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ORBIT SPACE OF SPLIT FREUDENTHAL VECTOR SPACE

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Abstract. A maximal compact subgroup of the noncompact exceptional Lie group of type $E_{7(7)}$ acts on the split Freudenthal $\boldsymbol{R}$-vector space. We give the orbit space of the split $\boldsymbol{R}$-vector space under this action.

1. Introduction

In [1], the author showed that any element P of the split Freudenthal $\boldsymbol{R}$-vector space $\mathfrak{P}'$ can be transformed to a diagonal form by some element of a maximal compact subgroup $(E_{7(7)})_K$ of the noncompact exceptional Lie group $E_{7(7)}$. In this paper we show that this diagonal form is uniquely determined by $P \in \mathfrak{P}'$ independent of the choice of $\alpha = \varphi(A) \in (E_{7(7)})_K (\cong SU(8)/\boldsymbol{Z}_2)$, $A \in SU(8)$ under the certain conditions of parameters. As a result, the orbit space $\mathfrak{P}'/SU(8)$ of $\mathfrak{P}'$ with respect to $SU(8)$ is given as follows:

$$\mathfrak{P}'/SU(8) = \left\{ (\xi_0, \xi_1, \xi_2, \xi_3, \xi_4) \in \boldsymbol{R}^5 \,\middle|\, \begin{matrix} \xi_4 \le \xi_1 \le \xi_2 \le \xi_3 \\ 4{\xi_0}^2 \le (\xi_2 + \xi_3)(\xi_4 - \xi_1) \end{matrix} \right\}.$$

2. Preliminaries

2.1 Split Cayley algebra $\mathfrak{C}'$

We denote by $\boldsymbol{R}, \boldsymbol{C} = \{a_0 + a_1 e_1 \,|\, a_k \in \boldsymbol{R}\}\,({e_1}^2 = -1)$ and $\boldsymbol{H} = \{a_0 + a_1 e_1 + a_2 e_2 + a_3 e_3 \,|\, a_k \in \boldsymbol{R}\}\,({e_k}^2 = -1, e_1 e_2 = e_3, e_2 e_3 = e_1, e_3 e_1 = e_2, e_k e_l = -e_l e_k\,(k \ne l))$, the fields of real numbers, complex numbers and quaternions, respectively. Let $\mathfrak{C}' = \boldsymbol{H} \oplus \boldsymbol{H} e'$ be the split Cayley algebra over $\boldsymbol{R}$ with the multiplication

$$(a + be')(c + de') = (ac + \overline{d}b) + (b\overline{c} + da)e', \quad a + be', c + de' \in \mathfrak{C}',$$

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where $\overline{c}$, $\overline{d}$ are the conjugate elements of $c, d \in \boldsymbol{H}$, respectively. In $\mathfrak{C}'$, the conjugation is defined by $\overline{a + be'} = \overline{a} - be'$.

2.2 Split Jordan algebra $\mathfrak{J}'$

Let

$$\begin{aligned} \mathfrak{J}' &= \{X \in M(3, \mathfrak{C}') \,|\, X^* = X\} \\ &= \left\{ X = X(\xi_k, x_k) = \begin{pmatrix} \xi_1 & x_3 & \overline{x}_2 \\ \overline{x}_3 & \xi_2 & x_1 \\ x_2 & \overline{x}_1 & \xi_3 \end{pmatrix} \,\middle|\, \xi_k \in \boldsymbol{R}, x_k \in \mathfrak{C}' \right\} \end{aligned}$$

be the split Jordan algebra with the Jordan multiplication

$$X \circ Y = \frac{1}{2}(XY + YX).$$

In $\mathfrak{J}'$, we define the inner product (X, Y), the Freudenthal multiplication $X \times Y$ respectively by

$$(X, Y) = \mathrm{tr}(X \circ Y),$$
$$X \times Y = \frac{1}{2}(2X \circ Y - \mathrm{tr}(X)Y - \mathrm{tr}(Y)X + (\mathrm{tr}(X)\mathrm{tr}(Y) - (X, Y))E),$$

where E is the unit matrix.

