Yokohama Mathematical
Journal Vol. 58, 2012

SOME RESULTS ON LP-SASAKIAN MANIFOLDS WITH A COEFFICIENT α

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(Received February 11, 2012)

Abstract. The object of the present paper is to study an LP-Sasakian manifold with a coefficient α and several interesting results are obtained on that manifold. Also locally ϕ-symmetric and ϕ-conformally flat LP-Sasakian manifolds with a coefficient α have been studied. Also it is proved that a 3-dimensional LP-Sasakian manifold with a constant coefficient α satisfies cyclic parallel Ricci tensor if and only if it is locally ϕ-symmetric.Finally we give some examples of 3-dimensional LP-Sasakian manifolds with a coefficient α.

1. Introduction

In 1989, Matsumoto [10] introduced the notion of LP-Sasakian manifolds. Then Mihai and Rosca [12] introduced the same notion independently and they obtained several results in this manifold. LP-Sasakian manifolds have been studied by several authors ([1], [5], [11]). In a recent paper De, Shaikh and Sengupta [4] introduced the notion of LP-Sasakian manifolds with a coefficient α which generalizes the notion of LP-Sasakian manifolds. Lorentzian para-Sasakian manifold with a coefficient α have been studied by De et al ([2], [3]). Recently, T.Ikawa and his coauthors ([7], [8]) studied Sasakian manifolds with Lorentzian metric and obtained several results in this manifold. Motivated by the above studies we like to generalize LP-Sasakian manifold which is called an LP-Sasakian manifold with a coefficient α. In [2] it is shown that if a Lorentzian manifold admits a unit torse-forming vector field, then the manifold becomes an LP-Sasakian manifold with a coefficient α where α is a non-zero smooth function.

The paper is organized as follows.
In section 2, some preliminary results are recalled. After preliminaries in section 3, we prove that the Ricci operator Q commutes with ϕ. Then we study locally ϕ-symmetric LP-Sasakian Manifold with a coefficient α. In the next section, we study ϕ-conformally flat LP-Sasakian manifold with a coefficient α. In section 6, it is proved that a 3-dimensional LP-Sasakian manifold with a constant coefficient

2010 Mathematics Subject Classification: 53c15, 53c25
Key words and phrases: LP-Sasakian manifold with a coefficient α, ϕ-conformally flat manifold, locally ϕ-symmetric manifold,η- Einstein manifold, η- parallel Ricci tensor

α satisfies cyclic parallel Ricci tensor if and only if it is locally ϕ-symmetric. Finally we construct some examples of 3-dimensional LP-Sasakian manifolds with a coefficient α .

2. Preliminaries

Let M^n be an n-dimensional differentiable manifold endowed with a $(1,1)$ tensor field ϕ,a contravariant vector field ξ, a covariant vector field η and a Lorentzian metric g of type $(0,2)$ such that for each point $p \;\varepsilon M$, the tensor g_p:$T_pM \times T_pM \rightarrow \mathbb{R}$ inner product of signature $(-,+,+,.....,+)$, where T_pM denotes the tangent vector space of M at p and $\mathbb{R}$ is the real number space which satisfies

$$\phi^2(X) = X + \eta(X)\xi, \eta(\xi) = -1, \tag{2.1}$$

$$g(X,\xi) = \eta(X), g(\phi X, \phi Y) = g(X,Y) + \eta(X)\eta(Y) \tag{2.2}$$

for all vector fields X, Y. Then such a structure (ϕ, ξ, η, g) is termed as Lorentzian almost paracontact structure and the manifold M^n with the structure (ϕ, ξ, η, g) is called Lorentzian almost paracontact manifold [10]. In the Lorentzian almost paracontact manifold M^n, the following relations hold [10]:

$$\phi\xi = 0, \eta(\phi X) = 0, \tag{2.3}$$

$$\Omega(X,Y) = \Omega(Y,X), \tag{2.4}$$

where $\Omega(X,Y) = g(X, \phi Y)$.

In the Lorentzian almost paracontact manifold M^n, if the relations

$$\begin{aligned}(\nabla_Z\Omega)(X,Y) = \alpha[&(g(X,Z) + \eta(X)\eta(Z))\,\eta(Y)\\ &+(g(Y,Z) + \eta(Y)\eta(Z))\eta(X)],\end{aligned} \tag{2.5}$$

$$\Omega(X,Y) = \frac{1}{\alpha}(\nabla_X\eta)(Y), \tag{2.6}$$

hold where ∇ denotes the operator of covariant differentiation with respect to the Lorentzian metric g, and α is a non-zero scalar function then M^n is called an LP-Sasakian manifold with a coefficient α [4]. An LP-Sasakian manifold with a coefficient 1 is an LP-Sasakian manifold [10].

