho_v^{-1}$ of degree k. The signature $\mathrm{sign}(\pi_v)$ is defined as follows.

$$\mathrm{sign}(\pi_v) = \begin{cases} +1 & \text{if } \pi_v \text{ is an even permutation,} \\ -1 & \text{otherwise.} \end{cases}$$

Especially, we might denote $\mathrm{sign}_\rho(\pi_v)$ instead of $\mathrm{sign}(\pi_v)$ when we emphasize the basis ρ. We call $\prod_{v \in V(G)} \mathrm{sign}(\pi_v)$ the *signature of an edge-coloring φ (with respect to ρ)* and denoted by $\mathrm{sign}_\rho(\varphi)$. Figure 4.1 is an example of the signature of an edge-coloring.

Note that the signatures of edge-colorings φ_1 and φ_2 of G depend on the basis ρ, but the equivalence relation of φ_1 and φ_2 does not depend on a basis.

Proposition 4.2.1 *Let G be a k-regular k-edge-colorable graph. If k-edge-colorings φ_1 and φ_2 of G satisfy $\mathrm{sign}_\rho(\varphi_1) = \mathrm{sign}_\rho(\varphi_2)$ for a basis ρ, then $\mathrm{sign}_{\rho'}(\varphi_1) = \mathrm{sign}_{\rho'}(\varphi_2)$ for each basis ρ'.*

Proof. Let $v \in V(G)$ and let π_v^1 (resp. π_v^2) be the permutation of edge-colorings φ_1 (resp. φ_2) at v with respect to the basis ρ. Moreover, let τ_v^1 (resp. τ_v^2) be the permutation

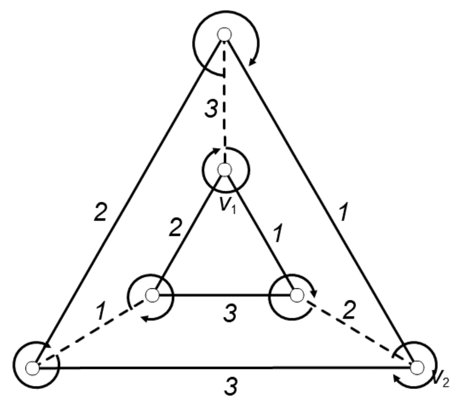

Figure 4.1: An example of the signature of a 3-edge-coloring φ with respect to the basis ρ. The cyclic arrow around each vertex v shows the order of edges incident to the vertex on ρ_v. For example, $\pi_{v_1} = \begin{pmatrix} 1 & 2 & 3 \\ 3 & 1 & 2 \end{pmatrix}$ and $\pi_{v_2} = \begin{pmatrix} 1 & 2 & 3 \\ 2 & 1 & 3 \end{pmatrix}$ and hence we have $\mathrm{sign}(\pi_{v_1}) = +1$ and $\mathrm{sign}(\pi_{v_2}) = -1$. In this situation, we have $\mathrm{sign}_\rho(\varphi) = (-1)^3(+1)^3 = -1$.

of edge-colorings φ_1 (resp. φ_2) at v with respect to the basis ρ'. By the definition of τ_v^1 and τ_v^2, we have $\tau_v^1 = \pi_v^1 \circ \rho \circ \rho'^{-1}$ and $\tau_v^2 = \pi_v^2 \circ \rho \circ \rho'^{-1}$. Thus $\mathrm{sign}_\rho(\pi_v^1) = \mathrm{sign}_\rho(\pi_v^2)$ if and only if $\mathrm{sign}_{\rho'}(\tau_v^1) = \mathrm{sign}_{\rho'}(\tau_v^2)$. $\square$

By Proposition 4.2.1, it suffices to consider only one basis in order to see the equivalence relation between the edge-colorings. In the following, we focus on graphs embedded on closed surfaces and consider the signatures of edge-colorings under a fixed basis, which depends on the embedding.

4.2.2 Planar graphs

Let G be a k-regular k-edge-colorable plane graph. Note that G has a perfect matching M, since in any k-edge-coloring of G, the set of edges with the same color forms a perfect matching. As a canonical basis, we consider the following: For every vertex v in G, we take the clockwise permutation on $E(v)$ which starts from the edge of M incident with v. We denote this basis by $\rho[M]$. For example, the dotted edges in Figure 4.1 represent the edges in a perfect matching M in the graph. Since each ρ_v is a clockwise permutation on $E(v)$ that starts from the elements of M, we have $\rho = \rho[M]$.

Note that the signature of a cyclic permutation of odd degree does not depend on the first element. Therefore, if k is odd, then $\mathrm{sign}_{\rho[M]}(\varphi) = \mathrm{sign}_{\rho[M']}(\varphi)$ for any perfect matchings M and M' of G. However, when k is even, $\mathrm{sign}_{\rho[M]}(\varphi)$ and $\mathrm{sign}_{\rho[M']}(\varphi)$ could be either same or different.

The signature of edge-colorings of cubic plane graphs for $\rho[M]$ can be expressed by only the order of the graph.

Theorem 4.2.2 ([14]) *Let G be a cubic plane graph and M be a perfect matching of G. Moreover, let φ be a k-edge-coloring of G. Then* $\mathrm{sign}_{\rho[\mathrm{M}]}(\varphi) = (-1)^{\frac{|G|}{2}}$.

By Theorem 4.2.2, the signature with respect to $\rho[M]$ of cubic plane graphs does not depend on edge-colorings. This indicates that Conjecture 4.1.1 holds for cubic planar graphs by Theorem 4.1.2. In general, the following is known.

Theorem 4.2.3 ([10]) *Let G be a planar k-regular k-edge-colorable graph. Then G satisfies* $\mathrm{sign}_\rho(\varphi_1) = \mathrm{sign}_\rho(\varphi_2)$ *for a basis ρ and k-edge-colorings φ_1, φ_2.*

4.2.3 Graphs on the projective plane

Next, we will focus on the projective planar case. Let G be a k-regular k-edge-colorable graph on the projective plane and let G^* be the dual of G. Since the projective plane is not an orientable surface, we cannot give a clockwise orientation around each vertex. Thus, we cannot directly define a canonical base, and hence we need some preliminary as follows.

Let D^* be an essential cycle of G^*. In this paper, we often regard D^* as its edge set. Let D be the set of edges in G which intersect an edge of D^*. If we cut the projective plane along the essential cycle D^*, then we obtain an open disk, which is orientable. We call the set D^* a *boundary* and D a *dual boundary* of G, respectively. In this situation, we get a clockwise permutation on $E(v)$ for each vertex v, which starts from the edge of a perfect matching M of G on the open disc. We denote this basis by $\rho[M, D]$. See Figure 5.2.

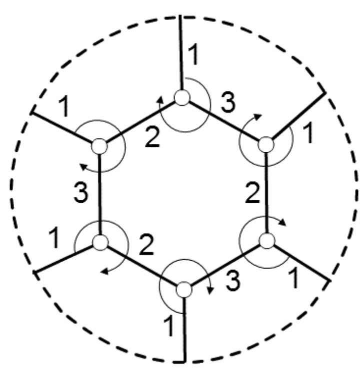

Figure 4.2: We obtain the projective plane by identifying the antipodal points of the dotted line, which represents a boundary D^*. The three edges intersecting with the dotted line D^* form the dual boundary D and the circle arrows denote the basis $\rho[\varphi^{-1}(1), D]$. The matching $\varphi^{-1}(1)$ satisfies $|\varphi^{-1}(1) \cap D| = 3$ and $|\varphi^{-1}(j) \cap D| = 0$ for $j = 2, 3$. Thus, $\#\{j; |\varphi^{-1}(1) \cap D| \equiv |\varphi^{-1}(j) \cap D| \pmod 2,\ 1 \leq j \leq 3\} = 1$ and hence this edge-coloring is of type-1 for $\varphi^{-1}(1)$.

In order to verify the signatures of edge-colorings by the topological conditions, we introduce the notion called *type* for edge-colorings. Let M be a perfect matching of G. If a k-edge-coloring φ satisfies

$$\#\{j; |M \cap D| \equiv |\varphi^{-1}(j) \cap D| \pmod 2, 1 \leq j \leq k\} = s,$$

then we say that φ is of *type-s for M*. Note that φ is of type-s for a perfect matching M for some $s \in \{0, 1, \ldots, k\}$, and φ cannot be of type-0 for $\varphi^{-1}(1)$. The case of $k = 3$ and $s = 1$ for $\varphi^{-1}(1)$ is represented in Figure 5.2.

Let e be an edge of a dual boundary D, and v be an end vertex of e. Moreover, let $(D')^*$ be the boundary obtained from the boundary D^* by the homotopic shift beyond v. Note that $D' = D \triangle E(v)$, where $\triangle$ denotes the symmetric difference between two sets. Since $|\varphi^{-1}(i) \cap D| \not\equiv |\varphi^{-1}(i) \cap D'| \pmod 2$ for each $i \in \{1, ..., k\}$, we have $|M \cap D| \equiv |\varphi^{-1}(i) \cap D| \pmod 2$ if and only if $|M \cap D'| \equiv |\varphi^{-1}(i) \cap D'| \pmod 2$. This implies that the type for M does not change even if we switch the dual boundary from D to D'. Since any two essential cycles on the projective plane can be transformed by a homotopic shift, we have the following.

Proposition 4.2.4 *Let G be a k-regular k-edge-colorable graph on the projective plane, let M be a perfect matching of G, and let φ be a k-edge-coloring of G. Then the type of φ for M does not depend on the choice of dual boundaries.*

Moreover, the type of an edge-coloring is related to the parity of $|D|$ as follows.

Proposition 4.2.5 *Let G be a k-regular k-edge-colorable graph on the projective plane and let D be a dual boundary of G. Let φ be an edge-coloring of type-s for $\varphi^{-1}(1)$ and φ' be an edge-coloring of type-s' for $\varphi'^{-1}(1)$ of G. Then both of the following hold.*

(i) If k is odd and $|\varphi^{-1}(1) \cap D| \not\equiv |\varphi'^{-1}(1) \cap D| \pmod 2$, then $s \not\equiv s' \pmod 2$.

(ii) If k is even, then $s \equiv s' \equiv |D| \pmod 2$.

Proof. Suppose that k is odd, and $|\varphi^{-1}(1) \cap D| \not\equiv |\varphi'^{-1}(1) \cap D| \pmod 2$. Without loss of generality, we may assume $|\varphi^{-1}(1) \cap D| \equiv 1 \pmod 2$ and $|\varphi'^{-1}(1) \cap D| \equiv 0 \pmod 2$. By the definition of the type, we have $\sum_{i=1}^{k} |\varphi^{-1}(i) \cap D| \equiv s \pmod 2$ and $\sum_{i=1}^{k} |\varphi'^{-1}(i) \cap D| \equiv k - s' \pmod 2$. Thus,

$$s \equiv \sum_{i=1}^{k} |\varphi^{-1}(i) \cap D| = |D| = \sum_{i=1}^{k} |\varphi'^{-1}(i) \cap D| \equiv k - s' \pmod 2.$$

Since k is odd, this implies $s \not\equiv s' \pmod 2$.

Next, suppose that k is even. If $|\varphi^{-1}(1) \cap D| \equiv 1 \pmod 2$, then $|D| = \sum_{i=1}^{k} |\varphi^{-1}(i) \cap D| \equiv s \pmod 2$. On the other hand, if $|\varphi^{-1}(1) \cap D| \equiv 0 \pmod 2$, then $|D| = \sum_{i=1}^{k} |\varphi^{-1}(i) \cap D| \equiv k - s \equiv s \pmod 2$, since k is even. In either case, we have $s \equiv |D|$

(mod 2), which directly shows (ii). $\square$

For cubic graphs on the projective plane, the signature of edge-colorings has already been verified in [22]. We can translate the statement in terms of the type of edge-colorings.