2.3 Split Freudenthal $\boldsymbol{R}$-vector space $\mathfrak{P}'$ and Lie group $E_{7(7)}$

We define the split Freudenthal $\boldsymbol{R}$-vector space $\mathfrak{P}'$ by

$$\mathfrak{P}' = \mathfrak{J}' \oplus \mathfrak{J}' \oplus \boldsymbol{R} \oplus \boldsymbol{R}$$

with the inner product

$$(P, Q) = (X, Z) + (Y, W) + \xi\zeta + \eta\omega,$$

for $P = (X, Y, \xi, \eta)$, $Q = (Z, W, \zeta, \omega) \in \mathfrak{P}'$. For $\phi \in \mathfrak{e}_{6(6)} (:= \{\phi \in \mathrm{Hom}_{\boldsymbol{R}}(\mathfrak{J}') \,|\, (\phi X, X \times X) = 0\})$, $A, B \in \mathfrak{J}'$ and $\nu \in \boldsymbol{R}$, we define an $\boldsymbol{R}$-linear mapping $\Phi(\phi, A, B, \nu) : \mathfrak{P}' \to \mathfrak{P}'$ by

$$\Phi(\phi, A, B, \nu) \begin{pmatrix} X \\ Y \\ \xi \\ \eta \end{pmatrix} = \begin{pmatrix} \phi X - \frac{1}{3}\nu X + 2B \times Y + \eta A \\ 2A \times X - {}^t\phi Y + \frac{1}{3}\nu Y + \xi B \\ (A, Y) + \nu\xi \\ (B, X) - \nu\eta \end{pmatrix},$$

where ${}^t\phi \in \mathfrak{e}_{6(6)}$ is the transpose of ϕ with respect to the inner product (X, Y): $({}^t\phi X, Y) = (X, \phi Y)$. Next, for $P = (X, Y, \xi, \eta)$, $Q = (Z, W, \zeta, \omega) \in \mathfrak{P}'$, we define an $\boldsymbol{R}$-linear mapping $P \times Q : \mathfrak{P}' \to \mathfrak{P}'$ by

$$P \times Q = \Phi(\phi, A, B, \nu), \qquad \begin{cases} \phi = -\frac{1}{2}(X \vee W + Z \vee Y), \\ A = -\frac{1}{4}(2Y \times W - \xi Z - \zeta X), \\ B = \ \frac{1}{4}(2X \times Z - \eta W - \omega Y), \\ \nu = \ \frac{1}{8}((X, W) + (Z, Y) - 3(\xi\omega + \zeta\eta)), \end{cases}$$

where $X \vee W \in \mathfrak{e}_{6(6)}$ is defined by $(X \vee W)U = \frac{1}{2}(W, U)X + \frac{1}{6}(X, W)U - 2W \times (X \times U)$ for $U \in \mathfrak{J}'$. Then the noncompact simple Lie group of type $E_{7(7)}$ is given by

$$E_{7(7)} = \{\, \alpha \in \mathrm{Iso}_{\boldsymbol{R}}(\mathfrak{P}') \,|\, \alpha(P \times Q)\alpha^{-1} = \alpha P \times \alpha Q \,\}$$

([2], [3]).

3. Orbit space $\mathfrak{P}'/SU(8)$ of $\mathfrak{P}'$ with respect to $SU(8)$

3.1 Orbit space $\mathfrak{P}'/SU(8)$

We shall recall the following groups:

$$\begin{aligned} O(n) &= \{\, A \in M(n, \boldsymbol{R}) \,|\, {}^tAA = E \,\}, \\ U(n) &= \{\, A \in M(n, \boldsymbol{C}) \,|\, A^*A = E \,\}, \\ SU(n) &= \{\, A \in U(n) \,|\, \det A = 1 \,\}, \end{aligned}$$