If a vector field V satisfies the equation of the following form:

$$\nabla_X V = \beta X + T(X)V,$$

where β is a non-zero scalar function and T is a covariant vector field, then V is called a torse-forming vector field [15].

In the Lorentzian manifold M^n, if we assume that ξ is a unit torse-forming vector field,then we have the equation:

$$(\nabla_X \eta)(Y) = \alpha[g(X,Y) + \eta(X)\eta(Y)], \tag{2. 7}$$

where α is a non-zero scalar function. Especially, if η satisfies

$$(\nabla_X \eta)(Y) = \epsilon[g(X,Y) + \eta(X)\eta(Y)], \quad \epsilon^2 = 1 \tag{2. 8}$$

then M^n is called an LSP-Sasakian manifold[10]. In particular, if α satisfies (2. 7) and the equation of the following form:

$$\nabla_X \alpha = d\alpha(X) = \sigma\eta(X), \tag{2. 9}$$

where σ is a smooth function and η is the $1-$ form, then ξ is called a concircular vector field.

Let us consider an LP-Sasakian manifold M^n (ϕ, ξ, η, g) with a coefficient α. Then we have the following relations [4]:

$$\eta(R(X,Y)Z) = (\alpha^2 - \sigma)[g(Y,Z)\eta(X) - g(X,Z)\eta(Y)], \tag{2. 10}$$

$$S(X,\xi) = (n-1)(\alpha^2 - \sigma)\eta(X), \tag{2. 11}$$

$$R(X,Y)\xi = (\alpha^2 - \sigma)[\eta(Y)X - \eta(X)Y], \tag{2. 12}$$

$$(\nabla_X \phi)(Y) = \alpha[g(X,Y)\xi + 2\eta(X)\eta(Y)\xi + \eta(Y)X], \tag{2. 13}$$

for all vector fields X, Y, Z, where R, S denote respectively the curvature tensor and the Ricci tensor of the manifold.

Now we state the following result which will be needed in the later section.

Lemma 2.1. ([4]) *In a Lorentzian almost paracontact manifold $M^n(\phi, \xi, \eta, g)$ with its structure (ϕ, ξ, η, g) satisfying $\Omega(X, Y) = \frac{1}{\alpha}(\nabla_X \eta)(Y)$, where α is a non-zero scalar, the vector field ξ is torse-forming if and only if $\psi^2 = (n-1)^2$ holds good.*

3. Fundamental results of LP-Sasakian manifold with a coefficient α

In this section we begin with the following:

Theorem 3.1. *Let (M^n, g) be an LP-Sasakian manifold with a coefficient α. Then the Ricci operator Q commutes with ϕ.*

Proof. We assume that X, Y, Z are (local) vector fields such that $(\nabla X)_P = (\nabla Y)_P = (\nabla Z)_P = 0$, for a fixed point P of M^n.

By the Ricci identity for ϕ, that is,

$$\begin{aligned} R(X, Y)\phi Z - \phi R(X, Y)Z = (\nabla_X \nabla_Y \phi)Z \\ - (\nabla_Y \nabla_X \phi)Z - (\nabla_{[X,Y]}\phi)Z, \end{aligned} \tag{3.1}$$

we have at the point P

$$\begin{aligned} R(X, Y)\phi Z - \phi R(X, Y)Z = \nabla_X(\nabla_Y \phi)Z \\ - \nabla_Y(\nabla_X \phi)Z. \end{aligned} \tag{3.2}$$

Using (2. 13),it follows that

$$\begin{aligned} \nabla_Y(\nabla_X \phi)Z = {} & \sigma\eta(Y)[(g(X, Z) + \eta(X)\eta(Z))\xi \\ & + (X + \eta(X)\xi)\eta(Z)] + \alpha^2[2g(X, Y)\eta(Z)\xi \\ & + 6\eta(X)\eta(Y)\eta(Z)\xi + 2g(Y, Z)\eta(X)\xi \\ & + Xg(Y, Z) + X\eta(Y)\eta(Z) + Yg(X, Z) \\ & + g(X, Z)\eta(Y)\xi + 2Y\eta(X)\eta(Z)] \end{aligned} \tag{3.3}$$