Theorem 4.2.6 ([22]) *Let G be a cubic 3-edge-colorable graph on the projective plane, M be a perfect matching, and D be a dual boundary. Then*

$$\mathrm{sign}_{\rho[\varphi^{-1}(1),D]}(\varphi) = \begin{cases} (-1)^{\frac{|G|}{2}+|D|} & \textit{if } \varphi \textit{ is of type-3 for } \varphi^{-1}(1), \\ (-1)^{\frac{|G|}{2}+|D|+1} & \text{otherwise.} \end{cases}$$

As we have described above, the signature of edge-colorings does not depend on the choice of the perfect matching when the graph is odd regular, say, $\mathrm{sign}_{\rho[M,D]}(\varphi) = \mathrm{sign}_{\rho[\varphi^{-1}(1),D]}(\varphi)$ for any perfect matching M and an edge-coloring φ. Thus, if we fix a dual boundary D and a 3-edge-coloring φ, then Theorem 4.2.6 gives the signature $\mathrm{sign}_{\rho[M,D]}(\varphi)$ for any perfect matching M.

However, for an even regular case, in order to see the difference between the edge-colorings φ and φ', it is not sufficient to obtain the signatures under the basis $\rho[\varphi^{-1}(1), D]$ and $\rho[\varphi'^{-1}(1), D]$, By considering the types of 4-edge-colorings in 4-regular graphs, we calculate the signatures of 4-edge-colorings with respect to the basis $\rho[M, D]$, where M is a fixed perfect matching of G.

Theorem 4.2.7 *Let G be a 4-regular 4-edge-colorable graph on the projective plane, let M be a perfect matching of G, and let D be a dual boundary. For a 4-edge-coloring φ of G, the following holds.*

$$\mathrm{sign}_{\rho[M,D]}(\varphi) = \begin{cases} (-1)^{\frac{|G|}{2}+|M\cap D|} & \text{if } \varphi \text{ is of type-0, 3 or 4 for } M, \\ (-1)^{\frac{|G|}{2}+|M\cap D|+1} & \text{otherwise.} \end{cases}$$

Theorem 4.2.7 has an application for the List Coloring Conjecture, which is mentioned in Section 5. Extending Theorem 4.2.7 to k-edge-colorings of k-regular graphs on the projective plane, we also have the following theorem.

Theorem 4.2.8 *For $k \geq 3$, let G be a k-regular k-edge-colorable graph, let M be a perfect matching of G, and let D be a dual boundary. Then for a k-edge-coloring φ of type-s for M, where $s \in \{0, ..., k\}$, both of following holds.*

(i) If $k \equiv 0, 3 \pmod 4$,

then $\mathrm{sign}_{\rho[M,D]}(\varphi) = \begin{cases} (-1)^{\frac{|G|}{2}+|M\cap D|} & \text{if } k-s \equiv 0, 1 \pmod 4, \\ (-1)^{\frac{|G|}{2}+|M\cap D|+1} & \text{otherwise.} \end{cases}$

(ii) If $k \equiv 1, 2 \pmod 4$,

then $\mathrm{sign}_{\rho[M,D]}(\varphi) = \begin{cases} +1 & \text{if } k-s \equiv 0, 1 \pmod 4, \\ -1 & \text{otherwise.} \end{cases}$

Since Theorem 4.2.8 can be shown similarly to the proof of Theorem 4.2.7 in the next section, we leave its proof for the readers.

4.3 Proof of Theorem 4.2.7

In order to prove Theorem 4.2.7, we first consider that the signatures of edge-colorings with respect to a certain base, which depends on the edge-colorings. Recall that any 4-edge-coloring φ cannot be of type-0 for $\varphi^{-1}(1)$.

Lemma 4.3.1 *Let G be a 4-regular 4-edge-colorable graph on the projective plane and let D be a dual boundary. For a 4-edge-coloring φ of G, the following holds.*

$$\mathrm{sign}_{\rho[\varphi^{-1}(1),D]}(\varphi) = \begin{cases} (-1)^{\frac{|G|}{2}+|\varphi^{-1}(1)\cap D|} & \text{if } \varphi \text{ is of type-3 or 4 for } \varphi^{-1}(1), \\ (-1)^{\frac{|G|}{2}+|\varphi^{-1}(1)\cap D|+1} & \text{otherwise.} \end{cases}$$

We define a notation used in the proof of Lemma 4.3.1. For a 4-edge-coloring in a 4-regular graph G and for distinct $i, j \in \{1, 2, 3, 4\}$, we denote by $C(i, j)$ the subgraph induced by the edges of color i or j. Note that each vertex of $C(i, j)$ has degree exactly 2 and hence $C(i, j)$ is a 2-factor, i.e. a spanning 2-regular subgraph of G. Since $C(i, j)$ consists of vertex-disjoint cycles, we regard it also as a set of such cycles. Note that any edge of color 1 is contained in a cycle in $C(1, j)$ for $j \in \{2, 3, 4\}$.

Proof of Lemma 4.3.1. Let G be a 4-regular graph, let D be a dual boundary, and let φ be a 4-edge-coloring of G. We give a partition of $\varphi^{-1}(1)$ into S and NS as follows. (The notation S and NS stands for "singular" and "non-singular" as in [13].)

Definition 4.3.2 *For an edge $e = uv \in \varphi^{-1}(1)$, the sets S and NS are defined as follows.*

(i) If $e \notin D$, then $\begin{cases} e \in NS & \text{if } \mathrm{sign}_{\rho[M,D]}(\pi_u) \cdot \mathrm{sign}_{\rho[M,D]}(\pi_v) = 1, \\ e \in S & \text{otherwise.} \end{cases}$

(ii) If $e \in D$, then $\begin{cases} e \in NS & \text{if } \mathrm{sign}_{\rho[M,D]}(\pi_u) \cdot \mathrm{sign}_{\rho[M,D]}(\pi_v) = -1, \\ e \in S & \text{otherwise.} \end{cases}$

By Definition 4.3.2, we have

$$\begin{aligned}
\mathrm{sign}_{\rho[\varphi^{-1}(1),D]}(\varphi) &= \prod_{v\in V(G)} \mathrm{sign}_{\rho[\varphi^{-1}(1),D]}(\pi_v) \\
&= \prod_{uv\in\varphi^{-1}(1)} \left(\mathrm{sign}_{\rho[\varphi^{-1}(1),D]}(\pi_u)\cdot \mathrm{sign}_{\rho[\varphi^{-1}(1),D]}(\pi_v)\right) \\
&= (-1)^{|(S-D)|+|D\cap NS|} \\
&= (-1)^{|(S-D)|+|S\cap D|+|S\cap D|+|NS\cap D|} \\
&= (-1)^{|\varphi^{-1}(1)|-|NS|+|\varphi^{-1}(1)\cap D|} \\
&= (-1)^{\frac{|G|}{2}+|\varphi^{-1}(1)\cap D|+|NS|}. \qquad (4.1)
\end{aligned}$$

Thus, it suffices to focus on the parity of $|NS|$. For $e \in \varphi^{-1}(1)$, $\mathrm{Int}_e(\varphi)$ denotes the number of pairs $\{i,j\}$ such that $C(1,i)$ and $C(1,j)$ intersect transversally at e, see Figure 4.3. We have its relation to S and NS as follows.

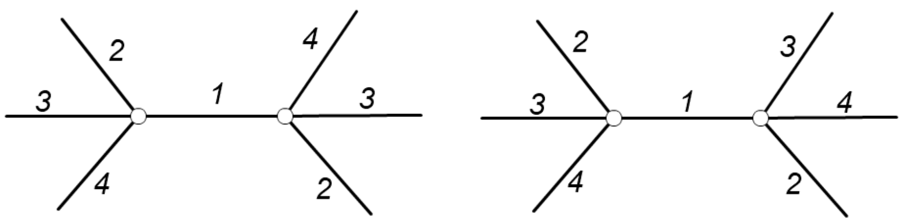

Figure 4.3: In the left, for any $2 \le i < j \le 4$, the 2-factors $C(1,i)$ and $C(1,j)$ intersect transversally at e, where e is the middle edge of color 1. In the right, the 2-factors $C(1,2)$ and $C(1,3)$ (and also $C(1,2)$ and $C(1,4)$) intersect transversally at e, while the ones $C(1,3)$ and $C(1,4)$ intersect at e but not transversally.

Claim 4 *For every $e \in \varphi^{-1}(1)$, we have $e \in NS$ if and only if* $\mathrm{Int}_e(\varphi) \equiv 1 \pmod 2$.

Proof. Let $e = uv$. Suppose first $e \notin D$ and $e \in NS$. If π_u and π_v are both identity mappings (see the left of Figure 4.3), then $C(1,i)$ and $C(1,j)$ intersect transversally at e for all the pairs $\{i,j\}$ with $2 \le i < j \le 4$, and hence $\mathrm{Int}_e(\varphi) = 3 \equiv 1 \pmod 2$. Thus, we consider the other cases. By the definition of NS, $\mathrm{sign}(\pi_u)$ and $\mathrm{sign}(\pi_v)$ are both positive or negative. Thus, we obtain the product $\mathrm{sign}(\pi_u)\cdot\mathrm{sign}(\pi_v)$ from the product of the identity mappings by transpositions an even number of times in total. Since any transposition changes the number of pairs $\{i,j\}$ such that $C(1,i)$ and $C(1,j)$ intersect transversally at e by one, we have $\mathrm{Int}_e(\varphi) \equiv 1 \pmod 2$.

Suppose next $e \notin D$ and $e \in S$. Without loss of generality, we may assume that $\mathrm{sign}(\pi_u) = 1$ and $\mathrm{sign}(\pi_v) = -1$. In this case, we obtain the product $\mathrm{sign}(\pi_u)\cdot\mathrm{sign}(\pi_v)$ from the product of the identity mappings by transpositions an odd number of times. By the same way, we have $\mathrm{Int}_e(\varphi) \equiv 0 \pmod 2$. Therefore, Claim 4 holds when $e \notin D$.

The same arguments work even when $e \in D$, which proves Claim 4. ■

By Claim 4, we directly obtain

$$|NS| \equiv \sum_{e\in\varphi^{-1}(1)} \mathrm{Int}_e\big(\varphi\big) \pmod 2. \tag{4.2}$$

Thus, in order to discuss the parity of $|NS|$, we focus on $\sum_{e\in\varphi^{-1}(1)} \mathrm{Int}_e\big(\varphi\big)$. To think this in detail, for $2 \le i < j \le 4$, we denote by $\mathrm{Int}_{i,j}\big(\varphi\big)$ the number of edges in $\varphi^{-1}(1)$ at which $C(1,i)$ and $C(1,j)$ intersect transversally. Since any pairs $C(1,i)$ and $C(1,j)$ intersect at only edges in $\varphi^{-1}(1)$, we have

$$\sum_{e\in\varphi^{-1}(1)} \mathrm{Int}_e\big(\varphi\big) = \sum_{2\le i<j\le 4} \mathrm{Int}_{i,j}\big(\varphi\big), \tag{4.3}$$

in which both sides represent the total number of transversal intersections between $C(1,i)$ and $C(1,j)$ for all possible pairs $\{i,j\}$ with $2 \le i < j \le 4$. As in Proposition 4.1.5, this number is closely related to the topology of cycles in $C(1,i)$'s, which will be handled in the next claim.

Claim 5 *For $i \in \{2,3,4\}$, both of the following hold.*

(i) If φ satisfies $|\varphi^{-1}(1) \cap D| \equiv |\varphi^{-1}(i) \cap D| \pmod 2$, then every cycle in $C(1,i)$ is contractible.

(ii) If φ satisfies $|\varphi^{-1}(1) \cap D| \not\equiv |\varphi^{-1}(i) \cap D| \pmod 2$, then there exists exactly one cycle in $C(1,i)$ which is essential.