$\boldsymbol{R}$-vector spaces:

$$\begin{aligned} \mathfrak{J}(4, \boldsymbol{H}) &= \{\, X \in M(4, \boldsymbol{H}) \,|\, X^* = X \,\}, \\ \mathfrak{J}(4, \boldsymbol{H})_0 &= \{\, X \in \mathfrak{J}(4, \boldsymbol{H}) \,|\, \mathrm{tr}(X) = 0 \,\}, \\ \mathfrak{S}(n, \boldsymbol{C}) &= \{\, S \in M(n, \boldsymbol{C}) \,|\, {}^tS = -S \,\}, \end{aligned}$$

and $\boldsymbol{R}$-isomorphism: $\chi : \mathfrak{P}' \to \mathfrak{S}(8, \boldsymbol{C})$,

$$\chi(X, Y, \xi, \eta) = k\Big(gX - \frac{\xi}{2}E\Big)J_4 + e_1 k\Big(g(\gamma Y) - \frac{\eta}{2}E\Big)J_4,$$

where $k : M(4, \boldsymbol{H}) \to M(8, \boldsymbol{C})$ is the naturally extended mapping of $k : \boldsymbol{H} = \boldsymbol{C} \oplus \boldsymbol{C}e_2 \to M(2, \boldsymbol{C})$,

$$k(a + be_2) = \begin{pmatrix} a & b \\ -\bar{b} & \bar{a} \end{pmatrix},$$

$g : \mathfrak{J}' \to \mathfrak{J}(4, \boldsymbol{H})_0$,

$$g\left(\begin{pmatrix} \xi_1 & a_3 + b_3 e' & \overline{a}_2 - b_2 e' \\ \overline{a}_3 - b_3 e' & \xi_2 & a_1 + b_1 e' \\ a_2 + b_2 e' & \overline{a}_1 - b_1 e' & \xi_3 \end{pmatrix}\right) = \begin{pmatrix} \lambda_1 & b_1 & b_2 & b_3 \\ \overline{b}_1 & \lambda_2 & a_3 & \overline{a}_2 \\ \overline{b}_2 & \overline{a}_3 & \lambda_3 & a_1 \\ \overline{b}_3 & a_2 & \overline{a}_1 & \lambda_4 \end{pmatrix},$$

$$\lambda_1 = \frac{1}{2}(\xi_1 + \xi_2 + \xi_3), \ \ \lambda_2 = \frac{1}{2}(\xi_1 - \xi_2 - \xi_3),$$
$$\lambda_3 = \frac{1}{2}(\xi_2 - \xi_3 - \xi_1), \ \ \lambda_4 = \frac{1}{2}(\xi_3 - \xi_1 - \xi_2),$$

$J_4 = \operatorname{diag}(J, J, J, J) \in M(8, \boldsymbol{C}), J = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$ and $\gamma : \mathfrak{J}' \to \mathfrak{J}'$, $\gamma X(\xi_k, a_k + b_k e') = X(\xi_k, a_k - b_k e')$ ([1], [2], [3]).

Hereafter, we regard $U(n-1) \subset U(n)$: $U(1) \subset U(2) \subset \cdots \subset U(n-1) \subset U(n)$, identifying $A \in U(n-1)$ with $\begin{pmatrix} 1 & 0 \\ 0 & A \end{pmatrix} \in U(n)$.

LEMMA 1. *For a given vector* ${}^t(x_1, x_2, x_3, \dots, x_n) \in \boldsymbol{C}^n$, *there exists* $D \in U(n-1) \subset U(n)$ *satisfying*

$$D \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ \vdots \\ x_n \end{pmatrix} = \begin{pmatrix} x_1 \\ s \\ 0 \\ \vdots \\ 0 \end{pmatrix}, \quad s \in \boldsymbol{R}.$$

(*Note that the first term* x_1 *leaves invariant under the action of* D.)

Proof. For $\boldsymbol{x} = {}^t(x_2, x_3, \dots, x_n) \in \boldsymbol{C}^{n-1}$, let $\boldsymbol{y} = {}^t(s, 0, \dots, 0) \in \boldsymbol{C}^{n-1}$, where $s = ||\boldsymbol{x}|| := \sqrt{|x_2|^2 + \cdots + |x_n|^2}$. We may assume that $s > 0$. Since the unitary group $U(n-1)$ acts transitively on the sphere $S^{2n-3} = \{\boldsymbol{z} \in \boldsymbol{C}^{n-1} \mid ||\boldsymbol{z}|| = s\}$, there exists $D \in U(n-1)$ such that $D\boldsymbol{x} = \boldsymbol{y}$. This $D \in U(n-1) \subset U(n)$ is the required matrix.

