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WEIGHTED SHARING OF THREE VALUES BY LINEAR DIFFERENTIAL POLYNOMIALS

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Abstract. In the paper we prove two uniqueness theorems for linear differential polynomials which improve a recent result of Fang-Lahiri.

1. Introduction, Definitions and Results

Let f be a noncnstant meromorphic function defined in the open complex plane $\mathbb{C}$. If for some $a \in \mathbb{C} \cup \{\infty\}$, f and g have the same set of a-points with the same multiplicities, we say that f and g share the value a CM (counting multiplicities) and if we do not consider the multiplicities then f, g are said to share the value a IM (ignoring multiplicities). We do not explain the standard notations and definitions of the value distribution theory as these are available in [3].

DEFINITION 1. We denote by $N(r,a;f \mid= 1)$ the counting function of simple a-points of f for $a \in \mathbb{C} \cup \{\infty\}$.

DEFINITION 2. Let p be a positive integer and $a \in \mathbb{C} \cup \{\infty\}$. We denote by $\overline{N}(r,a;f \mid\geq p)$ the counting function of those distinct a-points of f whose multiplicities are not less than p.

In [4] the following result is proved.

THEOREM A ([4]). *Let f and g be two nonconstant meromorphic functions and $a_1, a_2, \ldots, a_n (a_n \neq 0)$ be finite complex numbers. If*

(i) *f and g share ∞ CM,*

(ii) *F and G share 0,1 CM, where $F = \sum\limits_{i=1}^{n} a_i f^{(i)}$ and $G = \sum\limits_{i=1}^{n} a_i g^{(i)}$,*

(iii) $\frac{\sum\limits_{a\neq\infty} \delta(a;f)}{1+n(1-\Theta(\infty;f)} - \frac{3(1-\Theta(\infty;f))}{2\sum\limits_{a\neq\infty}\delta(a;f)} > \frac{1}{2}$, *where* $\sum\limits_{a\neq\infty} \delta(a;f) > 0$,

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then either (a) $F \equiv G$ *or* (b) $FG \equiv 1$. *If, further, f has at least one pole or F has at least one zero, the case* (b) *does not arise.*

Following question is asked in [1] : Is it possible in any way to relax the nature of sharing the values and to weaken the condition on deficiencies in Theorem A?

To answer this question the notion of weighted sharing of values is used in [1] which we explain in the following definition.

DEFINITION 3 ([5, 6]). Let k be a nonnegative integer or infinity . For $a \in \mathbb{C} \cup \{\infty\}$ we denote by $E_k(a; f)$ the set of all a-points of f where an a-point of multiplicity m is counted m times if $m \leq k$ and $k+1$ times if $m > k$. If $E_k(a; f) = E_k(a; g)$, we say that f, g share the value a with weight k.

The definition implies that if f , g share a value a with weight k then z_o is an a-point of f with multiplicity $m(\leq k)$ if and only if it is an a-point of g with multiplicity $m(\leq k)$ and z_o is an a-point of f with multiplicity $m(> k)$ if and only if it is an a-point of g with multiplicity $n(> k)$ where m is not necessarily equal to n.

We write f, g share (a, k) to mean that f, g share the value a with weight k. Clearly if f, g share (a, k) then f, g share (a, p) for all integers p, $0 \leq p < k$. Also we note that f, g share a value a IM or CM if and only if f, g share $(a,0)$ or (a,∞) respectively.

Improving Theorem A in [1] following result is proved.

THEOREM B ([1]). *Let f and g be two nonconstant meromorphic functions and $a_1, a_2, \ldots, a_n (a_n \neq 0)$ be finite complex numbers. If*

(i) *f and g share (∞, ∞),*

(ii) *F and G share $(0, 1)$, $(1, \infty)$, where $F = \sum_{i=1}^{n} a_i f^{(i)}$ and $G = \sum_{i=1}^{n} a_i g^{(i)}$,*

(iii) $\sum_{a \neq \infty} \delta(a; f) > \frac{1}{2}$,

then either (a) $F \equiv G$ *or* (b) $FG \equiv 1$. *If, further, f has at least one pole or F has at least one zero, the case* (b) *does not arise.*

The purpose of the paper is to investigate the possibility of further reducing the weight of sharing values in Theorem B. We prove in the paper the following two results which improve Theorem B.