Using (3. 3), from (3. 2) we have

$$\begin{aligned} R(X, Y)\phi Z - \phi R(X, Y)Z = {} & (\alpha^2 - \sigma)[g(X, Z)\eta(Y) - g(Y, Z)\eta(X)]\xi \\ & + (\alpha^2 - \sigma)\eta(Z)[X\eta(Y) - Y\eta(X)]. \end{aligned} \tag{3.4}$$

Replacing X, Y by ϕX, ϕY respectively in (3. 4) and taking the inner product on both sides by ϕW we get

$$g(R(\phi X, \phi Y)\phi Z, \phi W) = g(\phi R(\phi X, \phi Y)Z, \phi W). \tag{3.5}$$

Now

$$\begin{aligned} g(\phi R(\phi X,\phi Y)Z,\phi W) &= g(R(\phi X,\phi Y)Z,W) \\ &= g(R(Z,W)\phi X,\phi Y) \\ &= g(\phi R(Z,W)X,\phi Y)+(\alpha^2-\sigma)[g(Z,X)g(W,\phi Y) \\ &\quad - g(W,X)g(Z,\phi Y)]+(\alpha^2-\sigma) \\ &\quad [g(Z,\phi Y)\eta(X)\eta(W)-g(W,\phi Y)\eta(X)\eta(Z)]. \end{aligned}$$

Therefore from (3. 4) we have

$$\begin{aligned} g(R(\phi X,\phi Y)\phi Z,\phi W) &= g(R(X,Y)Z,W)+(\alpha^2-\sigma) \\ &\quad [\eta(X)\eta(Z)g(W,Y)-\eta(W)\eta(X)g(Y,Z)] \\ &\quad +(\alpha^2-\sigma)[X\eta(Z)-g(X,Z)\xi]\eta(Y). \end{aligned} \tag{3. 6}$$

From (3. 6) it follows that

$$\begin{aligned} \phi R(\phi X,\phi Y)\phi Z &= R(X,Y)Z+(\alpha^2-\sigma) \\ &\quad \eta(X)[\eta(Z)Y-g(Y,Z)\xi]+(\alpha^2-\sigma) \\ &\quad [X\eta(Z)-g(X,Z)\xi]\eta(Y). \end{aligned} \tag{3. 7}$$

We now consider the following two cases:

Case (i): If $n=2m+1$, let $\{e_i,\phi e_i,\xi\}$, $i=1,2,.....,m$ be an orthonormal frame at any point of the manifold. Then putting $Y=Z=e_i$ in (3. 7) and taking summation over i and using $\eta(e_i)=0$, we get

$$\sum_{i=1}^{m}\epsilon_i\phi R(\phi X,\phi e_i)\phi e_i=\sum_{i=1}^{m}\epsilon_i R(X,e_i)e_i-m(\alpha^2-\sigma)\eta(X)\xi, \tag{3. 8}$$

where $\epsilon_i=g(e_i,e_i)$.

Again setting $Y=Z=\phi e_i$ in (3. 7)and taking summation over i and using $\eta.\phi=0$, we get

$$\sum_{i=1}^{m}\epsilon_i\phi R(\phi X,e_i)e_i=\sum_{i=1}^{m}\epsilon_i R(X,\phi e_i)\phi e_i-m(\alpha^2-\sigma)\eta(X)\xi. \tag{3. 9}$$

Adding (3. 8) and (3. 9) and using the definition of the Ricci tensor, we obtain

$$\phi(Q\phi X-R(\phi X,\xi)\xi)=QX-R(X,\xi)\xi-2m(\alpha^2-\sigma)\eta(X)\xi.$$

Using (2. 12) and $\phi\xi=0$ in the above relation, we have

$$\phi Q\phi X=QX-2m(\alpha^2-\sigma)\eta(X)\xi.$$

Operating both sides by ϕ and using (2. 1),symmetry of Q and $\phi\xi = 0$ we get $\phi Q = Q\phi$.

Case (ii): If $n = 2m+2$, let $\{e_i, \phi e_i, \xi\}$, $i = 1, 2, ..., m+1$ be an orthonormal frame at any point of the manifold. Then putting $Y = Z = e_i$ in (3. 7) and taking summation over i and using $\eta(e_i) = 0$, we get

$$\sum_{i=1}^{m+1} \epsilon_i \phi R(\phi X, \phi e_i)\phi e_i = \sum_{i=1}^{m+1} \epsilon_i R(X, e_i)e_i - (m+1)(\alpha^2 - \sigma)\eta(X)\xi, \tag{3. 10}$$

where $\epsilon_i = g(e_i, e_i)$.