LEMMA 2. *Any complex skew-symmetric matrix* $X \in \mathfrak{S}(n, \boldsymbol{C})$ *is transformed to the following canonical form by some unitary matrix* $A \in U(n)$:

$$A X\, {}^t\!A = \begin{pmatrix} (0) & & & \\ & R_1 & & \\ & & \ddots & \\ & & & R_{[n/2]} \end{pmatrix}, \quad (0) = \begin{cases} 0 & \textit{if } n \textit{ is odd}, \\ \textit{empty} & \textit{if } n \textit{ is even}, \end{cases}$$
$$R_k = \begin{pmatrix} 0 & r_k \\ -r_k & 0 \end{pmatrix}, \quad 0 \le r_1 \le \cdots \le r_{[n/2]}.$$

Moreover, this form is uniquely determined by X independent of the choice of $A \in U(n)$.

Proof. For the first column vector $\boldsymbol{x} = {}^t(0, x_{21}, \dots, x_{n1})$ of X, construct $D_1 \in U(n-1) \subset U(n)$ of Lemma 1. Then X is transformed to the form

$$X_1 = D_1 X\, {}^t D_1 = \begin{pmatrix} 0 & * & \cdots & * \\ s_2 & * & \cdots & * \\ 0 & * & \cdots & * \\ & & \cdots & \\ 0 & * & \cdots & * \end{pmatrix}, \quad s_2 \in \boldsymbol{R}.$$

Since X_1 is also skew-symmetric, the first row vector of X_1 is $(0, -s_2, 0, \dots, 0)$. Therefore, X_1 is of the form

$$X_1 = \begin{pmatrix} 0 & -s_2 & 0 & \cdots & 0 \\ s_2 & & & & \\ 0 & & & & \\ \vdots & & Y & & \\ 0 & & & & \end{pmatrix}, \quad s_2 \in \boldsymbol{R},\ Y \in M(n-1, \boldsymbol{C}),\ {}^t Y = -Y.$$

For the complex skew-symmetric matrix $Y \in M(n-1, \boldsymbol{C})$, applying the similar process as above, we can obtain that X_1 is transformed by some unitary matrix U to a real skew-symmetric tridiagonal form

$$X_2 = U X_1 {}^t U = \begin{pmatrix} 0 & -s_2 & & & & & \\ s_2 & 0 & -s_3 & & & & \\ & s_3 & 0 & -s_4 & & & \\ & & s_4 & 0 & & & \\ & & & & \cdots & & \\ & & & & & 0 & -s_n \\ & & & & & s_n & 0 \end{pmatrix}.$$

Since X_2 is a real skew-symmetric matrix, as is well known, X_2 can be transformed to the canonical form of the lemma by some orthogonal matrix $O \in O(n) \subset U(n)$. Thus, the first half of the lemma is proved.

We shall now show the uniqueness of the canonical form. For the sake of brevity, we suppose $n = 2m$ and

$$A X\, {}^t A = \operatorname{diag}\left(\begin{pmatrix} 0 & r_1 \\ -r_1 & 0 \end{pmatrix}, \dots, \begin{pmatrix} 0 & r_m \\ -r_m & 0 \end{pmatrix}\right), \quad r_k \geq 0.$$

Then, we have

$$A(XX^*)A^{-1} = AX^tA\overline{A}X^*A^* = (AX^tA)(AX^tA)^* = \mathrm{diag}({r_1}^2, {r_1}^2, \dots, {r_m}^2, {r_m}^2).$$

Therefore, ${r_1}^2, \dots, {r_m}^2$ are the eigenvalues of Hermitian matrix XX^*. Thus ${r_1}^2, \dots, {r_m}^2$, namely, $r_1, \dots, r_m$ are uniquely determined by X. (Note that $r_k \geq 0$.) The case $n = 2m+1$ is also proved in exactly the same way.