Proof. Without loss of generality, we may assume $i = 2$. Suppose that $|\varphi^{-1}(1) \cap D| \equiv |\varphi^{-1}(2) \cap D| \pmod 2$. Then the total number of points at which $C(1,2)$ intersects with the boundary D^* is $|\varphi^{-1}(1) \cap D| + |\varphi^{-1}(2) \cap D|$, that is, $C(1,2)$ intersects D^* an even number of times by the assumptions. Since the cycle D^* is essential, it follows from Proposition 4.1.5 that the number of essential cycles in $C(1,2)$ is even. Moreover, since two distinct cycles in $C(1,2)$ cannot intersect with each other, the number of essential cycles in $C(1,2)$ is at most one by Proposition 4.1.5. Thus, every cycle in $C(1,2)$ is contractible and hence (i) holds. By the similar discussion, we also obtain (ii). ■

We are ready to discuss $\sum_{2\le i<j\le 4} \mathrm{Int}_{i,j}\big(\varphi\big)$, depending on the type of the edge-coloring φ for $\varphi^{-1}(1)$.

Case 1: φ is of type-4 for $\varphi^{-1}(1)$
By Claim 5, each cycle of $C(1,i)$ is contractible for any $i \in \{2,3,4\}$. Thus, Proposition 4.1.5 implies that $\mathrm{Int}_{i,j}\big(\varphi\big)$ is even for $2 \le i < j \le 4$, which concludes that $|NS|$ is even

by Equalities (4.2) and (4.3).

Case 2: φ is of type-3 for $\varphi^{-1}(1)$.
Without loss of generality, we may assume

$$|\varphi^{-1}(1) \cap D| \equiv |\varphi^{-1}(2) \cap D| \equiv |\varphi^{-1}(3) \cap D| \not\equiv |\varphi^{-1}(4) \cap D| \pmod 2.$$

By Claim 5, any cycles of $C(1,2)$ and $C(1,3)$ are contractible but exactly one of $C(1,4)$ is essential. By Proposition 4.1.5, $\mathrm{Int}_{i,j}(\varphi)$ is even for $2 \leq i < j \leq 4$, which concludes that $|NS|$ is even by Equalities (4.2) and (4.3).

Case 3: φ is of type-2 for $\varphi^{-1}(1)$.
Without loss of generality, we may assume

$$|\varphi^{-1}(1) \cap D| \equiv |\varphi^{-1}(2) \cap D| \not\equiv |\varphi^{-1}(3) \cap D| \equiv |\varphi^{-1}(4) \cap D| \pmod 2.$$

By Claim 5, every cycle in $C(1,2)$ is contractible but one of $C(1,i)$ is essential for $i \in \{3,4\}$. By Proposition 4.1.5, $\mathrm{Int}_{2,i}(\varphi)$ is even for $i \in \{3,4\}$ and $\mathrm{Int}_{3,4}(\varphi)$ is odd, which concludes that $|NS|$ is odd by Equalities (4.2) and (4.3).

Case 4: φ is of type-1 for $\varphi^{-1}(1)$. In this case, we have

$$|\varphi^{-1}(1) \cap D| \not\equiv |\varphi^{-1}(2) \cap D| \equiv |\varphi^{-1}(3) \cap D| \equiv |\varphi^{-1}(4) \cap D| \pmod 2.$$

By the similar discussion, $\mathrm{Int}_{i,j}(\varphi)$ is odd for $2 \leq i < j \leq 4$, and hence $|NS|$ is odd.

By Cases 1–4 together with Equality (4.1), the proof of Lemma 4.3.1 is completed. □

Next, we proof the Theorem 4.2.7.

Proof of Theorem 4.2.7. We divide the proof into two cases.

Case 1: φ is not of type-0 for M.
In this case, there exists $i \in \{1,2,3,4\}$ such that $|M \cap D| \equiv |\varphi^{-1}(i) \cap D| \pmod 2$. By symmetry, we may assume that $i = 1$.

We show this case by the induction on $M \setminus \varphi^{-1}(1)$. If $M = \varphi^{-1}(1)$, then we obtain the desired result from Lemma 4.3.1. Thus, we may assume $M \neq \varphi^{-1}(1)$, and hence there exists a cycle C consisting of edges in M and $\varphi^{-1}(1)$ alternately. By the similar discussions as Claim 5 in the proof of Lemma 4.3.1, the assumption $|M \cap D| \equiv |\varphi^{-1}(1) \cap D| \pmod 2$ implies that C is contractible. Let $M' = M \triangle E(C)$. Since C is contractible, Proposition 4.1.5 implies that C intersect D an even number of times. Thus, $|M' \cap D| \equiv |M \cap D| \pmod 2$, and hence the type of φ for M' does not change from the type for M. Since $M' \setminus \varphi^{-1}(1) = (M \setminus \varphi^{-1}(1)) \setminus E(C)$, it follows from the induction hypothesis that it suffices to show $\mathrm{sign}_{\rho[M,D]}(\varphi) = \mathrm{sign}_{\rho[M',D]}(\varphi)$.

Since C is contractible, C separates the projective plane into the inside and the outside. Let A be the set of vertices v in C such that the numbers of edges incident with v from the inside of C is odd, and let $B = V(C) - A$. Since G is 4-regular, each vertex in A is incident with an odd number of edges from the outside of C, and each vertex in B is incident with an even number of edges from both the inside and the outside of C. (See Figure 4.4.) This condition and the definition of π_v imply that for a vertex v in C, $\mathrm{sign}_{\rho[M,D]}(\pi_v) \neq \mathrm{sign}_{\rho[M',D]}(\pi_v)$ if and only if $v \in B$. Thus, $\mathrm{sign}_{\rho[M,D]}(\varphi) = \mathrm{sign}_{\rho[M',D]}(\varphi)$ if and only if $|B|$ is an even integer.

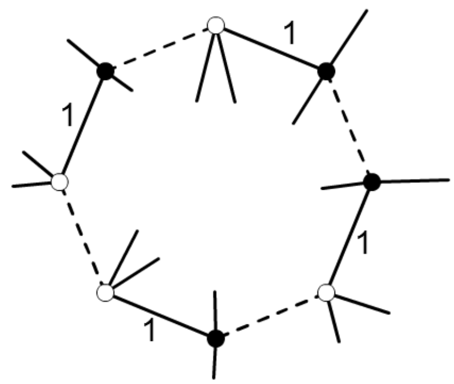

Figure 4.4: The contractible cycle C, where the bold (resp. dotted) edges denote the elements of $\varphi^{-1}(1)$ (resp. M). The black (resp. white) vertices belong to A (resp. B).

Since C is an even cycle, $|A| + |B| \equiv 0 \pmod 2$. Moreover, since G is 4-regular, it follows from applying Handshaking lemma to the subgraph induced by C and its inside that $|A| \equiv 0 \pmod 2$. Thus, we have $|B| \equiv 0 \pmod 2$. This completes the proof of Case 1.

Case 2: φ is of type-0 for M.
For $1 \leq i \leq 4$, let $C(i, M)$ denotes the set of cycles consisting of the edges of color i or edges in M. In this case, each $i \in \{1, 2, 3, 4\}$ satisfies $|M \cap D| \not\equiv |\varphi^{-1}(i) \cap D| \pmod 2$, and hence there exists exactly one cycle C_i in $C(i, M)$ which is essential. By Proposition 4.1.5, C_i and C_j intersect transversally an odd number of times for $1 \leq i < j \leq 4$.

Let $M' = M \triangle E(C_1)$. Since C_1 is essential, Proposition 4.1.5 implies that C_1 intersects D an odd number of times. Thus, $|M' \cap D| \not\equiv |M \cap D| \pmod 2$, and hence φ is of type-4 for M'. By Case 1, we already have

$$\mathrm{sign}_{\rho[M',D]}(\varphi) = (-1)^{\frac{|G|}{2} + |M' \cap D|} = (-1)^{\frac{|G|}{2} + |M \cap D| + 1}.$$

Therefore, it suffices to show $\mathrm{sign}_{\rho[M,D]}(\varphi) \neq \mathrm{sign}_{\rho[M',D]}(\varphi)$.

We contract all edges of M and obtain a 6-regular graph, say G' on the projective plane. We denote by $[uv]$ the vertex in G' corresponding to the contraction of $uv \in M$.

From the essential cycle C_i, we obtain an essential cycle in G', say C_i', by contracting all edges in $M \cap E(C_i)$. Note that all edges in C_i' are colored with i. Let A' be the set of vertices $[uv]$ in C_1' such that the number of edges incident with $[uv]$ from one side of C_1' is odd, and let $B' = V(C_1') - A'$. Since G' is 6-regular, if $[uv]$ is incident with an odd number of edges from one side of C_1', then it is incident with an odd number of edges from the other side, too. (See Figure 4.5.)

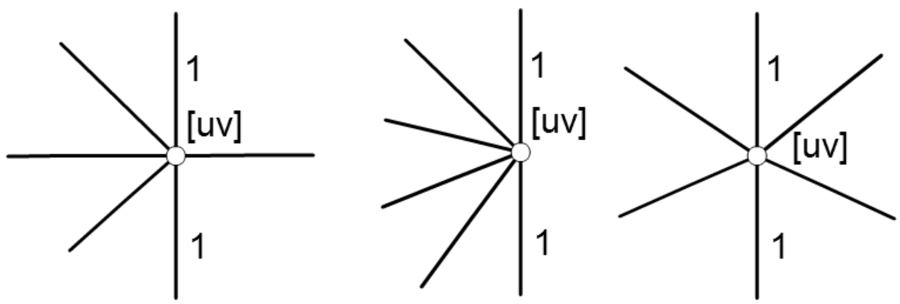

Figure 4.5: In the left, the vertex $[uv]$ belongs to A', while $[uv]$ belong to B' in the center and right.

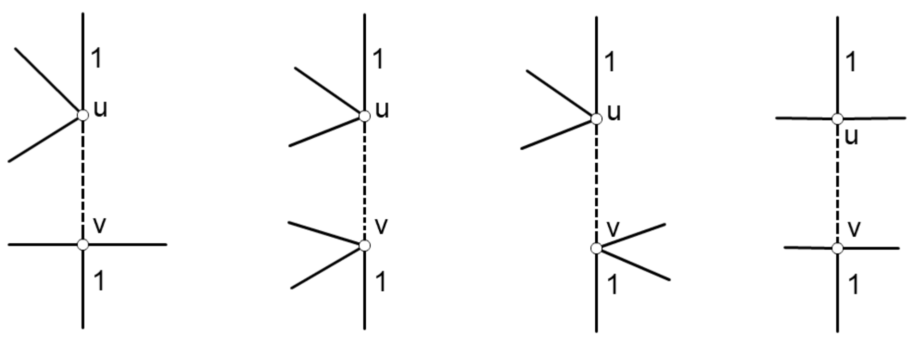

Figure 4.6: The first one represent the situation of an edge uv in M such that $[uv] \in A'$, while the other three represent the case $[uv] \in B'$.

It is easy to see that $[uv] \in A'$ if and only if an odd number of $C(2, M), C(3, M)$ and $C(4, M)$ intersect with C_1' transversally at $[uv]$. By Proposition 4.1.5, C_1' and $C(i, M)$ intersect transversally an odd number of times for $i \in \{2, 3, 4\}$, which implies $|A'| \equiv 1 \pmod 2$.

On the other hand, for each vertex $[uv]$ in C_1', we see that

$$\mathrm{sign}_{\rho[M,D]}(\pi_u) \cdot \mathrm{sign}_{\rho[M,D]}(\pi_v) \neq \mathrm{sign}_{\rho[M',D]}(\pi_u) \cdot \mathrm{sign}_{\rho[M',D]}(\pi_v)$$

if and only if $[uv] \in A'$ (See Figure 4.6). These directly imply $\mathrm{sign}_{\rho[M,D]}(\varphi) \neq \mathrm{sign}_{\rho[M',D]}(\varphi)$, which completes the proof of Theorem 4.2.7. $\square$

4.4 Application to the List Coloring Conjecture

In the remaining of this paper, for an ease of notation, we always denote by $\varphi^{-1}(1)$ the set of edges of color 1 under the edge-coloring we are dealing with.

