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ON SOLUTIONS OF $x'' = t^{\alpha\lambda-2}x^{1+\alpha}$ IN THE UNSETTLED CASES

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Abstract. As a continuation work, we show asymptotic behaviour of all positive solutions of the differential equation written in the title, in terms of getting the analytical expressions of the solution in the neighbourhoods of the ends of its domain. This is done in all unsettled cases.

1. Introduction

Using the method of [3, 4], we show the asymptotic behaviour of all positive solutions of

$$x'' = t^{\alpha\lambda-2}x^{1+\alpha} \ ({}' = d/dt) \tag{E}$$

in [5, 6, 7, 10, 11, 13, 14], where α, λ are real parameters and t, x are positive variables. However the cases

$$\begin{aligned} \mathrm{I}: &\ \alpha < \lambda_0,\ \lambda < -1 \text{ or } \ \alpha < \lambda_0,\ \lambda > 0 \ (\lambda_0 = -(2\lambda+1)^2/4\lambda(\lambda+1)) \\ \mathrm{II}: &\ \alpha < 0,\ -1 < \lambda < 0 \end{aligned}$$

are not yet treated. On the other hand, the techniques of the previous papers suffice for treating these cases. So we consider these in this paper and in the proofs we state only their outlines.

2. On the case I

In this section, we consider the case I. The transformation

$$(t, x) \to (1/t, x/t)$$

of [2] reduces the case $\lambda < -1$ to the case $\lambda > 0$. So, suppose

$$\alpha < \lambda_0,\ \lambda > 0.$$

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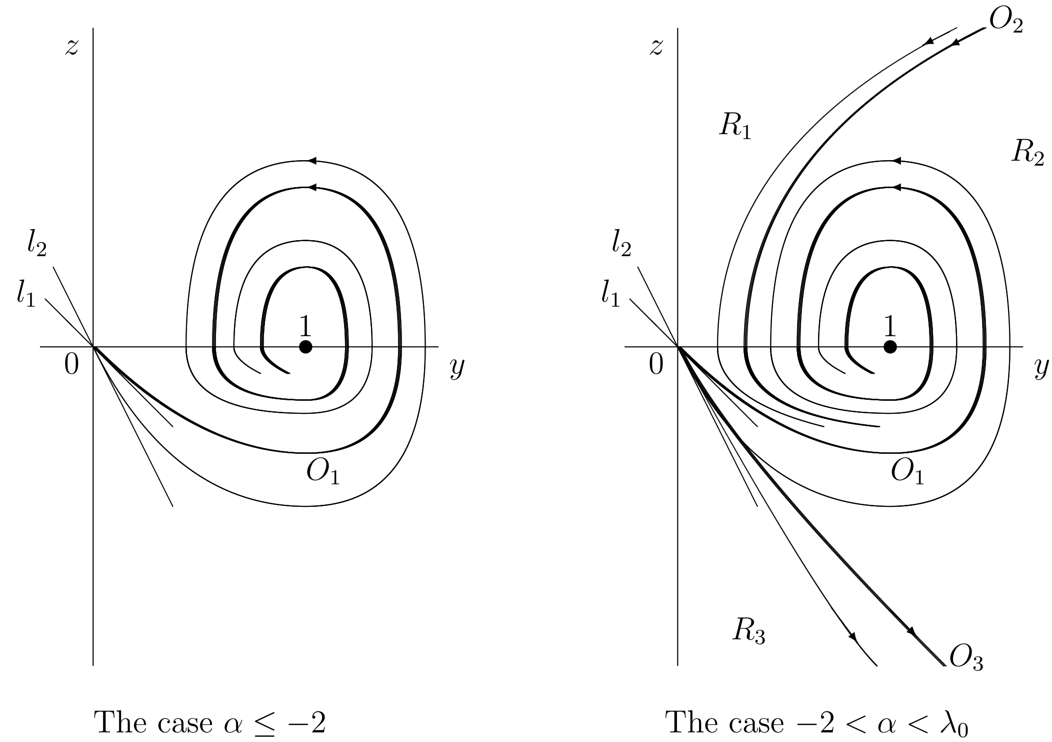


(l_1: the line $z = \alpha\lambda y$, l_2: the line $z = \alpha(\lambda+1)y$)

Figure 1

Note $\alpha < 0$ from this.