PROPOSITION 3. *Any complex skew-symmetric matrix $X \in \mathfrak{S}(n, \boldsymbol{C})$ is transformed to the following canonical form by some special unitary matrix $A \in SU(n)$. Moreover, this form is uniquely determined by X independent of the choice of $A \in SU(n)$.*
(1) *In the case* $\det X = 0$,

$$AX\,{}^tA = \begin{pmatrix} (0) & & & \\ & R_1 & & \\ & & \ddots & \\ & & & R_{[n/2]} \end{pmatrix}, \quad \begin{array}{l} (0) = \begin{cases} 0 & \textit{if } n \textit{ is odd}, \\ \textit{empty} & \textit{if } n \textit{ is even}, \end{cases} \\ R_k = \begin{pmatrix} 0 & r_k \\ -r_k & 0 \end{pmatrix}, \quad 0 \leq r_1 \leq \cdots \leq r_{[n/2]}. \end{array}$$

(2) *In the case* $\det X \neq 0$,

$$AX\,{}^tA = \begin{pmatrix} R_1 & & & \\ & R_2 & & \\ & & \ddots & \\ & & & R_{n/2} \end{pmatrix}, \quad \begin{array}{l} R_1 = \begin{pmatrix} 0 & c \\ -c & 0 \end{pmatrix}, \quad c \in \boldsymbol{C}, \\ R_k = \begin{pmatrix} 0 & r_k \\ -r_k & 0 \end{pmatrix}, \quad k = 2, \dots, n/2, \\ 0 < |c| \leq r_2 \leq \cdots \leq r_{n/2}. \end{array}$$

Proof. For a given $X \in \mathfrak{S}(n, \boldsymbol{C})$, let AX^tA be expressed in the canonical form of Lemma 2. Now, constructing a matrix $A' \in M(n, \boldsymbol{C})$ by $A' = \begin{pmatrix} a & \\ & E \end{pmatrix} A$, where $a = 1/\det A \in \boldsymbol{C}$, we obtain that $A' \in SU(n)$ and $A'X^tA'$ has the canonical form of the proposition. (Note that $|a| = 1$.) The remainder of the proof is the uniqueness of the form above independent of the choice of $A \in SU(n)$.

In the case $\det X = 0$, it is obvious from Lemma 2.

In the case $\det X \neq 0$, we may assume that there exists some matrix $A \in U(n), n = 2m$ such that

$$AX\,{}^tA = \mathrm{diag}(R_1, \dots, R_m) \ (\underset{\text{put}}{=} R\,), \quad R_k = \begin{pmatrix} 0 & r_k \\ -r_k & 0 \end{pmatrix}, \quad 0 < r_1 \leq \cdots \leq r_m.$$

Note that the matrix R is uniquely determined by $X \in \mathfrak{S}(n, \boldsymbol{C})$. To prove the uniqueness of the form (2), we shall first show that if $AX\,{}^tA = R =$

$BX\,{}^tB$, $A, B \in U(n)$ then $\det A = \det B$. To do this, it is sufficient to show that if $CR\,{}^tC = R$, $C \in U(n)$ then $\det C = 1$. Now, let

$$\begin{aligned} R &= \operatorname{diag}(R_1, \dots, R_m) \\ &= \operatorname{diag}(\underbrace{a_1 J, \dots, a_1 J}_{m_1}, \underbrace{a_2 J, \dots, a_2 J}_{m_2}, \dots, \underbrace{a_l J, \dots, a_l J}_{m_l}) \\ &= \operatorname{diag}(a_1 J_{m_1}, a_2 J_{m_2}, \dots, a_l J_{m_l}), \quad a_i \neq a_j \quad (i \neq j), \end{aligned}$$

where $J_k = \operatorname{diag}(\underbrace{J, \dots, J}_{k})$, and C be partitioned into 2×2 block matrices:

$$C = \Big(C_{ij}\Big)_{1 \le i,j \le m}, \quad C_{ij} \in M(2, \boldsymbol{C}).$$