Again setting $Y = Z = \phi e_i$ in (3.7) and taking summation over i and using $\eta.\phi = 0$, we get

$$\sum_{i=1}^{m+1} \epsilon_i \phi R(\phi X, e_i)e_i = \sum_{i=1}^{m+1} \epsilon_i R(X, \phi e_i)\phi e_i - (m+1)(\alpha^2 - \sigma)\eta(X)\xi. \tag{3. 11}$$

Adding (3. 10) and (3. 11) and then proceeding similarly as in Case (i) we can easily obtain $\phi Q = Q\phi$.This proves the theorem. □

Proposition 3.1. *In an LP-Sasakian manifold with a coefficient α the relation*

$$S(\phi X, \phi Y) = (n-1)(\alpha^2 - \sigma)g(X, Y) + S(X, Y) \tag{3. 12}$$

holds.

Proof. We have $S(X, Y) = g(QX, Y)$.
Then

$$\begin{aligned} S(\phi X, \phi Y) &= g(Q\phi X, \phi Y) \\ &= g(\phi QX, \phi Y), since Q\phi = \phi Q \\ &= g(QX, Y) + \eta(QX)\eta(Y) \\ &= S(X, Y) + S(X, \xi)\eta(Y). \end{aligned}$$

Using (2. 11) we get from above

$$S(\phi X, \phi Y) = S(X, Y) + (n-1)(\alpha^2 - \sigma)g(X, Y).$$

□

Definition 3.1. The Ricci tensor S of an LP-Sasakian manifold with a coefficient α is said to be η-parallel if it satisfies

$$(\nabla_X S)(\phi Y, \phi Z) = 0, \tag{3.13}$$

for all vector fields X, Y and Z.

This notion was introduced in the context of Sasakian manifolds by M. Kon [9].

Differentiating (3. 12) covariantly with respect to Z we get

$$(\nabla_Z S)(\phi X, \phi Y) = (\nabla_Z S)(X, Y) + (n-1)[2\alpha d\alpha(Z) - d\sigma(Z)]g(X, Y). \tag{3.14}$$

Hence we can state the following:

Corollary 3.1. *In an LP-Sasakian manifold with a coefficient α, η-parallelity of the Ricci tensor and the Ricci-symmetry are equivalent provided α, $\sigma =$ constant.*

4. Locally ϕ-symmetric LP-Sasakian Manifold with a coefficient α

Definition 4.1. An LP-Sasakian manifold with a coefficient α (M^n, g) is said to be locally ϕ-symmetric if

$$\phi^2((\nabla_W R)(X, Y)Z) = 0 \tag{4.1}$$

for all vector fields W,X,Y,Z orthogonal to ξ. This notion was introduced for Sasakian manifolds by Takahashi [14].

Let us consider an LP-Sasakian manifold with a coefficient α (M^n, g) which is locally ϕ-symmetric. Then by using (2. 1)in (4. 1) we have

$$(\nabla_W R)(X, Y)Z + \eta((\nabla_W R)(X, Y)Z)\xi = 0, \tag{4.2}$$

for any X, Y, Z, W orthogonal to ξ. It follows from (4. 2) that

$$(\nabla_W R)(X, Y)Z - g((\nabla_W R)(X, Y)\xi, Z)\xi = 0,$$

which yields by virtue of (2. 12) that

$$(\nabla_W R)(X, Y)Z = \alpha(\alpha^2 - \sigma)[g(X, Z)g(W, Y) - g(W, X)g(Y, Z)]\xi, \tag{4.3}$$

for any X, Y, Z, W orthogonal to ξ. Next, if the relation (4. 3) holds, it follows by $\phi\xi = 0$ that (4. 1) holds and hence the manifold is locally ϕ-symmetric. Thus we can state the following:

Theorem 4.1. *An LP-Sasakian manifold with a coefficient α, (M^n, g) is locally ϕ-symmetric if and only if the relation (4.3) holds for all horizontal vector fields X, Y, Z, W on M.*

5. ϕ-conformally flat LP-Sasakian Manifold with a coefficient α

Definition 5.1. An LP-Sasakian manifold with a coefficient α (M^n, g) $(n > 3)$ is said to be ϕ-conformally flat if it satisfies

$$\phi^2(C(\phi X, \phi Y)\phi Z) = 0 \tag{5.1}$$

for any vector field X, Y, Z in T_pM where C is the Weyl conformal curvature tensor defined by

$$\begin{aligned} C(X,Y)Z = R(X,Y)Z \\ &- \frac{1}{n-2}[g(Y,Z)QX - g(X,Z)QY + S(Y,Z)X - S(X,Z)Y] \\ &+ \frac{r}{(n-1)(n-2)}[g(Y,Z)X - g(X,Z)Y]. \end{aligned}$$

The notion of ϕ-conformally flat for K-contact manifolds was first introduced by G. Zhen [16]. In a recent paper [13] Chian Ozgur studied ϕ-conformally flat Lorentzian Para-Sasakian Manifold.