First, use Saito's transformation

$$y = \psi(t)^{-\alpha} x^{\alpha}, \ z = ty' \tag{T}$$

where $\psi(t) = \{\lambda(\lambda+1)\}^{1/\alpha} t^{-\lambda}$ is a particular solution of (E) (see [3, 4]). Then (E) is transformed into a first order rational differential equation

$$\frac{dz}{dy} = \frac{(\alpha-1)z^2 + \alpha(2\lambda+1)yz + \alpha^2\lambda(\lambda+1)y^2(y-1)}{\alpha yz} \tag{R}$$

and using a parameter s, we rewrite this as a two dimensional autonomous system

$$\frac{dy}{ds} = \alpha yz, \ \frac{dz}{ds} = (\alpha-1)z^2 + \alpha(2\lambda+1)yz + \alpha^2\lambda(\lambda+1)y^2(y-1) \tag{S}$$

whose critical points are $(0,0)$, $(1,0)$. Note $y > 0$, for $t > 0$, $x > 0$.

As shown below, the phase portrait of (S) is drawn as in Figure 1. Here $(1,0)$ is a spiral point, O_1, O_2, O_3 are unique orbits such that O_1 is tangent to the line

$z = \alpha\lambda y$, and O_2, O_3 satisfy $z = O(y^{3/2})$ as $y \to \infty$. In the phase portrait of the case $-2 < \alpha < \lambda_0$, we denote as R_1 the region which the z axis, O_1 and O_2 surround, as R_2 the region which O_1, O_2 and O_3 surround, and as R_3 the region which the z axis and O_3 surround. Note that the case $-2 < \alpha < \lambda_0$ arises if $\lambda > (-1+\sqrt{2})/2$.

Given the initial condition

$$x(t_0) = A,\ x'(t_0) = B\ (t_0 > 0,\ A > 0,\ B \in \mathbb{R}) \tag{I}$$

of (E), from applying (T) to the solution $x = x(t)$ of the initial value problem (E), (I) we get the solution $z = z(y)$ of (R) with

$$z(y_0) = z_0$$

and the orbit (y, z) of (S) passing (y_0, z_0), where

$$y_0 = \psi(t_0)^{-\alpha}A^{\alpha},\ z_0 = \alpha y_0\left(\lambda + \frac{t_0 B}{A}\right).$$

Now, take (t_0, A, B) of (I) and determine (y_0, z_0). Then we state our theorems as follows: First, suppose $\alpha \le -2$.

Theorem 1. (i) *If $(y_0, z_0) \in O_1$, then $x(t)$ is defined for $0 < t < \infty$. In the neighbourhood of $t = 0$, $x(t)$ is represented as*

$$x(t) = \psi(t)\left[1 + \sum_{k=1}^{\infty} t^{k\delta_1}\sum_{l=0}^{k}\left\{x_{kl}^{(1)}\cos(l\delta_2\log t) + x_{kl}^{(2)}\sin(l\delta_2\log t)\right\}\right] \tag{1}$$

where $x_{kl}^{(1)}$, $x_{kl}^{(2)}$ are constants such that

$$x_{k0}^{(1)} = 0 \text{ if } k = 2m-1,\ x_{kl}^{(1)} = x_{kl}^{(2)} = 0 \text{ if } k - l = 2m-1$$
$$(m = 1, 2, \cdots)$$

and

$$\delta_1 = \frac{2\lambda+1}{2}(>0),\ \delta_2 = \frac{\sqrt{-(2\lambda+1)^2 - 4\alpha\lambda(\lambda+1)}}{2}(>0).$$

Also, in the neighbourhood of $t = \infty$, $x(t)$ is represented as

$$x(t) = K\left(1 + \sum_{n=1}^{\infty} x_n t^{\alpha\lambda n}\right)\ (K, x_n : \text{constants}). \tag{2}$$

(ii) *If* $(y_0, z_0) \notin O_1$, *then* $x(t)$ *is defined for* $0 < t < \infty$. *In the neighbourhood of* $t = 0$, $x(t)$ *is represented as (1), and in the neighbourhood of* $t = \infty$, *as*

$$\begin{aligned} x(t) &= Lt\left(1 + \sum_{m+n>0} x_{mn} t^{\alpha(\lambda+1)m-n}\right) \quad \text{if } -1/\alpha(\lambda+1) \notin \mathbb{N} \\ x(t) &= Lt\left\{1 + \sum_{k=1}^{\infty} t^{\alpha(\lambda+1)k} p_k(\log t)\right\} \quad \text{if } -1/\alpha(\lambda+1) \in \mathbb{N} \end{aligned} \tag{3}$$

where L, x_{mn} *are constants,* $x_{0n} = 0$ *if* $n = 2, 3, \cdots$, *and* p_k *are polynomials with* $\deg p_k \leq [-\alpha(\lambda+1)k]$.