Then, since $CR\,{}^tC = R$, i.e., $CR = R\overline{C}$, we have

$$C_{ij} R_j = R_i \overline{C}_{ij} \qquad \text{for } 1 \le i, j \le m.$$

Therefore, noting that the conditions $C_{ij} R_j = R_i \overline{C}_{ij}$ $(R_k \neq 0)$ imply

$$\begin{cases} \text{if } R_i \neq R_j \text{ then } C_{ij} = 0, \\ \text{if } R_i = R_j \text{ then } C_{ij} J = J \overline{C}_{ij}, \end{cases}$$

we obtain

$$C = \operatorname{diag}(C_1, C_2, \dots, C_l), \quad C_k J_{m_k} = J_{m_k} \overline{C}_k, \quad C_k \in U(2m_k).$$

In particular,

$$C_k \in Sp(m_k, \boldsymbol{C}) \cap U(2m_k) \ (\underset{\text{put}}{=} Sp(m_k)\ ),$$

where $Sp(m_k, \boldsymbol{C}) = \{A \in M(2m_k, \boldsymbol{C}) | A J_{m_k}{}^t A = J_{m_k}\}$. Besides, we have $\det C_k = \pm 1$ from $C_k J_{m_k}{}^t C_k = J_{m_k}$. Furthermore, the sign of $\det C_k$ is constant, since, as is well known, the symplectic group $Sp(m_k)$ is connected. Then we obtain $\det C_k = 1$, that is, $\det C = 1$, since $E \in Sp(m_k)$ and $\det E = 1$. Next, for $X \in \mathfrak{S}(n, \boldsymbol{C}), n = 2m$, suppose

$$AX\,{}^tA = \operatorname{diag}\left(\begin{pmatrix} 0 & c \\ -c & 0 \end{pmatrix}, \begin{pmatrix} 0 & r_2 \\ -r_2 & 0 \end{pmatrix}, \dots, \begin{pmatrix} 0 & r_m \\ -r_m & 0 \end{pmatrix}\right), \ 0 < |c| \le r_2 \le \dots \le r_m,$$

$$BX\,{}^tB = \operatorname{diag}\left(\begin{pmatrix} 0 & d \\ -d & 0 \end{pmatrix}, \begin{pmatrix} 0 & s_2 \\ -s_2 & 0 \end{pmatrix}, \dots, \begin{pmatrix} 0 & s_m \\ -s_m & 0 \end{pmatrix}\right), \ 0 < |d| \le s_2 \le \dots \le s_m,$$

$c, d \in \boldsymbol{C}$, $A, B \in SU(n)$. Then we have

$$\begin{pmatrix} \overline{c}/|c| & 0 \\ 0 & E \end{pmatrix} AX\,{}^tA\,{}^t\!\begin{pmatrix} \overline{c}/|c| & 0 \\ 0 & E \end{pmatrix} = \operatorname{diag}\left(\begin{pmatrix} 0 & |c| \\ -|c| & 0 \end{pmatrix}, \begin{pmatrix} 0 & r_2 \\ -r_2 & 0 \end{pmatrix}, \dots, \begin{pmatrix} 0 & r_m \\ -r_m & 0 \end{pmatrix}\right),$$

$$\begin{pmatrix} \overline{d}/|d| & 0 \\ 0 & E \end{pmatrix} BX\,{}^tB \,{}^t\!\begin{pmatrix} \overline{d}/|d| & 0 \\ 0 & E \end{pmatrix} = \operatorname{diag}\left(\begin{pmatrix} 0 & |d| \\ -|d| & 0 \end{pmatrix}, \begin{pmatrix} 0 & s_2 \\ -s_2 & 0 \end{pmatrix}, \dots, \begin{pmatrix} 0 & s_m \\ -s_m & 0 \end{pmatrix}\right).$$

Hence we get

$$|c| = |d|,\ r_2 = s_2, \dots, r_m = s_m,$$

since the above form is unique for X by Lemma 2. Then, by the result of the first half, we obtain

$$\det\left(\begin{pmatrix} \overline{c}/|c| & 0 \\ 0 & E \end{pmatrix} A\right) = \det\left(\begin{pmatrix} \overline{d}/|d| & 0 \\ 0 & E \end{pmatrix} B\right), \text{ that is, } c = d.$$

We have thus completed the proof of the proposition.