THEOREM 1. *Let f and g be two nonconstant meromorphic functions and $a_1, a_2, \ldots, a_n (a_n \neq 0)$ be finite complex numbers. If*

(i) *f and g share $(\infty, 1)$,*

(ii) *F and G share* $(0,1)$, $(1,6)$, *where* $F = \sum_{i=1}^{n} a_i f^{(i)}$ *and* $G = \sum_{i=1}^{n} a_i g^{(i)}$,

(iii) $\sum_{a \neq \infty} \delta(a; f) > \frac{1}{2}$,

then either (a) $F \equiv G$ *or* (b) $FG \equiv 1$. *If, further, f has at least one pole or F has at least one zero, the case* (b) *does not arise.*

THEOREM 2. *Let f and g be two nonconstant meromorphic functions and $a_1, a_2, \dots, a_n (a_n \neq 0)$ be finite complex numbers, where $n \geq 2$. If*

(i) *f and g share* $(\infty, 0)$,

(ii) *F and G share* $(0,1)$, $(1,6)$, *where* $F = \sum_{i=1}^{n} a_i f^{(i)}$ *and* $G = \sum_{i=1}^{n} a_i g^{(i)}$,

(iii) $\sum_{a \neq \infty} \delta(a; f) > \frac{1}{2}$,

then either (a) $F \equiv G$ *or* (b) $FG \equiv 1$. *If, further, f has at least one pole or F has at least one zero, the case* (b) *does not arise.*

Considering $f = \exp(z) - (1/2)^n \exp(2z)$, $g = (-1)^n \exp(-z) - (-1/2)^n \exp(-2z)$, $F = f^{(n)}$ and $G = g^{(n)}$ we see that the condition (iii) of Theorems 1 and 2 is sharp.

We now give two more definitions.

DEFINITION 4 ([2]). We put for $a \in \mathbb{C} \cup \{\infty\}$

$$T_o(r, f) = \int_1^r \frac{T(t, f)}{t} dt,$$

$$N_o(r, a; f) = \int_1^r \frac{N(t, a; f)}{t} dt,$$

$$m_o(r, a; f) = \int_1^r \frac{m(t, a; f)}{t} dt,$$

$$S_o(r, f) = \int_1^r \frac{S(t, f)}{t} dt \text{ etc.}$$

DEFINITION 5 ([2]). For a meromorphic function f we put

$$\delta_0(a; f) = 1 - \limsup_{r \to \infty} \frac{N_0(r, a; f)}{T_0(r, f)} = \liminf_{r \to \infty} \frac{m_0(r, a; f)}{T_0(r, f)}.$$

2. Lemmas

In this section we present some lemmas which will be needed in the sequel.

LEMMA 1 ([2]). *For a meromorphic function f*

$$\lim_{r\to\infty}\frac{S_0(r,f)}{T_0(r,f)}=0$$

through all values of r.

LEMMA 2 ([9]). *If f is a meromorphic function and $a\in\mathbb{C}\cup\{\infty\}$ then $\delta(a;f)\le\delta_0(a;f)$.*

LEMMA 3 ([8]). *If f, g share $(0,0)$, $(1,0)$, $(\infty,0)$ then*

$$(i)\ T(r,f)\le 3T(r,g)+S(r,f),$$
$$(ii)\ T(r,g)\le 3T(r,f)+S(r,g).$$

Lemma 3 shows that $S(r,f)=S(r,g)$ and we denote them by $S(r)$.

LEMMA 4 ([10]). *Let f, g share $(0,0)$, $(1,0)$, $(\infty,0)$ and*

$$H=\left(\frac{f''}{f'}-\frac{2f'}{f-1}\right)-\left(\frac{g''}{g'}-\frac{2g'}{g-1}\right).$$

If $H\equiv 0$ then f, g share $(0,\infty)$, $(1,\infty)$, (∞,∞).

LEMMA 5 ([7]). *If f, g share $(1,1)$ and $H\not\equiv 0$ then*

$$\begin{aligned}N(r,1;f\mid=1)&=N(r,1;g\mid=1)\\&\le N(r,H)+S(r,f)+S(r,g).\end{aligned}$$

LEMMA 6. *Let f, g share $(0,1)$, $(1,m)$, (∞,k) and f, g have no pole of multiplicity less than n. If $f\not\equiv g$ and $(m-1)\{m(\lambda-1)-1\}>(1+m)^2$, where $\lambda=\max\{n,1+k\}$, then for $a=0,1,\infty$ we get $\overline{N}(r,a;f\mid\ge 2)=\overline{N}(r,a;g\mid\ge 2)=S(r)$.*

Proof. We prove $\overline{N}(r,a;f\mid\ge 2)=S(r)$ for $a=0,1,\infty$ because the other can similarly be proved. We consider the following cases.