Definition 5.2. An LP-Sasakian manifold with a coefficient α is said to be an η-Einstein manifold if the Ricci tensor S satisfies the condition

$$S(X,Y) = ag(X,Y) + b\eta(X)\eta(Y),$$

where a, b are smooth functions.

First let (5.1) holds. Then we have

$$g(C(\phi X, \phi Y)\phi Z, \phi W) = 0.$$

Hence using the definition of conformal curvature tensor, the above relation implies that

$$\begin{aligned} \widetilde{R}(\phi X, \phi Y, \phi Z, \phi W) = \frac{1}{(n-2)}[S(\phi Y, \phi Z)g(\phi X, \phi W) \\ &- S(\phi X, \phi Z)g(\phi Y, \phi W) + g(\phi Y, \phi Z)S(\phi X, \phi W) \\ &- g(\phi X, \phi Z)S(\phi Y, \phi W)] - \frac{r}{(n-1)(n-2)} \\ &[g(\phi Y, \phi Z)g(\phi X, \phi W) - g(\phi X, \phi Z)g(\phi Y, \phi W)], \end{aligned} \tag{5.2}$$

where $\widetilde{R}(\phi X, \phi Y, \phi Z, \phi W) = g(R(\phi X, \phi Y)\phi Z, \phi W)$.

Now using (3. 6) and (3. 12) in (5. 2) we have

$$
\begin{aligned}
g(R(X,Y)Z,W) + (\alpha^2-\sigma)[\eta(X)\eta(Z)g(W,Y)
&- g(Y,Z)\eta(W)\eta(X) + g(X,W)\eta(Y)\eta(Z)\\
&- g(X,Z)\eta(Y)\eta(W)] = \frac{1}{(n-2)}\\
&[[S(Y,Z) + (n-1)(\alpha^2-\sigma)g(Y,Z)]\\
&[g(X,W) + \eta(X)\eta(W)]\\
&- [S(X,Z) + (n-1)(\alpha^2-\sigma)g(X,Z)]\\
&[g(Y,W) + \eta(Y)\eta(W)]\\
&+ [S(X,W) + (n-1)(\alpha^2-\sigma)g(X,W)]\\
&[g(Y,Z) + \eta(Y)\eta(Z)]\\
&- [S(Y,W) + (n-1)(\alpha^2-\sigma)g(Y,W)]\\
&[g(X,Z) + \eta(X)\eta(Z)]]\\
&- \frac{r}{(n-1)(n-2)}[[g(Y,Z) + \eta(Y)\eta(Z)]\\
&[g(X,W) + \eta(X)\eta(W)]\\
&- [g(X,Z) + \eta(X)\eta(Z)][g(Y,W) + \eta(Y)\eta(W)]]. \qquad (5.3)
\end{aligned}
$$

Taking an orthonormal frame field and contracting over X and W in (5. 3), it follows that

$$
\begin{aligned}
S(Y,Z) = [\frac{r}{(n-1)} - (\alpha^2-\sigma)]g(Y,Z)&\\
+ [\frac{r}{(n-1)} - n(\alpha^2-\sigma)]\eta(Y)\eta(Z).& \qquad (5.4)
\end{aligned}
$$

It is known [4] that if an LP-Sasakian manifold with a coefficient α is η-Einstein, then the Ricci tensor S is of the form

$$
\begin{aligned}
S(Y,Z) = [\frac{r}{n-1} - \alpha^2 - \frac{\psi\sigma}{n-1}]g(Y,Z)&\\
+ [\frac{r}{n-1} - n\alpha^2 - \frac{n\psi\sigma}{n-1}]\eta(Y)\eta(Z).& \qquad (5.5)
\end{aligned}
$$

By virtue of (5. 4) and (5. 5) we get

$$
\begin{aligned}
[\sigma + \frac{\psi\sigma}{n-1}]g(X,Z)&\\
+ [n\sigma + \frac{n\psi\sigma}{n-1}]\eta(X)\eta(Z) = 0.& \qquad (5.6)
\end{aligned}
$$