In this section, we introduce an application of the signature to list-edge-colorings. Let G be a graph. A map $L : E(G) \to 2^{\mathbb{N}}$ is called a *list* of G. If G has an edge-coloring φ such that $\varphi(e) \in L(e)$ for any $e \in E(G)$, we say that G is L*-list-edge-colorable* and such an edge-coloring φ is called an L-edge-coloring. If G is L-list-edge-colorable for any list L that satisfies $|L(e)| \geq k$ for any $e \in E(G)$, we say that G is k-list-edge-colorable. By Theorems 4.2.3 and 4.1.2, Conjecture 4.1.1 holds for k-regular k-edge-colorable planar graphs. We consider its analogy to graphs on the projective plane.

Let G be a k-regular k-edge-colorable graph on the projective plane. By Theorem 4.2.7, the signature depends on only types of edge-colorings if a dual boundary D is fixed. When G is a cubic graph, we have the following theorem. (Recall that the type does not depend on a perfect matching if a graph is odd-regular.)

Theorem 4.4.1 ([22]) *Let G be a cubic 3-edge-colorable graph on the projective plane. Then G has an edge-coloring of type-3 if and only if the dual triangulation G^* is 4-vertex-colorable.*

Therefore, it follows from Theorem 4.2.6 that if G^* is not 4-vertex-colorable, then all 3-edge-colorings of G has the same signature. Thus, together with Theorem 4.1.2, we conclude the following.

Theorem 4.4.2 ([22]) *Let G be a cubic 3-edge-colorable graph on the projective plane. If G^* is not vertex 4-colorable, then G is 3-list-edge-colorable.*

To look for the analogy of Theorem 4.4.1, we now focus on the existence of edge-colorings of type-k for $\varphi^{-1}(1)$. Let G be a k-regular k-edge-colorable graph on the projective plane. A $\mathbb{Z}_2^{k-1}$*-face-coloring* is a map from $F(G)$ to $\mathbb{Z}_2^{k-1}$, where $\mathbb{Z}_2^{k-1}$ is the direct product of $(k-1)$ copies of $\mathbb{Z}_2$, and $F(G)$ is the set of all faces of G. Let Ω_{k-1} be the standard basis of the $(k-1)$-dimensional space $\mathbb{Z}_2^{k-1}$ together with the all-ones vector, that is,

$$\Omega_k = \big\{(1, 0, \ldots, 0), (0, 1, \ldots, 0), \ldots, (0, \ldots, 0, 1), (1, 1, \ldots, 1)\big\}.$$

We say that a $\mathbb{Z}_2^{k-1}$-face-coloring ψ of G is *bad* if the following holds:

> For each vertex $v \in V(G)$, if $f_1, f_2, ..., f_k \in F(G)$ are the faces incident to v in the clockwise order around v, then
>
> $$\{\psi(f_i) \oplus \psi(f_{i+1}); 1 \leq i \leq k\} = \Omega_{k-1},$$
>
> where the subscript of f_i is taken modulo k and $\oplus$ denotes the sum on $\mathbb{Z}_2^{k-1}$.

(See the right of Figure 4.7 for the case of $k = 4$.) In the following, we will show that there exists a one-to-one-correspondence between k-edge-colorings of type-k for $\varphi^{-1}(1)$ and bad $\mathbb{Z}_2^{k-1}$-face-colorings up to the permutation of colors.

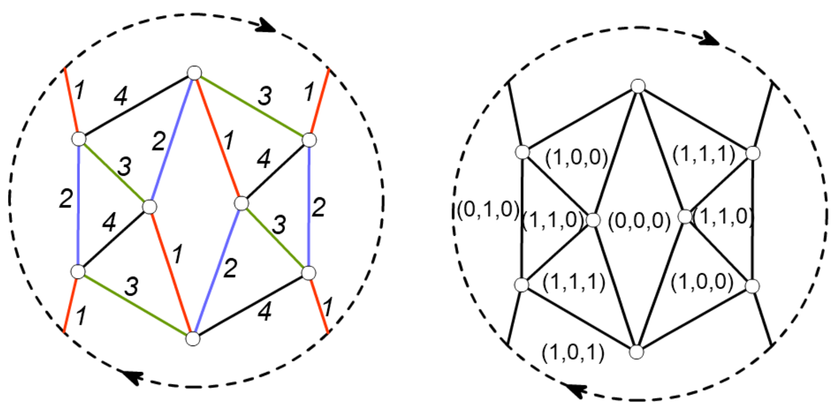

Figure 4.7: For a 4-regular graph on the projective plane, a 4-edge-coloring of type-4 for $\varphi^{-1}(1)$ in the left and a bad $\mathbb{Z}_2^3$-face-coloring in the right. They correspond to each other in the sense of Theorem 4.4.3.

Theorem 4.4.3 *Let G be a k-regular k-edge-colorable graph on the projective plane. Then G has an edge-coloring of type-k for $\varphi^{-1}(1)$ if and only if G has a bad $\mathbb{Z}_2^{k-1}$-face-coloring.*

Proof. First we show the only if part. Let φ be an edge-coloring of type-k for $\varphi^{-1}(1)$. Let $i \in \{2, \ldots, k\}$. Recall that $C(1, i)$ denotes the 2-factor consisting of the edges of color either 1 or i. Since φ is of type-k for $\varphi^{-1}(1)$, all cycles in $C(1, i)$ are contractible, and hence we can color the regions of $C(1, i)$ properly by two colors 0 and 1. For $f \in F(G)$, if f is colored by 0, then let $a_{i-1} = 0$: Otherwise, let $a_{i-1} = 1$. Let $\psi(f) = (a_1, \ldots, a_{k-1})$. Then each face receives the element of $\mathbb{Z}_2^{k-1}$ by ψ.

Let e be an edge, and let f and f' be the two faces incident to e. If $\varphi(e) = i \in \{2, \ldots, k\}$, then $\psi(f) \oplus \psi(f') = (b_1, \ldots, b_{k-1})$ where $b_{i-1} = 1$ and $b_j = 0$ for $j \in \{1, 2, \ldots, k-1\} - \{i-1\}$. On the other hand, if $\varphi(e) = 1$, then $\psi(f) \oplus \psi(f') = (1, 1, \ldots, 1)$. Thus $\psi(f) \oplus \psi(f')$ is equal to one element of Ω_{k-1}. Moreover, since φ is a k-edge-coloring, the vectors $\psi(f_i) \oplus \psi(f_{i+1})$ are distinct for each i. This proved the only if part.

Next, we will show the if part. Suppose that G has a bad $\mathbb{Z}_2^{k-1}$-face-coloring. Then we define a mapping φ from $E(G)$ to $\{1, 2, \ldots, k\}$ as follows. For an edge e in G, let f and f' be the faces incident to e. Then

$$\varphi(e) = \begin{cases} 1 & \text{if } \psi(f) \oplus \psi(f') = (1, \ldots, 1), \\ i & \text{if } \psi(f) \oplus \psi(f') = (b_1, \ldots, b_{k-1}), \\ & \text{where } b_{i-1} = 1 \text{ and } b_j = 0 \text{ for } j \in \{1, 2, \ldots, k-1\} - \{i-1\}. \end{cases}$$

By the definition of ψ, the edges incident to a vertex receive pairwise distinct colors, and hence φ is a k-edge-coloring. Suppose that φ is not of type-k for $\varphi^{-1}(1)$. Then, there exists $i \in \{2, \ldots, k\}$ such that $|\varphi^{-1}(1) \cap D| \not\equiv |\varphi^{-1}(i) \cap D| \pmod 2$. This implies that there exists exactly one essential cycle C_i in $C(1, i)$. Let $e \in E(C_i)$ with $\varphi(e) = 1$ and let $W = f_1 f_2 \cdots f_m$ be an essential cycle of G^* so that f_1 and f_m are incident to e. Moreover, let $\psi(f_1) = (a_1, \ldots, a_{k-1})$ and $\psi(f_m) = (a'_1, \ldots, a'_{k-1})$. Since any cycle in $C(1, i)$ except for C_i is contractible, a_{i-1} and a'_{i-1} must be the same if we trace W from f_1 to f_m. This contradicts $\psi(f_1) \oplus \psi(f_m) = (1, \cdots, 1)$. □

If $k = 3$, then Theorem 4.4.3 coincides with Theorem 4.4.1. Then, we focus on the 4-regular case as a next step. In particular, we have the following.

Corollary 4.4.4 *Let G be a 4-regular projective plane graph and D be a dual boundary of G. If $|D|$ is even and G does not have a bad $\mathbb{Z}_2^3$-face-coloring, then G is 4-list-edge-colorable.*

Proof. By the definition of types, recall that G has no edge-colorings of type-0 for $\varphi^{-1}(1)$. By Theorem 4.4.3, G has no edge-colorings of type-4 for $\varphi^{-1}(1)$. Thus, it follows from Proposition 4.2.5 that all edge-colorings φ of G are of type-2 for $\varphi^{-1}(1)$. Together with Theorems 4.2.7 and 4.1.2, this implies that G is 4-list-edge-colorable. □

Thus, Corollary 4.4.4 gives a new class of graphs for which the List Coloring Conjecture is true. We conclude this paper by showing that there exist infinitely many graphs G that satisfy the assumptions of Corollary 4.4.4.

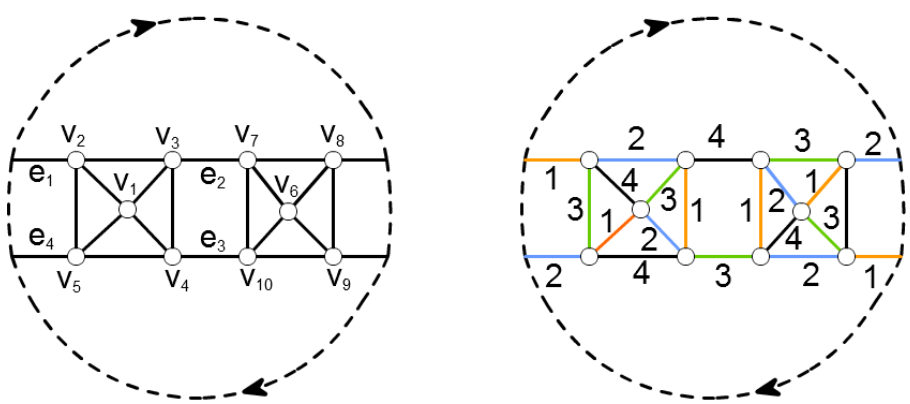

Figure 4.8: The left is an example of a graph having no bad $\mathbb{Z}_2^3$-face-colorings. The right is a 4-edge-coloring of type-2 for $\varphi^{-1}(1)$ in the same graph.

Proposition 4.4.5 *There exist infinitely many 4-regular 4-edge-colorable graphs G on the projective plane such that G does not have bad $\mathbb{Z}_2^3$-face-colorings.*

Proof. Let G be the graph in Figure 4.8, and let φ be a 4-edge-coloring of G. Since each $\varphi^{-1}(i)$ is a perfect matching of G and the subgraph induced by $\{v_1, v_2, v_3, v_4, v_5\}$ has an odd number of vertices, the edges e_1, e_2, e_3 and e_4 receive pairwise distinct colors by φ. This implies that φ must be type-2. (In fact there is a 4-edge-coloring of type-2 as in the right of Figure 4.8.)

The argument in the previous paragraph holds if a graph contains a 4-edge-coloring and the subgraph induced by $\{v_1, v_2, v_3, v_4, v_5\}$ such that two of the four edges e_1, e_2, e_3 and e_4 intersect with the boundary D^*. Thus, we can construct infinitely many graphs with desired properties. $\square$

Chapter 5

Dominating set

In this chapter, we introduce the domination. The domination number is also one of the important invariants of graphs and we consider this by applying the coloring methods.