Case I. Let $\lambda<\infty$. If $m=\infty$ then we can choose a sufficiently large positive integer m for which the given condition holds. So we may suppose, without loss of generality, m to be finite. Let $\phi_1=\frac{f'}{f(f-1)}-\frac{g'}{g(g-1)}=\left(\frac{f'}{f-1}-\frac{g'}{g-1}\right)-\left(\frac{f'}{f}-\frac{g'}{g}\right)$, $\phi_2=\frac{f'}{f-1}-\frac{g'}{g-1}$ and $\phi_3=\frac{f'}{f}-\frac{g'}{g}$.

We suppose that $\overline{N}(r,a;f) \neq S(r)$ because otherwise the case is trivial. Since $f \not\equiv g$, it follows that $\phi_i \not\equiv 0$ for $i = 1,2,3$. Now

$$\begin{aligned}\overline{N}(r,0;f \mid\geq 2) &\leq N(r,0;\phi_2) \leq T(r,\phi_2) + O(1)\\ &= N(r,\infty;\phi_2) + S(r)\\ &\leq \overline{N}(r,1;f \mid\geq 1+m) + \overline{N}(r,\infty;f \mid\geq \lambda) + S(r). \qquad (1)\end{aligned}$$

Again

$$\begin{aligned}m\overline{N}(r,1;f \mid\geq 1+m) &\leq (m-1)\overline{N}(r,1;f \mid\geq 1+m) + \overline{N}(r,1;f \mid\geq 2)\\ &\leq N(r,0;\phi_3)\\ &\leq N(r,\infty;\phi_3) + S(r)\\ &\leq \overline{N}(r,0;f \mid\geq 2) + \overline{N}(r,\infty;f \mid\geq \lambda) + S(r). \qquad (2)\end{aligned}$$

Also

$$\begin{aligned}(\lambda-1)\overline{N}(r,\infty;f \mid\geq \lambda) &\leq (\lambda-2)\overline{N}(r,\infty;f \mid\geq \lambda) + \overline{N}(r,\infty;f \mid\geq 2)\\ &\leq N(r,0;\phi_1)\\ &\leq N(r,\infty;\phi_1) + S(r)\\ &\leq \overline{N}(r,1;f \mid\geq 1+m) + \overline{N}(r,0;f \mid\geq 2) + S(r). \qquad (3)\end{aligned}$$

From (1) and (2) we get

$$(m-1)\overline{N}(r,0;f \mid\geq 2) \leq (m+1)\overline{N}(r,\infty;f \mid\geq \lambda) + S(r). \qquad (4)$$

From (2), (3) and (4) we obtain

$$\begin{aligned}\{m(\lambda-1)-1\}\overline{N}(r,\infty;f \mid\geq \lambda) &\leq (m+1)\overline{N}(r,0;f \mid\geq 2) + S(r)\\ &\leq \frac{(1+m)^2}{m-1}\overline{N}(r,\infty;f \mid\geq \lambda) + S(r)\end{aligned}$$

i.e., $[(m-1)\{m(\lambda-1)-1\} - (1+m)^2] \times \overline{N}(r,\infty;f \mid\geq \lambda) \leq (m-1)S(r)$

i.e., $\overline{N}(r,\infty;f \mid\geq \lambda) = S(r)$.

So from (4) we get $\overline{N}(r,0;f \mid\geq 2) = S(r)$ and from (2) we get $\overline{N}(r,1;f \mid\geq 2) = S(r)$ and from (3) we get $\overline{N}(r,\infty;f \mid\geq 2) = S(r)$.

Case II. Let $\lambda = \infty$. We now consider the following two subcases.

Subcase (i) Let $n = \infty$. Then f and g are entire functions. Hence proceeding as Case I we get $\overline{N}(r,0;f \mid\geq \lambda) \leq \overline{N}(r,1;f \mid\geq 1+m) + S(r)$, and

$$\begin{aligned}m\overline{N}(r,1;f \mid\geq 1+m) &\leq (m-1)\overline{N}(r,1;f \mid\geq 1+m) + \overline{N}(r,1;f \mid\geq 2)\\ &\leq \overline{N}(r,0;f \mid\geq 2) + S(r).\end{aligned}$$

So we see that $\overline{N}(r, 0; f \mid\geq 2) = S(r)$ and $\overline{N}(r, 1; f \mid\geq 2) = S(r)$.

Subcase (ii) Let $n < \infty$. Then $k = \infty$ and so it is possible to chose a sufficiently large finite $k(> n)$ such that for $\lambda_0 = 1 + k$ the condition of the lemma holds. Hence in this case the result can be proved in the line of Case I. This proves the lemma.