Putting $Z = \xi$ in (5. 6) we obtain

$$\eta(Y)\sigma[\psi + (n-1)] = 0, \tag{5. 7}$$

which gives

$$\psi^2 = (n-1)^2. \tag{5. 8}$$

Hence by Lemma 2.1 we conclude that ξ is torse-forming. Thus we can state the following:

THEOREM 5.1. *In a ϕ-conformally flat LP-Sasakian manifold with a coefficient α ,the characteristic vector field ξ is a torse-forming vector field.*

6. 3-dimensional lp-sasakian manifold with a constant coefficient α

Let us consider a 3-dimensional LP-Sasakian Manifold with a constant coefficient α. In a 3-dimensional Riemannian manifold we have

$$\begin{aligned} R(X,Y)Z = {} & g(Y,Z)QX - g(X,Z)QY + S(Y,Z)X - S(X,Z)Y \\ & - \frac{r}{2}[g(Y,Z)X - g(X,Z)Y], \end{aligned} \tag{6. 1}$$

where Q is the Ricci operator, that is, $g(QX,Y) = S(X,Y)$ and r is the scalar curvature of the manifold.

Since α is constant and the dimension of the manifold is 3, equation (2. 10) and (2. 11) reduces to

$$\eta(R(X,Y)Z) = \alpha^2[g(Y,Z)\eta(X) - g(X,Z)\eta(Y)], \tag{6. 2}$$

$$S(X,\xi) = 2\alpha^2\eta(X). \tag{6. 3}$$

From (6. 2) we get

$$R(X,Y)\xi = \alpha^2[\eta(Y)X - \eta(X)Y]. \tag{6. 4}$$

Putting $Z = \xi$ in (6. 1) and using (6. 4) we have

$$\eta(Y)QX - \eta(X)QY = (\frac{r}{2} - \alpha^2)[\eta(Y)X - \eta(X)Y]. \tag{6. 5}$$

Putting $Y = \xi$ in (6. 5) and using (2. 1) and (6. 3), we get

$$QX = \frac{1}{2}[(r - 2\alpha^2)X + (r - 6\alpha^2)\eta(X)\xi], \tag{6. 6}$$

that is,

$$S(X,Y) = \frac{1}{2}[(r-2\alpha^2)g(X,Y) + (r-6\alpha^2)\eta(X)\eta(Y)]. \tag{6.7}$$

Using (6.6) in (6.1), we get

$$\begin{aligned} R(X,Y)Z = {} & (\frac{r-4\alpha^2}{2})[g(Y,Z)X - g(X,Z)Y] + (\frac{r-6\alpha^2}{2})[g(Y,Z)\eta(X)\xi \\ & - g(X,Z)\eta(Y)\xi + \eta(Y)\eta(Z)X - \eta(X)\eta(Z)Y]. \end{aligned} \tag{6.8}$$

THEOREM 6.1. *A 3-dimensional LP-Sasakian manifold with a constant coefficient α is locally ϕ-symmetric if and only if the scalar curvature r is constant.*

Proof. Differentiating (6.8) covariently with respect to W, we get

$$\begin{aligned} (\nabla_W R)(X,Y)Z = {} & \frac{dr(W)}{2}[g(Y,Z)X - g(X,Z)Y] \\ & - \frac{dr(W)}{2}[g(Y,Z)\eta(X)\xi - g(X,Z)\eta(Y)\xi \\ & + \eta(Y)\eta(Z)X - \eta(X)\eta(Z)Y] \\ & -(\frac{r-6\alpha^2}{2})[g(Y,Z)(\nabla_W\eta)(X)\xi - g(X,Z)(\nabla_W\eta)(Y)\xi \\ & + g(Y,Z)\eta(X)\nabla_W\xi - g(X,Z)\eta(Y)\nabla_W\xi \\ & + (\nabla_W\eta)(Y)\eta(Z)X + \eta(Y)(\nabla_W\eta)(Z)X \\ & - (\nabla_W\eta)(X)\eta(Z)Y - \eta(X)(\nabla_W\eta)(Z)Y]. \end{aligned} \tag{6.9}$$

Now taking W, X, Y, Z are horizontal vector fields, that is, W, X, Y, Z are orthogonal to ξ, then we get from the above

$$\phi^2(\nabla_W R)(X,Y)Z = -\frac{dr(W)}{2}[g(Y,Z)X - g(X,Z)Y]. \tag{6.10}$$

Hence from the definition (4.1) the above Theorem follows. □

THEOREM 6.2. *A 3-dimensional LP-Sasakian manifold with a constant coefficient α satisfies cyclic parallel Ricci tensor if and only if $r = 6\alpha^2$.*