We have known the following proposition.

PROPOSITION 4 ([1], [2], [3]). *A maximal compact subgroup* $(E_{7(7)})_K$ *of* $E_{7(7)}$:

$$(E_{7(7)})_K = \{\, \alpha \in E_{7(7)} \mid \langle \alpha P,\ \alpha Q\rangle = \langle P,\ Q\rangle \,\},$$

where the positive definite inner product $\langle P, Q\rangle$ *is defined by* $\langle P, Q\rangle = (P, \gamma Q)$ $(\gamma(X, Y, \xi, \eta) = (\gamma X, \gamma Y, \xi, \eta))$, *is isomorphic to the group* $SU(8)/\boldsymbol{Z}_2$ *by the isomorphism induced from the homomorphism* $\varphi : SU(8) \to (E_{7(7)})_K \subset E_{7(7)}$,

$$\varphi(A)P = \chi^{-1}(A\,\chi(P)\,{}^tA), \qquad P \in \mathfrak{P}'.$$

By using Propositions 3 and 4, we can prove the following key lemma.

LEMMA 5. *Any element* $P \in \mathfrak{P}'$ *can be transformed to the following canonical form by some* $\alpha = \varphi(A) \in (E_{7(7)})_K$, $A \in SU(8)$:

$$\alpha P = \left(\begin{pmatrix} \xi_1 & 0 & 0 \\ 0 & \xi_2 & 0 \\ 0 & 0 & \xi_3 \end{pmatrix}, \begin{pmatrix} \xi_0 & 0 & 0 \\ 0 & \xi_0 & 0 \\ 0 & 0 & \xi_0 \end{pmatrix}, \xi_4, -\xi_0\right), \qquad \begin{array}{c} \xi_4 \le \xi_1 \le \xi_2 \le \xi_3, \\ 4{\xi_0}^2 \le (\xi_2 + \xi_3)(\xi_4 - \xi_1). \end{array}$$

Moreover, this form is uniquely determined by X *independent of the choice of* $\alpha \in (E_{7(7)})_K$.

Proof. For a given element $P \in \mathfrak{P}'$, $\chi(P) \in \mathfrak{S}(8, \boldsymbol{C})$ is transformed to the following form by some $A \in SU(8)$:

$$A\chi(P)\,{}^tA = \begin{pmatrix} R_1 & & & \\ & R_2 & & \\ & & R_3 & \\ & & & R_4 \end{pmatrix}, \qquad \begin{array}{l} R_1 = \begin{pmatrix} 0 & r_1 + r_0 e_1 \\ -r_1 - r_0 e_1 & 0 \end{pmatrix}, \\ R_k = \begin{pmatrix} 0 & r_k \\ -r_k & 0 \end{pmatrix}, \quad k = 2, 3, 4, \\ \sqrt{{r_0}^2 + {r_1}^2} \le r_2 \le r_3 \le r_4, \ r_k \in \boldsymbol{R}, \end{array}$$

from Proposition 3. Then we have

$$\begin{aligned}
&\chi^{-1}(A\chi(P)\,{}^tA)\\
&= \left(\frac{1}{2}\begin{pmatrix} r_1+r_2-r_3-r_4 & 0 & 0\\ 0 & r_1-r_2+r_3-r_4 & 0\\ 0 & 0 & r_1-r_2-r_3+r_4\end{pmatrix},\right.\\
&\quad \left.\frac{1}{2}\begin{pmatrix} r_0 & 0 & 0\\ 0 & r_0 & 0\\ 0 & 0 & r_0\end{pmatrix}, -\frac{1}{2}(r_1+r_2+r_3+r_4), -\frac{1}{2}r_0\right),
\end{aligned} \tag{3.1}$$

from straightforward calculation. Moreover, its form is uniquely determined by $P \in \mathfrak{P}'$ under the condition

$$\sqrt{{r_0}^2+{r_1}^2} \le r_2 \le r_3 \le r_4. \tag{3.2}$$

Now, put

$$\text{(the right-hand side of (3.1))} = \left(\begin{pmatrix} \xi_1 & 0 & 0\\ 0 & \xi_2 & 0\\ 0 & 0 & \xi_3\end{pmatrix}, \begin{pmatrix} \xi_0 & 0 & 0\\ 0 & \xi_0 & 0\\ 0 & 0 & \xi_0\end{pmatrix}, \xi_4, -\xi_0\right).$$