5.1 Introduction

For $v \in V(G)$, let $N(v)$ denote the set of vertices which are adjacent to v. In particular, we call the set $N[v] = \{v\} \cup N(v)$ the *closed neighborhood* of v. Moreover, for $S \subset V(G)$, let $N(S)$ denote the *neighborhood* of S, i.e., the set of vertices adjacent to a vertex of S in G. For $S, T \subset V(G)$, we say that S *dominates* T if $T \subset S \cup N(S)$. If $D \subset V(G)$ dominates $V(G)$, then D is called a *dominating set* of G. The *domination number* of G is the minimum cardinality over all dominating sets of G and denoted by $\gamma(G)$.

A *disk triangulation* is a 2-connected plane graph such that every face except for the infinite face is triangular. Matheson and Tarjan proved the following theorem by an elegant coloring method:

Theorem 5.1.1 (Matheson and Tarjan [21]) *Let G be a disk triangulation with n vertices. Then $\gamma(G) \leq \lfloor \frac{n}{3} \rfloor$.*

They constructed a disk triangulation with n vertices in which any dominating sets have cardinality at least $\lfloor \frac{n}{3} \rfloor$, and hence the estimation in Theorem 5.1.1 is best possible. The examples they constructed are *maximal outerplanar graphs*, (i.e., a 2-connected plane graph such that there is a single face f containing all vertices on the boundary cycle, and that every face other than f is triangular), and so they have asked what happens if every face is triangular:

Conjecture 5.1.2 (Matheson and Tarjan [21]) *Let G be a planar triangulation with n vertices. If n is sufficiently large, then $\gamma(G) \leq \lfloor \frac{n}{4} \rfloor$.*

They constructed a plane triangulation G with n vertices satisfying $\gamma(G) = \lfloor \frac{n}{4} \rfloor$ for any large n. but the conjecture is still open so far. For this conjecture, Plummer, Ye and Zha [23] proved that every 4-connected plane triangulation with $n \geq 26$ vertices satisfies $\gamma(G) \leq \lfloor 5n/16 \rfloor$. In addition, King and Pelsmajer [20] proved that every plane triangulation G of maximum degree 6 with n vertices satisfies that $\gamma(G) \leq \lfloor \frac{n}{4} \rfloor$.

Let us focus on maximal outerplanar graphs. By Theorem 5.1.1, every maximal outerplanar graph G with n vertices has domination number at most $\lfloor \frac{n}{3} \rfloor$. This result is easily obtained by a proper 3-coloring, as follows: A maximal outerplanar graph is known to have a proper 3-coloring $c : V(G) \to \{1, 2, 3\}$. Observe that for $i = 1, 2, 3$, the set $c^{-1}(i)$ dominates G where $c^{-1}(i)$ is the set of vertices colored by i for the coloring c. Hence for some $i \in \{1, 2, 3\}$, we have $|c^{-1}(i)| \leq \frac{n}{3}$ since $|c^{-1}(1)| + |c^{-1}(2)| + |c^{-1}(3)| = n$, and we are done. Moreover, there exists a maximal outerplanar graph each of whose dominating set requires $\lfloor \frac{n}{3} \rfloor$ vertices [21]. Campos and Wakabayashi [7] pointed out that maximal outerplanar graphs with a large domination number have many vertices of degree 2, and they (and Tokunaga independently)proved the following theorem.

Theorem 5.1.3 (Campos and Wakabayasi [7] and Tokunaga [28]) *Let G be a maximal outerplanar graph with n vertices and t vertices of degree 2. Then $\gamma(G) \leq \lfloor \frac{n+t}{4} \rfloor$, where the bound is sharp.*

In this thesis, we introduce an "annulus triangulation" and consider its domination number. An *annulus triangulation* is a 2-connected plane graph with two disjoint special faces f_1 and f_2 such that every face of G except for f_1 and f_2 are triangular, and that every vertex of G is contained in the boundary cycle of f_1 or f_2. We say f_1 and f_2 *holed face* and any other faces *facial 3-cycles*. The boundary cycle of f_1 and that of f_2 are called the *boundary* of G. This seems to be a natural extension of maximal outerplanar graphs.

Our main theorem is as follows:

Theorem 5.1.4 *Let G be an annulus triangulation with n vertices and t vertices of degree 2. If $n \geq 7$, then $\gamma(G) \leq \lfloor \frac{n+t+1}{4} \rfloor$, where this estimation is sharp.*

A big difference between maximal outerplanar graphs and annulus triangulations is that an annulus triangulation G is not necessarily 3-colorable, and that G might not have vertices of degree 2. In this thesis, we elaborate a coloring method in [21, 28] and prove Theorem 5.1.4. In Section 2, we will prove lemmas to show the main theorem, and in Section 3, we prove the main theorem.

5.2 Dominating k-set-assignment

Let G be a graph and k be a positive integer. A *k-coloring* is a map $c : V(G) \to \{1, 2, ..., k\}$, and c is *proper* if $c(x) \neq c(y)$ for any $xy \in E(G)$. A k-coloring c is said to be a *dominating k-coloring* if for any $i \in \{1, \ldots, k\}$, the vertex set $c^{-1}(i)$ is a dominating set of G. By the definition, we have the following:

Proposition 5.2.1 *If a graph G admits a dominating k-coloring, then $\gamma(G) \leq \lfloor \frac{|V(G)|}{k} \rfloor$.*
■

Proposition 5.2.1 is useful to prove that a maximal outerplanar graph G with n vertices has a dominating set with cardinality at most $\lfloor \frac{n}{3} \rfloor$, since every proper 3-coloring of G is a dominating 3-coloring of G, as is mentioned in the previous section.

Extending the notion of a dominating k-coloring of a graph G, we define a "dominating k-set-assignment", as follows: An assignment $f : V(G) \rightarrow 2^{\{1,\ldots,k\}}$ is a *dominating k-set-assignment* if for any $i \in \{1, \ldots, k\}$, the vertex set

$$D_f(i) = \{v \in V(G) \; : \; i \in f(v)\}$$

is a dominating set of G. It is easy to see that f is a dominating k-set-assignment if and only if every vertex v has all k colors in its closed neighborhood. Let

$$d_G(f) = \sum_{i=1}^{k} |D_f(i)|.$$

By the definition, we have:

Proposition 5.2.2 *If a graph G admits a dominating k-set-assignment f, then $\gamma(G) \leq \lfloor \frac{d_G(f)}{k} \rfloor$.*

Note that if $|f(v)| = 1$ for every vertex $v \in V(G)$ in Proposition 5.2.2, then the statement coincides with Proposition 5.2.1. In order to prove our theorem, we give the definition of a property called good. Let G be a graph embedded on the plane. We say a 4-set-assignment f of a graph G is *good* if f satisfies all of the following conditions,

(D1) for each vertex v of degree at least 3 except for at most one vertex u, $|f(v)| = 1$,

(D2) for each vertex w of degree 2 or the vertex u as above (if exists), $|f(w)| = |f(u)| = 2$, and

(D3) for every facial 3-cycle $C = xyz$ of G, there exist three distinct colors $i_1, i_2, i_3 \in \{1, \ldots, 4\}$ such that $i_1 \in f(x), i_2 \in f(y), i_3 \in f(z)$.

Note that if f is good, then we have $d_G(f) \leq n + t + 1$, where n is the number of vertices of G and t is the number of vertices of degree 2 in G. In particular, Tokunaga [28] proved Theorem 5.1.3 by constructing, for a maximal outerplanar graph, a good dominating 4-set-assignment with additional properties.

Proposition 5.2.3 ([28]) *Let G be a maximal outerplanar graph with n vertices and t vertices of degree 2. Then G has a good dominating 4-set assignment f such that*

(P1) there is no exception in (D1) and hence $d_G(f) = n + t$, and

(P2) for any 4-cycle $xyzw$ in G, the four colors $1, 2, 3, 4$ are contained in the four sets $f(x), f(y), f(z), f(w)$ bijectively.

Let G be an annulus triangulation and let C_1 and C_2 denote boundary components of G. An edge e is a *boundary edge* if e is contained in C_1 or C_2. An edge e is *trivial* if e is not a boundary edge but the endpoints of e are contained in the same boundary component. For example, the edge x_0x_1 in Figure 5.1 is a boundary edge and y_1y_3 is trivial. We usually represent an annulus triangulation G by a rectangle cutting G along a non-trivial and non-boundary edge x_0y_0, as in Figure 5.1. By identifying the arrows of both ends, we obtain the annulus triangulation.

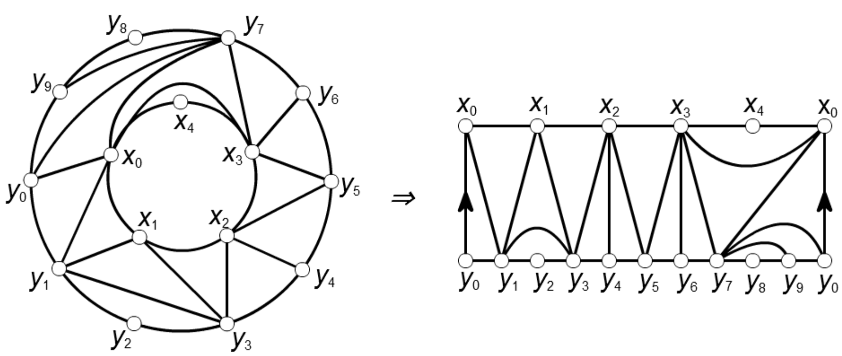

Figure 5.1: Different representations of an annulus triangulation.

Suppose that an annulus triangulation G has a trivial edge $e = xy$ whose endpoints are contained in C_1. Let P and P' be the two paths of G such that $V(P) \cup V(P') = V(C_1)$, that $V(P) \cap V(P') = \{x, y\}$, and that the cycle $P \cup \{e\}$ bounds a maximal outerplane subgraph D of G. We call D the *ear* of G separated by the edge xy. In particular, we say D is *maximal* if G has no trivial edge separating an ear including D as a proper subgraph. Removing an ear except for x and y decreases the number of trivial edges. So, repeating this operation, we finally get one with no trivial edges, which is called an *essential* subgraph of G and taken uniquely in G. See Figures 5.2 and 5.3. The graph drawn in Figure 3 is the essential subgraph of the graph in Figure 2.

In an essential annulus triangulation G, an edge e is called a *spoke* if an endpoint of e has degree 3. (We note that G has no vertex of degree less than 3 since G is essential.) An edge e is called a *frame edge* if e is neither a spoke nor boundary edge. The *frame* of G is the subgraph of G induced by the frame edges.

We first introduce two propositions for an annulus triangulation.

Proposition 5.2.4 *Let G be a non-essential annulus triangulation and let Y be a maximal ear of G separated by a trivial edge $e = xy$. Let G' be the annulus triangulation such that $G' \cup Y = G$ and $V(G') \cap V(Y) = \{x, y\}$ (See Figure 5.4). Then if G' admits a*

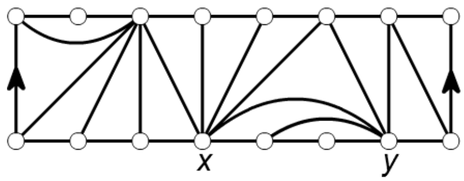

Figure 5.2: An annulus triangulation

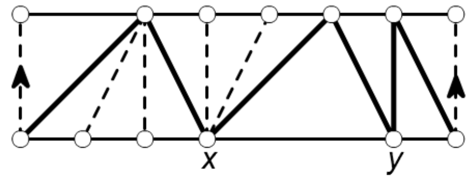

Figure 5.3: The thick edges are frame and the dotted ones are spoke

good dominating 4-set-assignment or if G' is isomorphic to the octahedron, then G has a good dominating 4-set-assignment.

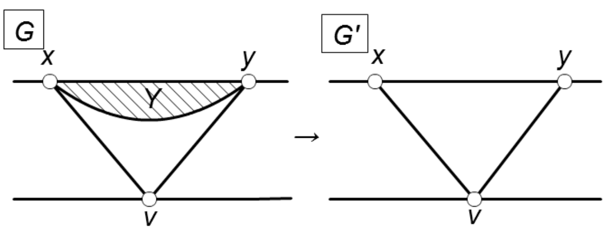

Figure 5.4: The ear reduction. The shaded area in the left figure is the maximal ear Y of G.