Proof. A. Gray [6] introduced two classes of Riemannian manifold determined by covariant derivative of Ricci tensor. The class A consisting of all Riemannian manifold whose Ricci tensor S is a Codazzi tensor, i.e.,

$$(\nabla_X S)(Y,Z) = (\nabla_Y S)(X,Z).$$

The class B consisting of all Riemannian manifolds whose Ricci tensor is cyclic parallel i.e.,

$$(\nabla_X S)(Y,Z) + (\nabla_Y S)(X,Z) + (\nabla_Z S)(X,Y) = 0. \tag{6.11}$$

A Riemannian manifold is said to satisfy cyclic parallel Ricci tensor if the Ricci tensor is non-zero and satisfies the condition (6.11). From (6.11) it follows that $r = constant$.

Differentiating (6.7) covariantly, we have

$$(\nabla_Z S)(X,Y) = \frac{1}{2}(r - 6\alpha^2)\{\eta(Y)(\nabla_Z \eta)X + \eta(X)(\nabla_Z \eta)Y\}, \tag{6.12}$$

since r =constant.

Applying (6.12) in (6.11) we have

$$\begin{aligned} \frac{1}{2}(r - 6\alpha^2)\ \{\ &\eta(Y)(\nabla_X \eta)Z + \eta(Z)(\nabla_X \eta)Y \\ &+ \eta(X)(\nabla_Y \eta)Z + \eta(Z)(\nabla_Y \eta)X \\ &+ \eta(X)(\nabla_Z \eta)Y + \eta(Y)(\nabla_Z \eta)X\} = 0. \end{aligned} \tag{6.13}$$

Taking a frame field we get from (6.13)

$$(r - 6\alpha^2)3\alpha\eta(X) = 0.$$

Here $\alpha \neq 0$, hence $r = 6\alpha^2$.

Conversely, if $r = 6\alpha^2$ then from (6.12) it follows that $(\nabla_Z S)(X,Y) = 0$ and hence the manifold satisfies cyclic parallel Ricci tensor.This completes the proof. $\square$

7. Examples

EXAMPLE 7.1. We consider the 3-dimensional manifold $M = \{(x,y,z)\varepsilon\mathbb{R}^3\}$, where (x,y,z) are standard coordinate of $\mathbb{R}^3$.

The vector fields

$$e_1 = e^{-z}(\frac{\partial}{\partial x} + y\frac{\partial}{\partial y}), \quad e_2 = e^{-z}\frac{\partial}{\partial y}, \quad e_3 = e^{-2z}\frac{\partial}{\partial z}$$

are linearly independent at each point of M.

Let g be the Lorentzian metric defined by

$$g(e_1, e_3) = g(e_1, e_2) = g(e_2, e_3) = 0,$$

$$g(e_1, e_1) = g(e_2, e_2) = 1,$$

$$g(e_3, e_3) = -1.$$

Let η be the 1-form defined by $\eta(Z) = g(Z, e_3)$ for any $Z\varepsilon\chi(M)$.

Let ϕ be the $(1, 1)$ tensor field defined by

$$\phi(e_1) = e_1, \quad \phi(e_2) = e_2, \quad \phi(e_3) = 0.$$

Then using the linearity of ϕ and g, we have

$$\eta(e_3) = -1,$$

$$\phi^2 Z = Z + \eta(Z)e_3,$$

$$g(\phi Z, \phi W) = g(Z, W) + \eta(Z)\eta(W),$$

for any $Z, W\varepsilon\chi(M)$.

Then for $e_3 = \xi$, the structure (ϕ, ξ, η, g) defines a Lorentzian paracontact structure on M.

Let ∇ be the Levi-Civita connection with respect to the Lorentzian metric g and R be the curvature tensor of g. Then we have

$$[e_1, e_2] = -e^{-z}e_2 \quad , [e_1, e_3] = e^{-2z}e_1 \quad and \quad [e_2, e_3] = e^{-2z}e_2.$$

Taking $e_3 = \xi$ and using Koszul's formula for the Lorentzian metric g, we can easily calculate

$$\nabla_{e_1}e_3 = e^{-2z}e_1, \quad \nabla_{e_1}e_2 = 0, \quad \nabla_{e_1}e_1 = e^{-2z}e_3,$$

$$\nabla_{e_2}e_3 = e^{-2z}e_2, \quad \nabla_{e_2}e_2 = e^{-2z}e_3 - e^{-z}e_1, \quad \nabla_{e_2}e_1 = e^{-2z}e_2,$$

$$\nabla_{e_3}e_3 = 0, \quad \nabla_{e_3}e_2 = 0, \quad \nabla_{e_3}e_1 = 0. \tag{7.1}$$

From the above it can be easily seen that $M^3(\phi, \xi, \eta, g)$ is an LP-Sasakian manifold with $\alpha = e^{-2z} \neq 0$.