Then we have

$$\begin{cases} r_0 = 2\xi_0, \\ r_1 = \frac{1}{2}(\xi_1+\xi_2+\xi_3-\xi_4), \quad r_2 = \frac{1}{2}(\xi_1-\xi_2-\xi_3-\xi_4), \\ r_3 = \frac{1}{2}(-\xi_1+\xi_2-\xi_3-\xi_4), \; r_4 = \frac{1}{2}(-\xi_1-\xi_2+\xi_3-\xi_4). \end{cases} \tag{3.3}$$

Substituting (3.3) into (3.2), we obtain the following inequalities

$$4{\xi_0}^2 \le (\xi_2+\xi_3)(\xi_4-\xi_1), \;\; \xi_2+\xi_3+\xi_4 \le \xi_1 \le \xi_2 \le \xi_3, \tag{3.4}$$

which are equivalent to (3.2). Further, noting that $0 \le 4{\xi_0}^2 \le (\xi_2+\xi_3)(\xi_4-\xi_1)$, $\xi_2+\xi_3+\xi_4 \le \xi_1$ imply $\xi_4-\xi_1 \le 0$, we can easily obtain the inequalities $4{\xi_0}^2 \le (\xi_2+\xi_3)(\xi_4-\xi_1)$, $\xi_4 \le \xi_1 \le \xi_2 \le \xi_3$ are equivalent to (3.4), i.e., (3.2). We have thus proved the lemma.

Consequently, we have the following theorem.

THEOREM 6. *The orbit space $\mathfrak{P}'/SU(8)$ of $\mathfrak{P}'$ with respect to $SU(8)$ of which the action is defined through $\varphi : SU(8) \to (E_{7(7)})_K \subset E_{7(7)}$, is given by*

$$\mathfrak{P}'/SU(8) = \left\{(\xi_0,\xi_1,\xi_2,\xi_3,\xi_4) \in \boldsymbol{R}^5 \,\middle|\, \begin{matrix} \xi_4 \le \xi_1 \le \xi_2 \le \xi_3 \\ 4{\xi_0}^2 \le (\xi_2+\xi_3)(\xi_4-\xi_1)\end{matrix}\right\},$$

with the cross section $\sigma : \mathfrak{P}'/SU(8) \to \mathfrak{P}'$,

$$\sigma(\xi_0, \xi_1, \xi_2, \xi_3, \xi_4) = (\mathrm{diag}(\xi_1, \xi_2, \xi_3), \mathrm{diag}(\xi_0, \xi_0, \xi_0), \xi_4, -\xi_0).$$

By exchanging the parameters λ_k for ξ_k in Theorem 6 such that $\lambda_k = \xi_k$ $(k = 0, 1, 2)$, $2\lambda_3 = \xi_2 + \xi_3$, $2\lambda_4 = \xi_4 - \xi_1$, we can also express the orbit space $\mathfrak{P}'/SU(8)$ in the following form:

$$\mathfrak{P}'/SU(8) = \left\{ (\lambda_0, \lambda_1, \lambda_2, \lambda_3, \lambda_4) \in \boldsymbol{R}^5 \,\middle|\, \begin{matrix} \lambda_1 \le \lambda_2 \le \lambda_3 \le 0 \\ {\lambda_0}^2 \le \lambda_3 \lambda_4 \end{matrix} \right\},$$

with the cross section $\sigma : \mathfrak{P}'/SU(8) \to \mathfrak{P}'$,

$$\sigma(\lambda_0, \lambda_1, \lambda_2, \lambda_3, \lambda_4) = (\mathrm{diag}(\lambda_1, \lambda_2, -\lambda_2 + 2\lambda_3), \mathrm{diag}(\lambda_0, \lambda_0, \lambda_0), \lambda_1 + 2\lambda_4, -\lambda_0).$$

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