Proof. Without loss of generality, we may assume $\deg_Y(y) \geq \deg_Y(x)$. First, we will show that the edge xy is incident to a facial 3-cycle in G'. Let f_1 and f_2 be two distinct holed faces such that the vertices x and y are on the boundary of f_1. The edge xy is incident to exactly two faces, say f_1 and f_3 in G. If $f_3 = f_1$, then the edge xy is a cut edge of G, which contradicts 2-connectivity of G. Moreover, if $f_3 = f_2$, e is on the boundary of both f_1 and f_2, which contradicts that f_1 and f_2 are disjoint with each other. Thus e is incident to a facial 3-cycle $f_3 = xyv$. Moreover, since Y is a maximal ear, the vertex v is on the boundary of f_2.
Next, we show that G has a good dominating 4-set-assignment. If G' has a good dominating 4-set-assignment f', without loss of generality, we may assume $1 \in f'(x)$ and $2 \in f'(y)$. On the other hand, if G' is isomorphic to the octahedron, then we let f' be as shown in Figure 5.5.

We divide the proof into two cases depending on $|V(Y)|$.

Case 1 Suppose $|V(Y)| \geq 4$.
Since Y is a maximal outerplane graph and $\deg_Y(y) \geq \deg_Y(x)$, we have $\deg_Y(y) \geq 3$. Thus we may assume that Y has a cycle $C = xyzw$ such that $wy \in E(Y)$. By Proposition 5.2.3, Y admits a good dominating 4-set-assignment f_Y such that $1 \in f_Y(x)$, $\{2\} = f_Y(y)$, $3 \in f_Y(z)$ and $4 \in f_Y(w)$.

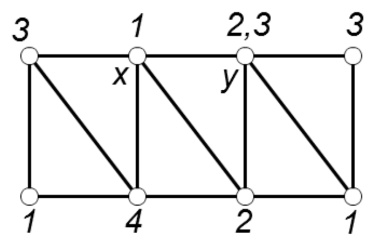

Figure 5.5: The 4-set-assignment f' of the octahedron.

We define the assignment f as

$$f(u) = \begin{cases} f'(u) & (u \in V(G')), \\ f_Y(u) & (u \in V(Y) - \{x, y\}). \end{cases}$$

By the construction of f, it is sufficient to prove that every vertex which is adjacent to x or y or which is x or y itself has all 4 colors in its closed neighborhood. We see that every vertex in $V(G') - \{y\}$ has all 4 colors in its closed neighborhood by f' in either case. Moreover, since y is contained in the cycle C in Y, y has all 4 colors in its closed neighborhood in Y. Thus every vertex in $V(G')$ has all 4 colors in its closed neighborhood for f. Next, we show that every vertex which is adjacent to x or y in Y has all 4 colors for f. Since $f_Y(y) \subset f(y)$ in either case, the vertices which are adjacent to y also have all 4 colors for f. Moreover, if $\deg_Y(x) = 2$, then $N_Y(x) = \{y, w\}$ and hence $f_Y(x) = \{1, 3\}$ by the assumptions and Proposition 5.2.3. Since $3 \in f_Y(z)$, the vertices which are adjacent to x in Y have all 4 colors for f. On the other hand, if $\deg_Y(x) \geq 3$, then we have $f_Y(x) = \{1\}$. In this case, we have $f_Y(x) \subset f(x)$ and hence the vertices which are adjacent to x have all 4 colors for f. Therefore, we see that f is a good dominating 4-set-assignment in G.

Case 2 Suppose $|V(Y)| = 3$.
In this case, Y is isomorphic to the complete graph K_3. Let $w \in V(Y)$ be the vertex which is neither x nor y. In this case, we get a good dominating 4-set-assignment f of G from f' such that

$$f(u) = \begin{cases} f'(u) & (u \in V(G')), \\ \{3, 4\} & (u = w). \end{cases} \qquad \blacksquare$$

Proposition 5.2.5 *Let G be an essential annulus triangulation and v be a vertex to which at least three consective spokes av, bv, cv are incident. Moreover, let G' be the graph obtained from G by removing the three edges av, bv, cv and smoothing the vertices a, b, c of degree 2, as shown in Figure 5.6. If G' is simple and admits a good dominating 4-set-assignment or if G' is isomorphic to the octahedron, then G admits a good dominating 4-set-assignment.*

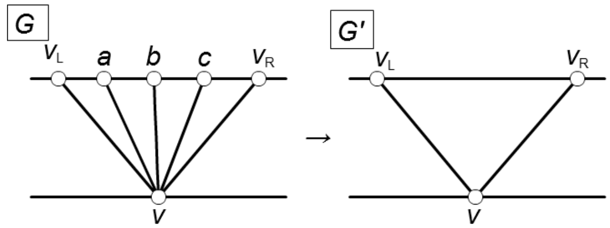

Figure 5.6: the spoke reduction

Proof. We devide the proof into two cases whether G' has a dominating 4-set-assignment f' or G is isomorphic to the octahedron.

Case 1 Suppose G' has a good dominating 4-set-assignment f'.

Clearly, G' has no vertices of degree 2. Let v_L (v_R respectively) be the vertex which is adjacent to v and a (v and c respectively) with $v_L \neq b$ ($v_R \neq b$ respectively) as in Figure 5.6. Without loss of generality, we may assume $1 \in f'(v_L), 2 \in f'(v)$ and $3 \in f'(v_R)$.

We define $f : V(G) \to 2^{\{1,2,3,4\}}$ as

$$f(z) = \begin{cases} f'(z) & (z \in V(G')), \\ \{3\} & (z = a), \\ \{4\} & (z = b), \\ \{1\} & (z = c). \end{cases}$$

We can easily check that all vertices except for v_L and v_R in G have all 4 colors in their closed neighborhoods and all facial cycles have distinct three colors. Suppose f is not a good dominating 4-set-assingment. By symmetry, we may assume v_L does not have four colors in its closed neighborhood. Since f' is a good dominating 4-set-assignment in G', we have $f(v_R) = \{3, 4\}$ and $N_G[v_L] \cap D_f(4) = \emptyset$. On the other hand, G has a facial cycle uv_Lv such that $u \neq v_R$. Since f' is good, we have $f(u) = \{3\}$. In this case, by exchanging the color of the vertices a and b in f, we obtain a good dominating 4-set-assignment in G.

Case 2 Suppose G' is isomorphic to the octahedron.

By symmetric, G is isomorphic the graph as shown in Figure 5.7 and we assign a 4-set assignment f to G as follows.

It is easy to see that f is a good dominating 4-set-assignment in G. ■

5.3 Domination number of annulus triangulations

For a graph G, a proper 4-coloring $c : V(G) \to \{1, 2, 3, 4\}$ is an *admissible 4-coloring* of G if every four vertices of G contained in a 4-cycle have four distinct colors. We can easily

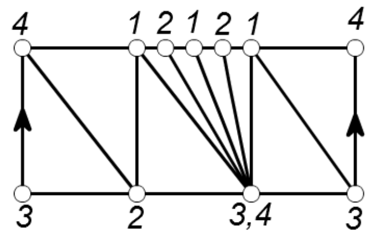

Figure 5.7: The graph G obtained from the octahedron by adding three spokes.

check that a 4-set-assignment f which includes an admissible coloring c (i.e. $c(v) \in f(v)$ for every $v \in V(G)$) satisfies the conditions $(D3)$ and $(P2)$. It is easy to see following.

Lemma 5.3.1 *Every maximal outerplane graph with $n \geq 4$ vertices has an admissible 4-coloring.*

Let G be a maximal outerplane graph and $c : V(G) \to \{1, 2, 3, 4\}$ be an admissible 4-coloring. Since every 4-cycle has all 4 colors, each color class $c^{-1}(i)$ dominates all vertices of degree at least 3. On the other hand, every vertex v of degree 2 has exactly one color i such that the set $c^{-1}(i)$ does not dominate v. In this case, the color i is the *missing color* for v.

The following is a key claim for the proof.

Theorem 5.3.2 *Let G be an annulus triangulation with n vertices which is not isomorphic to the octahedron. If G has no vertex of degree 2 or at least 7, then G has a good dominating 4-set-assignment.*

Proof. Let G be a minimum counterexample of Theorem 5.3.2.

Claim 6 *G does not have a vertex of degree 6.*

Proof. Suppose not. Let v be a vertex of degree 6 and let $v_1, v_2, v_3, v_4, v_5, v_6$ be the neighbors of v in this order with respect to the rotation of v, as in Figure 5.8. Let G' be the maximal outerplane graph obtained from G by removing v, v_3 and v_4, where we note that exactly one of v_1 and v_2, say x, has degree 2 in G' and so does exactly one of v_5 and v_6, say y. Since G is essential, G' has no vertex of degree 2 except for x and y, and hence by Lemma 5.3.1, G' has an admissible 4-coloring c such that each of x and y has a missing color. Without loss of generality, we may assume $c(v_1) = 1, c(v_2) = 2$ and the missing color of x is 4. Let $c(v_6) = a_1, c(v_5) = a_2$ and let the missing color of y be a_3. By Lemma 5.3.1, it is easy to see that a_1, a_2 and a_3 are distinct. Now we construct a good dominating 4-set-assignment f in G as follows.

Case 6.1 Suppose $a_3 \in \{1, 2, 3\}$.

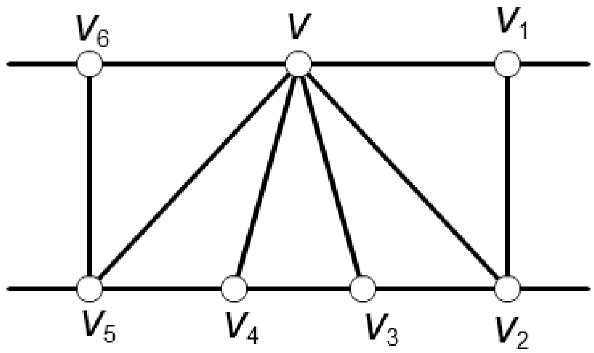

Figure 5.8: The vetex v and the neighbor of v

We let $b_3, b_4 \in \{1,2,3,4\}$ as follows.

$$b_4 \in \begin{cases} \{2\} & (a_2 \neq 2, a_3 \neq 2), \\ \{1,3\} - \{a_2\} & (otherwise). \end{cases}$$

$$b_3 \in \begin{cases} \{1,3\} - \{a_3\} & (a_2 \neq 2, a_3 \neq 2), \\ \{1,3\} - \{b_4\} & (otherwise). \end{cases}$$

Then we define an assignment f as

$$f(z) = \begin{cases} \{c(z)\} & (z \in V(G')), \\ \{a_3, 4\} & (z = v), \\ \{b_4\} & (z = v_4), \\ \{b_3\} & (z = v_3). \end{cases}$$

If $a_2 \neq 2$ and $a_3 \neq 2$, then we have $\{b_3, b_4, a_3\} = \{1,2,3\}$. Otherwise, we have $\{b_3, b_4\} = \{1,3\}$. In either case, we can easily check that every vertex has distinct four colors in its closed neighborhood and that f also satisfies good in G, which contradicts the assumption.

Case 6.2 Suppose $a_3 = 4$.
We assign

$$f(z) = \begin{cases} \{c(z)\} & (z \in V(G')), \\ \{4\} & (z = v), \\ \{1,2\} & (z = v_4), \\ \{3\} & (z = v_3). \end{cases}$$

In this case, it is easy to see this assignment f is also a good dominating 4-set-assignment. It contradicts the assumption. $\square$

Suppose G has a vertex of degree 5, say v. Let v_1, v_2, v_3, v_4, v_5 be the neighbors of v in this order with respect to the rotation of v, as in Figure 5.9. Let G' be the maximal outerplane graph obtained from G by removing v and v_3, where we note that exactly one of v_1 and v_2, say x, has degree 2 in G' and so does exactly one of v_4 and v_5, say y. By the assumption, G' has no vertex of degree 2 except for x and y, and hence by Lemma 5.3.1, G' has an admissible 4-coloring c such that each of x and y has a missing color. Without loss of generality, we may assume $c(v_1) = 1, c(v_2) = 2$ and the missing color of x is 4. Let $c(v_5) = a_1, c(v_4) = a_2$ and the missing color of y be a_3.