Next, suppose $-2 < \alpha < \lambda_0$.

THEOREM 2. (iii) *If* $(y_0, z_0) \in O_2$, *then* $x(t)$ *is defined for* $0 < t < \omega_+$ (ω_+: *a positive constant). In the neighbourhood of* $t = 0$, $x(t)$ *is represented as (1), and in the neighbourhood of* $t = \omega_+$, *as*

$$x(t) = \left\{\frac{2(\alpha+2)}{\alpha^2 \omega_+^{\alpha\lambda-2}}\right\}^{1/\alpha} (\omega_+ - t)^{-2/\alpha} \times \left\{1 + \sum_{n=1}^{\infty} x_n (\omega_+ - t)^n\right\} \quad (x_n : \text{constants}). \tag{4}$$

(iv) *If* $(y_0, z_0) \in R_1$, *then* $x(t)$ *is defined for* $0 < t < \omega_+$. *In the neighbourhood of* $t = 0$, $x(t)$ *is represented as (1), and in the neighbourhood of* $t = \omega_+$, *as*

$$x(t) = L(\omega_+ - t)\left\{1 + \sum_{j+k+l>0} d_{jkl}(\omega_+ - t)^j \times (\omega_+ - t)^{-(\alpha/2)k} (\omega_+ - t)^{((\alpha+2)/2)l}\right\} \quad (L, d_{jkl} : \text{constants}). \tag{5}$$

(v) *If* $(y_0, z_0) \in R_2$, *then (i), (ii) of Theorem 1 follow.*
(vi) *If* $(y_0, z_0) \in O_3$, *then* $x(t)$ *is defined for* $\omega_- < t < \infty$ (ω_-: *a positive constant). In the neighbourhood of* $t = \infty$, $x(t)$ *is represented as (3), and in the neighbourhood of* $t = \omega_-$, *as*

$$x(t) = \left\{\frac{2(\alpha+2)}{\alpha^2 \omega_-^{\alpha\lambda-2}}\right\}^{1/\alpha} (t - \omega_-)^{-2/\alpha} \times \left\{1 + \sum_{n=1}^{\infty} x_n (t - \omega_-)^n\right\} \quad (x_n : \text{constants}). \tag{6}$$

(vii) *If* $(y_0, z_0) \in R_3$*, then* $x(t)$ *is defined for* $\omega_- < t < \infty$*. In the neighbourhood of* $t = \infty$*,* $x(t)$ *is represented as (3), and in the neighbourhood of* $t = \omega_-$*, as*

$$x(t) = L(t-\omega_-)\left\{1 + \sum_{j+k+l>0} d_{jkl}(t-\omega_-)^j \right.$$
$$\left. \times\ (t-\omega_-)^{-(\alpha/2)k}(t-\omega_-)^{((\alpha+2)/2)l}\right\}\ (L, d_{jkl} : \textit{constants}). \quad (7)$$

For the proofs, let us first discuss the critical point $(1, 0)$ of (S), and follow the way of getting (1.4) of [8]. Then we see that $(1, 0)$ is a spiral point and the orbits reaching this point are expressed as

$$y = 1 + \alpha C e^{\mu s} + \alpha \bar{C} e^{\bar{\mu} s} + \cdots,\ z = \mu C e^{\mu s} + \bar{\mu}\bar{C} e^{\bar{\mu} s} + \cdots$$

where C is an arbitrary constant, $\mu = \alpha(\delta_1 + \delta_2 i)$, and $\cdots$ denote double power series of $e^{\mu s}$, $e^{\bar{\mu} s}$ converging in the neighbourhood of $s = -\infty$. As usual, $\cdots$ starts from a term with the greater degree than that of the previous term. Also, from this we have (1) in the neighbourhood of $t = 0$.

Let us next discuss the critical point $(0, 0)$, and adopt the way of [10]. Then we get the expressions of the orbits as follows: O_1 is expressed as

$$z = \alpha\lambda y(1 + \cdots)$$

and the other orbits, as

$$z = \alpha(\lambda + 1)y(1 + \cdots)$$

in the neighbourhood of $y = 0$, where $\cdots$ denote power series of y. O_2, O_3 in the case $-2 < \alpha < \lambda_0$ are expressed as

$$z = \pm\rho^{-1}y^{3/2}(1 + \cdots)$$

respectively in the neighbourhood of $y = \infty$, where

$$\rho = \frac{1}{\alpha}\sqrt{\frac{\alpha + 2}{2\lambda(\lambda + 1)}}$$

and $\cdots$ denote power series of $y^{-1/2}$, and the other orbits continuable to $y = \infty$ are represented as

$$z = Cy^{(\alpha-1)/\alpha}(1 + \cdots)$$

in the neighbourhood of $y = \infty$, where C is an arbitrary constant and $\cdots$ denotes a double power series of $y^{-1/2}$, $y^{(\alpha+2)/2\alpha}$. Also, we have the other analytical expressions (2) - (7) of $x(t)$ from these expressions.