Example 7.2. We consider the 3-dimensional manifold $M = \{(x, y, z)\varepsilon\mathbb{R}^3\}$, where (x, y, z) are standard coordinate of $\mathbb{R}^3$.

The vector fields

$$e_1 = e^z\frac{\partial}{\partial y}, \quad e_2 = e^z(\frac{\partial}{\partial x} + \frac{\partial}{\partial y}), \quad e_3 = \frac{\partial}{\partial z}$$

are linearly independent at each point of M.

Let g be the Lorentzian metric defined by

$$g(e_1, e_3) = g(e_1, e_2) = g(e_2, e_3) = 0,$$

$$g(e_1, e_1) = g(e_2, e_2) = 1,$$

$$g(e_3, e_3) = -1.$$

Let η be the 1-form defined by $\eta(Z) = g(Z, e_3)$ for any $Z \varepsilon \chi(M)$.

Let ϕ be the $(1, 1)$ tensor field defined by

$$\phi(e_1) = e_1, \quad \phi(e_2) = e_2, \quad \phi(e_3) = 0.$$

Then using the linearity of ϕ and g, we have

$$\eta(e_3) = -1,$$

$$\phi^2 Z = Z + \eta(Z)e_3,$$

$$g(\phi Z, \phi W) = g(Z, W) + \eta(Z)\eta(W),$$

for any $Z, W \varepsilon \chi(M)$.

Then for $e_3 = \xi$, the structure (ϕ, ξ, η, g) defines a Lorentzian paracontact structure on M.

Let ∇ be the Levi-Civita connection with respect to the Lorentzian metric g and R be the curvature tensor of g. Then we have

$$[e_1, e_2] = 0 \quad , [e_1, e_3] = -e_1 \quad and \quad [e_2, e_3] = -e_2.$$

Taking $e_3 = \xi$ and using Koszul's formula for the Lorentzian metric g, we can easily calculate

$$\nabla_{e_1} e_3 = -e_1, \quad \nabla_{e_1} e_2 = 0, \quad \nabla_{e_1} e_1 = -e_3,$$

$$\nabla_{e_2} e_3 = -e_2, \quad \nabla_{e_2} e_2 = -e_3, \quad \nabla_{e_2} e_1 = 0,$$

$$\nabla_{e_3} e_3 = 0, \quad \nabla_{e_3} e_2 = 0, \quad \nabla_{e_3} e_1 = 0. \tag{7.2}$$

From the above it can be easily seen that $M^3(\phi, \xi, \eta, g)$ is an LP-Sasakian manifold with a coefficient α. Here $\alpha = -1$.

With the help of the above results it can be easily verified that

$$R(e_1, e_2)e_3 = 0, \quad R(e_2, e_3)e_3 = -e_2, \quad R(e_1, e_3)e_3 = -e_1,$$

$$R(e_1, e_2)e_2 = e_1, \quad R(e_2, e_3)e_2 = -e_3, \quad R(e_1, e_3)e_2 = 0,$$

$$R(e_1, e_2)e_1 = -e_2, \quad R(e_2, e_3)e_1 = 0, \quad R(e_1, e_3)e_1 = -e_3.$$

From the above expressions of the curvature tensor we obtain

$$\begin{aligned} S(e_1, e_1) &= g(R(e_1, e_2)e_2, e_1) - g(R(e_1, e_3)e_3, e_1) \\ &= 2. \end{aligned}$$

Similarly we have

$$S(e_2, e_2) = 2$$

and

$$S(e_3, e_3) = -2.$$

Therefore,

$$r = S(e_1, e_1) + S(e_2, e_2) - S(e_3, e_3) = 6.$$

Hence the scalar curvature is constant. Thus the 3-dimensional LP-Sasakian manifold with a constant coefficient α is locally ϕ-symmetric. Therefore Theorem 6.1 is verified.

Also from the expression of the Ricci tensor we find that the manifold under consideration satisfies cyclic parallel Ricci tensor. Since $r = 6 = 6(\alpha^2)$ for $\alpha = -1$, therefore Theorem 6.2 holds.

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