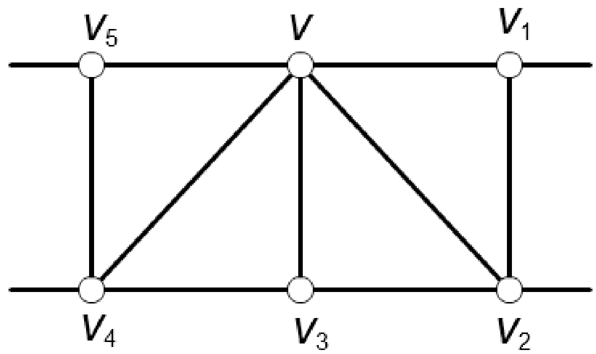

Figure 5.9: a vertex of degree 5

Claim 7 *If G has a vertex v of degree 5, then $a_2 = 4$ and $a_3 = 2$ where a_2 and a_3 are defined as above.*

Proof. Suppose not.

Case 7.1 Suppose $a_3 \in \{1, 2, 3\}$.
We let b_3 as follows.

$$b_3 \in \begin{cases} \{1,3\} - \{a_2\} & (a_2 \notin \{2,4\}), \\ \{1,3\} - \{a_3\} & (otherwise). \end{cases}$$

Then we define f as

$$f(z) = \begin{cases} \{c(z)\} & (z \in V(G')), \\ \{a_3, 4\} & (z = v), \\ \{b_3\} & (z = v_3), \end{cases}$$

If $a_2 \notin \{2, 4\}$, then b_3 is uniquely obtained and it is easy to see that f is a good dominating 4-set-assignment in G. If $a_2 = 2$, then we have $\{a_3, b_3\} = \{1, 3\}$ and it is easy to see that f is a good dominating 4-set-assignment in G. Moreover, if $a_2 = 4$ and $a_3 \neq 2$, then we have $\{a_3, b_3\} = \{1, 3\}$ and hence f is a good dominating 4-sat-assignment in G.

Case 7.2 Suppose $a_3 = 4$

We define f as

$$f(z) = \begin{cases} \{c(z)\} & (z \in V(G')), \\ \{4\} & (z = v), \\ \{1,3\} & (z = v_3), \end{cases}$$

In this case, we have f is a good dominating 4-set-assignment in G. □

Suppose G does not have a vertex of degree 4. In this case, every vertex in G has degree 3 or 5. Now we will show that G is uniquely obtained in this case. Since G is essential, we see that every vertex $v \in V(G)$ is an endpoint of a frame edge if and only if $\deg_G(v) = 5$. Let $C = x_0x_1...x_{k-1}$ and $C' = y_0y_1...y_{m-1}$ be two distinct boundary components in G. First, suppose that G has a boundary edge x_ix_{i+1} such that $\deg_G(x_i) = \deg_G(x_{i+1}) = 5$, where the subscript is taken modulo k. Since x_i and x_{i+1} are endpoints of the frame edges and $x_ix_{i+1} \in E(G)$, G has a vertex $y_j \in V(C')$ such that x_iy_j, y_jx_{i+1} are frame edges of G. Moreover, since y_j is endpoint of the frame edges and $x_ix_{i+1} \in E(G)$, we have $deg_G(y_j) = 4$. This contradicts the assumption. Next, suppose that G has a boundary edge x_ix_{i+1} such that $\deg_G(x_i) = \deg_G(x_{i+1}) = 3$. In this case, neither x_i nor x_{i+1} are endpoints of frame edges. Since they are endpoints of the spokes, they are adjacent to a common vertex $y \in V(C')$. This indicates that y has degree at least 6 and this fact contradicts the assumption. Thus, the vertices of degree 3 and ones of degree 5 appear alternatively in C and C'. Moreover, by counting the number of non-boundary edges, we have $2k = 2m$. This indicates that $|V(C)| = |V(C')|$. Without loss of generality, we may assume $\deg_G(x_0) = 3$ and $x_0y_0 \in E(G)$. Since G is an annulus triangulation, we have $\deg_G(y_0) = 5$ and hence $y_0x_1 \in E(G)$. Moreover, we see that $x_1y_1, x_1y_2 \in E(G)$ and that $y_2x_2 \in E(G)$ by the same reason as above. By repeating these argument, G is uniquely obtained as shown in Figure 5.10.

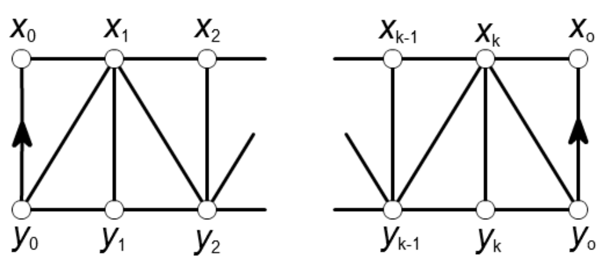

Figure 5.10: A situation without vertices of degree 4

Let $G' = G - \{x_1, y_1\}$. By lemma 5.3.1, G' has an admissible coloring c so that $c(x_2) = 1, c(y_2) = 2$ and missing color of x_2 is 4. We can easily check this coloring c satisfies that if $c(y_0) = 4$, then the missing color of x_0 is 1 and hence c does not satisfy Claim 7. Thus we may assume G has a vertex of degree 4. Next, we prove that G must be a 4-regular graph.

Claim 8 *G does not have a vertex of degree* 5*.*

Proof. Suppose not. Since G has a vertex of degree 4, G has a frame edge connecting a vertex of degree 5 and one of degree 4. Let v be a vertex of degree 5 and let v_1, v_2, v_3, v_4, v_5 be the neighbors of v in this order with respect to the rotation of v, as in Figure 5.9. We may assume that v_2 is a vertex of degree 4. Let u be the endpoint of the frame edge which is incident to v_1 with $u \neq v_2$. Since $G' = G - \{v, v_3\}$ is a maximal outerplanar graph, G' has an admissible coloring c by Lemma 5.3.1. Without loss of generality, we may assume $c(v_1) = 1, c(v_2) = 2$ and the missing color of v_2 is 4. By Claim 7, $c(v_4) = 4$ and hence $u \neq v_4$. Next, we construct a good dominating 4-set-assignment as follows.

Case 8.1 Suppose $\deg_G(v_1) = 4$. We define an assignment f as

$$f(z) = \begin{cases} \{c(z)\} & (z \in V(G') - \{v_2\}), \\ \{2\} & (z = v), \\ \{4\} & (z = v_2), \\ \{1, 3\} & (z = v_3). \end{cases}$$

In this case, it is easy to see that every vertex which is not adjacent to u and whose degree is at least 3 in G' has all 4 colors in its closed neighborhood by f. Moreover, since $u \neq v_4$ and any 4-cycles in G' except for the cycle bounded by v_2v_1wu have all 4 colors by Lemma 5.3.1, where w is a vertex which is adjacent to v_1 and u with $w \neq v_2$, we have $|N[u] \cap c^{-1}(2)| = 2$. Thus the vertex u also has all 4 colors in its closed neighborhood by f. Moreover, it is easy to see that every vertex $v_i (i \in \{1, 2, ..., 5\})$ has all 4 colors in its closed neighborhood by f. Thus f is a good dominating 4-set-assignment in G.

Case 8.2 Suppose $\deg_G(v_1) = 5$.

In this case, v_1 has one spoke v_1w'. If $u = v_4$, then it is easy to see that $N_{G'}[v_4] \cap N_{G'}[v_5] \cap c^{-1}(2) \neq \emptyset$, which contradicts Claim 7. Thus we conclude $u \neq v_4$. We define an assignment f as

$$f(z) = \begin{cases} \{c(z)\} & (z \in V(G') - \{v_2, w'\}), \\ \{2, 4\} & (z = v), \\ \{3\} & (z = v_2), \\ \{1\} & (z = v_3), \\ \{2\} & (z = w'). \end{cases}$$

By the similar argument as before, we conclude f is a good dominating 4-set-assignment. □

By Claim 6 and 8, the degree of every vertex in G is at most 4. Suppose that G has a vertex of degree 3, say v. Since G does not have a vertex of degree 2, v must be an endpoint of a spoke vw. This implies that the degree of w is at least 5, which is a contradiction. Thus G is a 4-regular graph. Let $C = x_0x_1...x_k$ and $C' = y_0y_1...y_m$ be two distinct boundary components of G. Without loss of generality, we may assume $x_0y_0, x_0y_1 \in E(G)$. We can easily check $|V(C)| = |V(C')|$ and G is uniquely obtained, as shown in Figure 5.11.

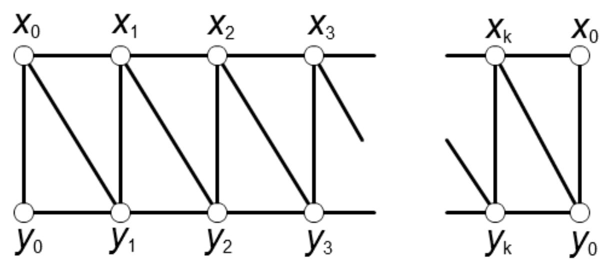

Figure 5.11: A 4-regular graph

Claim 9 $|V(G)|$ *is at most* 6.

Proof. Suppose not. Since $G' = G - \{x_1, y_1\}$ is a maximal outerplanar graph, G' has the admissible coloring c by Lemma 5.3.1. Without loss of generality, we may assume $c(x_2) = 1, c(y_2) = 2$ and $c(y_3) = 3$. If k is odd, then we see that $(c(x_0), c(y_0)) = (1, 2)$ and hence we can get a good dominating 4-set-assignment f in G naturally as follows.

$$f(z) = \begin{cases} \{c(z)\} & (z \in V(G')), \\ \{4\} & (z = x_1), \\ \{3\} & (z = y_1). \end{cases}$$

Otherwise, we see that $(c(x_0), c(y_0)) = (4, 3)$. In this case, we define f as

$$f(z) = \begin{cases} \{c(z)\} & (z \in V(G') - \{y_2\}), \\ \{2\} & (z = x_1), \\ \{1, 3\} & (z = y_1), \\ \{4\} & (z = y_2). \end{cases}$$

Since G is not isomorphic to the octahedron, we see that $k \geq 4$. Every vertex in $V(G') - \{x_0, x_2, y_2, y_3\}$ has all 4 colors in its closed neighborhood by f. Moreover, since $c(y_4) = 2$, the vertex y_3 also has all 4 colors in its closed neighborhood. It is easy to see that any other vertices have all 4 colors in their closed neighborhood. Thus f is a good dominating 4-set-assignment in G. Therefore, by Claim 6, 8 and 9, if G is not isomorphic to the octahedron, then G has a good dominating 4-set-assignment. ■

5.4 Proof of Theorem 5.1.4

Proof. By Proposition 5.2.2, it is sufficient to prove that G has a good domminating 4-set-assignment unless G is the octahedron. Let G be a counterexample as above with minimum cardinality. Suppose G is non-essential (i.e. G has a trivial edge xy). Let G' be a graph obtained by removing a maximal ear Y of G except for xy. It is easy to see that G' is an annulus triangulation. By the minimality of G, G' has a good dominating 4-set-assignment or that G' is isomorphic to the octahedron. On the other hand, by Proposition 5.2.4, we conclude that G has a good dominating 4-set-assignment, which contradicts the assumption. Thus we may assume G is essential.