To draw the phase portrait of (S) of the case $\alpha \leq -2$, note that there appears no periodic orbit in the phase plane of (S) from a lemma of [1] (see [12]). Also, to draw that of the case $-2 < \alpha < \lambda_0$, fix λ with $\lambda > (-1+\sqrt{2})/2$ so that $-2 < \alpha < \lambda_0$ is possible, let $(y_0(s,\alpha), z_0(s,\alpha))$ denote the unique orbit of (S) tending to $(0,0)$ with a tangent $l_1 : z = \alpha\lambda y$, and let $(y_1(s,\alpha), z_1(s,\alpha))$ denote the orbit of (S) reaching $(1,0)$ and identical with $(y_0(s,\alpha), z_0(s,\alpha))$ if $\alpha \leq -2$. Then $y_0(s,\alpha)$, $z_0(s,\alpha)$, $y_1(s,\alpha)$, $z_1(s,\alpha)$ are holomorphic in α, and from the monodromy theorem we have

$$(y_1(s,\alpha), z_1(s,\alpha)) = (y_0(s,\alpha), z_0(s,\alpha))$$

even if $-2 < \alpha < \lambda_0$. Therefore the phase portrait of (S) is drawn as in Figure 1.

Finally, note that if the solution $x(t)$ of (E), (I) gives (y,z) through (T), then (y,z) moves on all the orbit (see Lemma 2 of [6]). Then the proofs of Theorems 1, 2 are complete.

3. On the case II

Let us suppose the case II, namely

$$\alpha < 0, \ -1 < \lambda < 0.$$

As in Section 2, transform (E) into (R) through (T) and rewrite (R) as (S). Then from $\lambda(\lambda+1) < 0$, we have

$$y < 0$$

and from (T),

$$x = \{-\lambda(\lambda+1)\}^{1/\alpha} t^{-\lambda} (-y)^{1/\alpha}.$$

Also, the phase portrait of (S) is as in Figure 2.

Here the orbits tangent to the lines l_j $(j = 1,2)$ are represented as

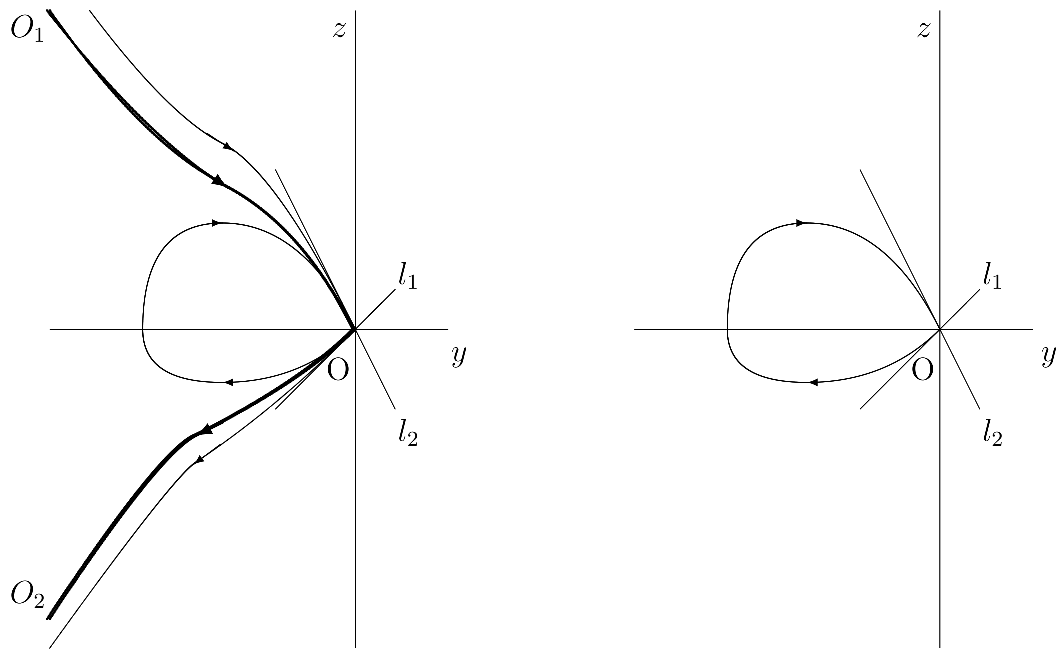
$$z = c_j y(1 + \cdots) \ (c_1 = \alpha(\lambda+1), \ c_2 = \alpha\lambda)$$