3. Proofs of the theorems

Proof of Theorem 1. Since f, g share $(\infty, 1)$ and F, G have only multiple poles, it follows that F, G share $(\infty, 2)$. So by Lemma 6 we get

$$\overline{N}(r, \infty; F) = \overline{N}(r, \infty; F \mid\geq 2) = S(r) \tag{5}$$

and

$$\overline{N}(r, \infty; G) = \overline{N}(r, \infty; G \mid\geq 2) = S(r), \tag{6}$$

where $S(r) = S(r, F) = S(r, G)$ by Lemma 3.

Let $b_1, b_2, \ldots, b_p$ be finite deficient values of f. Since $\sum_{n=1}^{p} m(r, b_n; f) \leq m(r, 0; F) + S(r, f)$, integrating we get $\sum_{n=1}^{p} m_0(r, b_n; f) \leq m_0(r, 0; F) + S_0(r, f)$ and so

$$\sum_{n=1}^{p} \frac{m_0(r, b_n; f)}{T_0(r, f)} \leq \frac{m_0(r, 0; F)}{T_0(r, F)} \cdot \frac{T_0(r, F)}{T_0(r, f)} + \frac{S_0(r, f)}{T_0(r, f)}.$$

Since $\overline{N}(r, \infty; F) = S(r, F) = S(r, f)$ implies $\overline{N}_0(r, \infty; F) = S_0(r, f)$, it follows from above that

$$\sum_{n=1}^{p} \frac{m_0(r, b_n; f)}{T_0(r, f)} \leq \frac{m_0(r, 0; F)}{T_0(r, F)} \cdot \frac{T_0(r, f) + S_0(r, f)}{T_0(r, f)} + \frac{S_0(r, f)}{T_0(r, f)}.$$

Hence in view of Lemma 1 we get $\sum_{n=1}^{p} \delta_0(b_n; f) \leq \delta_0(0; F)$ and since p is arbitrary, it follows that $\sum_{b \neq \infty} \delta_0(b; f) \leq \delta_0(0; F)$. So by Lemma 2 and condition (iii) we get

$$\delta_0(0; F) > \frac{1}{2}. \tag{7}$$

We put

$$\psi = \frac{F''}{F'} - \frac{2F'}{F - 1} - \frac{G''}{G'} + \frac{2G'}{G - 1}.$$

If possible suppose that $\psi \not\equiv 0$. Then by Lemmas 3 and 5 we get

$$N(r,1;F \mid= 1) = N(r,1;G \mid= 1) \leq N(r,\psi) + S(r). \tag{8}$$

Since F, G share $(1,6)$ by a simple computation, we see that if z_0 is a simple zero of $F-1$ and $G-1$ then $\psi(z_0) = 0$. Since the poles of ψ are all simple, we get in view of Lemma 6

$$N(r,\psi) \leq N_*(r,0;F') + N_*(r,0;G') + S(r), \tag{9}$$

where $N_*(r,0;F')$ is the counting function of those zeros of F' which are not the zeros of $F(F-1)$, a zero is counted according to its multiplicity and $N_*(r,0;G')$ is an analogous quantity.

By the second fundamental theorem we get in view of (5), (6), (8), (9) and Lemma 6

$$\begin{aligned} T(r,F) + T(r,G) &\leq \overline{N}(r,0;F) + \overline{N}(r,1;F) + \overline{N}(r,\infty;F) \\ &\quad + \overline{N}(r,0;G) + \overline{N}(r,1;G) + \overline{N}(r,\infty;G) \\ &\quad - N_*(r,0;F') - N_*(r,0;G') + S(r) \\ &\leq 2N(r,0;F) + T(r,G) + N(r,1;F \mid= 1) \\ &\quad - N_*(r,0;F') - N_*(r,0;G') + S(r) \\ &\leq 2N(r,0;F) + T(r,G) + S(r) \end{aligned}$$

and so $T(r,F) \leq 2N(r,0;F) + S(r,F)$.

On integration we get $T_0(r,F) \leq 2N_0(r,0;F) + S_0(r,F)$ so that $\delta_0(0;F) \leq 1/2$, which contradicts (7). Hence $\psi \equiv 0$. So by Lemma 4 F and G share $(0,\infty)$, $(1,\infty)$, (∞,∞). Now the theorem follows from Theorem B. This proves the theorem.

Proof of Theorem 2. The theorem can be proved in the line of the proof of Theorem 1 noting that F and G have no simple and double pole.

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