Suppose G has a vertex of degree at least 7. Then G has a vertex v such that v is an endpoint of at least three spokes av, bv, cv. Let G' be the graph obtained from G by removing the three edges av, bv, cv and smoothing the vertices a, b, c of degree 2. Let v_L (v_R respectively) be the vertex which is adjacent to v and a (v and c respectively) with $v_L \neq b$ ($v_R \neq b$ respectively) as in Figure 5.6. It is easy to see that G' is not simple if and only if $v_L v_R \in E(G)$ and $\deg_G(v) = 7$. If G' is simple, then G' has a good dominating 4-set-assignment or G' is isomorphic to the octahedron. By Proposition 5.2.5, G has a good dominating 4-set-assignment for either case, which contradicts the assumption. Therefore, we may assume that G is not simple, then G has the edge $v_L v_R$ and $\deg_G(v) = 7$. Since $v_L v_R \in E(G)$ and since G is a simple annulus triangulation, the structure of G is restricted as shown in Figure 5.12. The ? areas in Figure 5.12 may have some spokes.

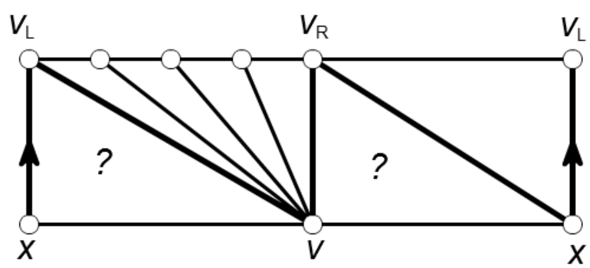

Figure 5.12: A situation of G such that G' is not simple.

By symmetry, we may assume $\deg_G(v_L) \leq \deg_G(v_R)$. If G satisfies that $\deg_G(v_R) \geq 8$

or that $\deg_G(v_R) = 7$ and $xv \notin E(G)$, then we obtain the simple annulus triangulation G'' by focusing on v_R instead of v. Thus G has a good dominating 4-set-assignment by the induction hypothesis, which contradicts the assumption. Moreover, if $deg_G(v_R) = 4$, then $deg_G(v_L) = 4$ and hence G has a multiple edge. Thus we have $5 \leq \deg_G(v_R) \leq 7$ We construct a good dominating 4-set-assignment in G depending on $\deg_G(v_R)$ as follows.

Case 1 Suppose that $\deg_G(v_R) = 7$ and $xv \in E(G)$.

We assign the 4-set-assignment as shown in Figure 5.13.

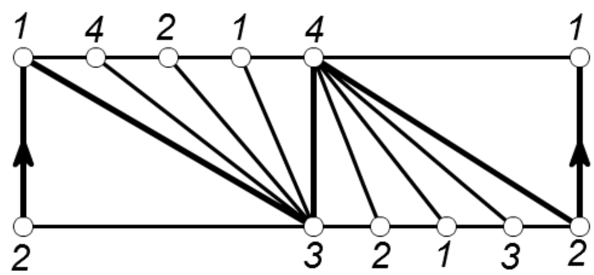

Figure 5.13: The degree of v_R is 7 and $xv \in E(G)$.

Case 2 Suppose that $deg_G(v_R) = 6$.

In this case, we see that $4 \leq \deg_G(v_L) \leq 6$. We assign the 4-set-assignment of G as in Figures 5.14 to 5.16.

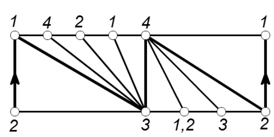

Figure 5.14: $\deg_G(v_L) = 4$

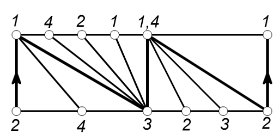

Figure 5.15: $\deg_G(v_L) = 5$

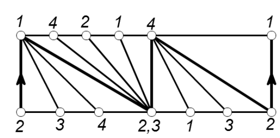

Figure 5.16: $\deg_G(v_L) = 6$

Case 3 Suppose that $\deg_G(v_R) = 5$.

In this case, we see that $4 \leq \deg_G(v_L) \leq 5$. We assign the 4-set-assignment as in Figures 5.17 and 5.18.

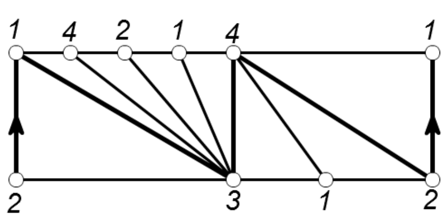

Figure 5.17: $\deg_G(v_L) = 4$

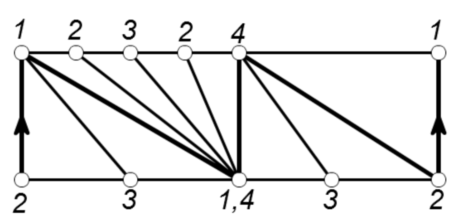

Figure 5.18: $\deg_G(v_L) = 5$

In either case, we see that each assignment as above is a good dominating 4-set-assignment in G, which contradicts the assumption. Thus we may assume that G does not have a verex of degree at least 7.

By Theorem 5.3.2, G has a good dominating 4-set-assignment. Thus in any cases except for the octahedron, we constructed a good dominating 4-set-assignment with $d_G(f) \leq n+t+1$. ■

In order to prove the sharpness of the theorem, we construct an annulus triangulation satisfying the equality of the estimation. See Figure 5.19. We show $\gamma(G) = 7$. Let A_i be the closed neighborhood of a_i, for $i = 1,2,3,4,5,6$. Then observe that $A_1, \ldots, A_6$ are pairwise disjoint. Thus we must have $\gamma(G) \geq 6$, since we have to choose at least one vertex from A_i for $i = 1,2,3,4,5,6$, in order to dominate a_i. Hence we suppose that G has a dominating set S with $|S| = 6$. It is trivial $|S \cap A_i| = 1$ for any i. Observe that b_1 is the only vertex in $\bigcup A_i$ adjacent to x and so $S \cap A_1 = \{b_1\}$. Next, in order to dominate the vertex c_1, we have $S \cap A_3 = \{b_3\}$. By the same reason, we have $b_5 \in S$ to dominate c_3. By any choice of three vertices in A_2, A_4, A_6, S does not dominate y. Hence $\gamma(G) > 6$.

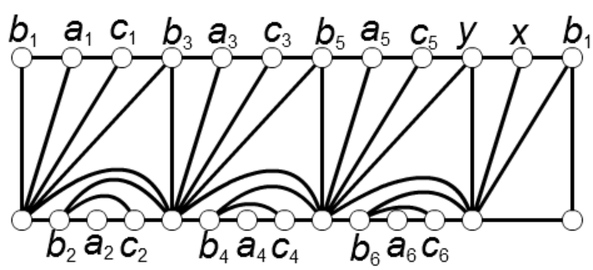

Figure 5.19: $n = 24, t = 3, \gamma(G) = 7$

By the similar discussion, we have an annulus triangulation with $\gamma(G) = \lfloor \frac{n+t+1}{4} \rfloor$ for some $n \geq N$, where N is a large constant.

Bibliography

[1] N. Alon, Combinatorial Nullstellensatz, *Combin. Probab Comput* **8** (1999) 7–29.

[2] N. Alon, Restricted Colorings of Graphs, *Surveys in Combinatorics* (1993), Walker (Ed.), London Mathematical Society Lecture Note Series 187, Cambridge University Press, 1993.

[3] N. Alon, M. Trasi, Coloring and orientations of graphs. *Combinatorica* **12** (1992), 125–134.

[4] B. Bollobas and A. J. Harris, List-Colourings of Graphs, *Graphs and Combinatorics* (1985), 115–127.

[5] K. Appel and W. Haken, The solution of the Four-Color-Map Problem, *Scientific American* **237** (1977), 108–121.

[6] A. Bernshteyn, A. Kostochka, On defferences between DP-coloring and list coloring, *arXiv:1705.04883* (2017).

[7] C.N. Campos, Y. Wakabayasi, On dominating sets of maximal outerplanar graphs, *Discrete Appl. Math.* **161** (2013), 330–335.

[8] W. Cushing, H.A. Kirestread, planar graphs are 1-relaxed 4-choosable, *European Journal of Combinatorics* **31**(2010) 1385–1397.

[9] Z. Dvořák, L. Postle, Correspondence coloring and its application to list coloring planar graphs without cycles of length 4 to 8, *Journal of Combinatorial Theory, Series B* **129** (2018), 38–54.

[10] M. Ellingham, L. Goddyn, List Edge Colourings of Some 1-Factorable Multigraphs. *Combinatorica* **16** (1996), no. 3, 343–352.

[11] N. Eaton, T.Hull, Defective List Colorings of Planar Graphs, *ulletin of the Institute of Combinatorics and its applications*, **25** (1999) 79–87.

[12] P. Erdős, A.L.Rubin, H. Taylor, *Proceedings of the West Coast Conference on Combinatorics, Graph Theory and Computing, Arcata, California, Congressus Numeratium XXVI.* (1979), 125–127.

[13] S. Fisk, *Geometric coloring theory,* Advances in Math. **324** (1977) 298–340.

[14] F. Jaeger, On the Penrose number of cubic diagrams, Discrete Math. **74** (1989), 85–97.

[15] J. Grytczuk, X.Zhu, The Alon-Tarsi number of a planar graph minus a matching, *Journal of Combinatorial Theory, Series B* **45** (2020) 511–520

[16] S. Gutner, The complexity of planar graph choosability, *Discrete Mathematics* **159** (1996), 119–130.

[17] W. He, W. Miao and Y. Shen, Another proof of the 5-choosability of K_5-minor-free graphs, *Discrete Math* **308** (2008) 4024–4026

[18] R. Kim, S. Kim and X. Zhu, The Alon-Tarsi number of subgraphs of a planar graph, arXiv:1906.01506

[19] T. Jensen, B.Toft, *Graph Coloring Problems, Wiley, New York* 1995.

[20] E.L.C. King and M. J. Pelsmajer, Dominating sets in plane triangulation, *Discrete Math.* **310** (2010), 2221-2230.

[21] L. Matheson, R. Tarjan, Dominating sets in planar graphs, *European J. Combin.* **17** (1996), 565–568.

[22] K. Ozeki, Kempe-equivalence of 3-edge-colorings in cubic graphs. *submitted.*

[23] M.D. Plummer, D. Ye, X. Zha, Dominating plane triangulation, *Discrete Appl. Math.* **211** (2016), 175–182.

[24] R. Skrekovski, List improper coloring s of planar graphs, *Combinatorics, Probability and Computing*, **8** (1994) 293–299.

[25] R. Skrekovski, Choosability of K_5-minor-free graphs, *Discrete Math* **190**(1998) 223–226.

[26] P.G. Tait, Remarks on the colorings of maps *Proc. R. Soc. Edinburgh.* **188** (1998), 183-203

[27] C. Thomassen, Every Planar Graph is 5-Choosable, *Journal of Combinatorial Theory, Series B* **62** (1994), 180–181.

[28] S. Tokunaga, Dominating sets of maximal outerplanar graphs, *Discrete Appl. Math.* **161** (2013), 3097–3099.

[29] V.G. Vizing, Vertex colorings with given colors (in Russian), *Metody Diskret. Analiz.*, **29** (1976), 3–10.

[30] M. Voigt, List colorings of planar graphs, *Discrete Mathematics* **120** (1993), 215–219.

[31] K. Wagner. Uber eine Eigenschaft der ebenen Komplexe. *Math. Ann* **114** (1937), 570–590

[32] W. Wang, J. Qian and T. Abe, Alon-Tarsi Number and Modulo Alon-Tarsi Number of Signed Graphs, *Graphs and Combinatorics*, **35**, (2019) 1051–1064.

[33] D.R. Woodall, Defective choosability of graphs in surfaces, *Discussiones Mathematicae Graph Theory* **31** (2011) 441–459

[34] X. Zhu, The Alon-Tarsi number of planar graphs, *J. Combin. Theory, Ser. B* **134** (2019) 354–358