where $\cdots$ denote double power series of

$$-y, \ (-y)^{(-1)^{j+1}/c_j}\{h \log(-y) + C\} \ (h, C \text{ are constants})$$

and $h = 0$ if $(-1)^{j+1}/c_j \notin \mathbb{N}$ (see [5, 9]). In the case $-2 < \alpha < 0$, O_1, O_2 are the unique orbits of (S) represented respectively as

$$z = \pm\rho y^{3/2}(1 + \cdots) \ \left(\rho = \frac{1}{\alpha}\sqrt{-\frac{\alpha+2}{2\lambda(\lambda+1)}}\right)$$



The case $-2 < \alpha < 0$ The case $\alpha \leq -2$

(l_1: the line $z = \alpha\lambda y$, l_2: the line $z = \alpha(\lambda+1)y$)

Figure 2

in the neighbourhood of $y = -\infty$, where $\cdots$ denotes power series of $(-y)^{-1/2}$. Also, the orbits lying above O_1 and below O_2 are expressed as

$$z = C(-y)^{(\alpha-1)/\alpha}(1 + \cdots)$$

in the neighbourhood of $y = -\infty$, where C is a constant and $\cdots$ denotes a double power series of $(-y)^{-1/2}$, $(-y)^{(\alpha+2)/2\alpha}$ (see [9]).

If $x(t)$ denotes the solution of (E), (I) again, then a point (y_0, z_0) is given in the phase plane of (S) through (T), where

$$y_0 = \psi(t_0)^{-\alpha}A^{\alpha}, \; z_0 = \alpha y_0\left(\lambda + \frac{t_0 B}{A}\right).$$

So, take (t_0, A, B) of (I) and determine (y_0, z_0). Then we state our theorems as follows: First, suppose $-2 < \alpha < 0$.

Theorem 3. (i) *If $(y_0, z_0) \in O_1$, then $x(t)$ is defined for $\omega_- < t < \infty$ (ω_-: a positive constant). In the neighbourhood of $t = \omega_-$, $x(t)$ is represented as (6) and in the neighbourhood of $t = \infty$, as (3).*

(ii) *If (y_0, z_0) lies above O_1, then $x(t)$ is defined for $\omega_- < t < \infty$. In the neighbourhood of $t = \omega_-$, $x(t)$ is represented as (7) and in the neighbourhood of $t = \infty$, as (3).*
(iii) *If $(y_0, z_0) \in O_2$, then $x(t)$ is defined for $0 < t < \omega_+$ (ω_+: a positive constant). In the neighbourhood of $t = \omega_+$, $x(t)$ is represented as (4) and in the neighbourhood of $t = 0$, as*

$$x(t) = K\left(1 + \sum_{m+n>0} x_{mn} t^{\alpha\lambda m + n}\right) \quad \text{if } 1/\alpha\lambda \notin \mathbb{N}$$
$$x(t) = K\left(1 + \sum_{k=1}^{\infty} t^{\alpha\lambda k} p_k(\log t)\right) \quad \text{if } 1/\alpha\lambda \in \mathbb{N} \tag{8}$$

where K, x_{mn} are constants, $x_{0n} = 0$ $(n = 2, 3, \cdots)$, and p_k are polynomials with $\deg p_k \leq [\alpha\lambda k]$.
(iv) *If (y_0, z_0) lies below O_2, then $x(t)$ is defined for $0 < t < \omega_+$. In the neighbourhood of $t = \omega_+$, $x(t)$ is represented as (5) and in the neighbourhood of $t = 0$, as (8).*
(v) *If (y_0, z_0) lies between O_1 and O_2, then $x(t)$ is defined for $0 < t < \infty$. In the neighbourhood of $t = 0$, $x(t)$ is represented as (8) and in the neighbourhood of $t = \infty$, as (3).*

Finally, we state the following:

THEOREM 4. *If $\alpha \leq -2$, then the conclusion of (v) of Theorem 3 follows.*

The proofs of Theorems 3, 4 are the same as those of Theorems 1, 2, and omitted.

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