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ACHROMATIC NUMBERS OF MAXIMAL OUTERPLANAR GRAPHS

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Abstract. A *complete k-coloring* of a graph G is a map from the vertices of G to k colors such that any two adjacent vertices get different colors and that any two different colors appear on the two endpoints of some edge. The *achromatic number* of G is the largest k such that G has a complete k-coloring. In this paper, we give a lower bound for the achromatic numbers of maximal outerplanar graphs.

1. Introduction

We consider only finite, simple, undirected graphs in this paper. We denote the vertex set and edge set of a graph G by $V(G)$ and $E(G)$, respectively. A *k-coloring* of G is a color-assignment $c: V(G) \to \{1, \ldots, k\}$ such that any two adjacent vertices of G get different colors. We say that G is *k-colorable* if G has a k-coloring. The *chromatic number* of G, denoted by $\chi(G)$, is the smallest k such that G is k-colorable.

Let G be a graph. A k-coloring $c: V(G) \to \{1, \ldots, k\}$ is said to be *complete* if for any two distinct colors $i, j \in \{1, \ldots, k\}$, there exists at least one edge whose two endpoints are colored by i and j respectively. We say that G is *complete k-colorable* if G has a complete k-coloring. The *achromatic number* of G, denoted by $\psi(G)$, is defined as the largest k such that G is complete k-colorable [2]. We have $\chi(G) \leq \psi(G)$ since any $\chi(G)$-coloring of G is always complete. (If in some $\chi(G)$-coloring, some two colors were not adjacent, then these two colors could be the same, contrary to that G was colored by $\chi(G)$ colors.)

Harary, Hedetniemi and Prins [3] have shown that G has a complete k-coloring for any k with $\chi(G) \leq k \leq \psi(G)$. In [6] and [4, 5], general upper and lower bounds of $\psi(G)$ of a graph G have been given. Recently, when G is a tree with bounded maximum degree, a lower bound of $\psi(G)$ has been given in [1]. It seems to be difficult to give a good bound of $\psi(G)$ for a given family of graphs.

A graph G is said to be *outerplanar* if G is embeddable on the plane so

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that every vertex of G lies on the boundary cycle of the infinite region. An outerplanar graph G is said to be *maximal* if G has an outerplanar embedding on the plane such that each finite face is bounded by a cycle of length 3. In this paper, we shall give a lower bound for achromatic numbers of outerplanar graphs, as follows:

THEOREM 1. *Let G be a maximal outerplanar graph with $n \geq 4$ vertices. If G has $m+1$ vertices of degree 2, then*

$$\psi(G) \geq \max\left\{2\left\lfloor\sqrt{\tfrac{n-m}{2}+\tfrac{1}{16}}-\tfrac{1}{4}\right\rfloor+1, 2\left\lfloor\sqrt{\tfrac{n-m+1}{2}}\right\rfloor\right\}.$$

It is easy to see that the set of vertices of degree 2 in a maximal outerplanar graph G is independent if $n \geq 4$ and hence $m+1 \leq \lfloor\frac{n}{2}\rfloor$. On the other hand, if G is complete N-colorable, then $|E(G)| = 2n-3 \geq N(N-1)/2$. Solving this for N, we have an upper bound for N. Therefore, we obtain the following corollary from Theorem 1.

COROLLARY 2. *Let G be a maximal outerplanar graph with $n \geq 4$ vertices. Then:*

$$\max\left\{2\left\lfloor\sqrt{\tfrac{n}{4}+\tfrac{9}{16}}-\tfrac{1}{4}\right\rfloor+1, 2\left\lfloor\sqrt{\tfrac{n}{4}+1}\right\rfloor\right\} \leq \psi(G) \leq \left\lfloor\sqrt{4n-\tfrac{23}{4}}+\tfrac{1}{2}\right\rfloor$$

2. Proof of theorems

Let G and H be two graphs. A *homomorphism* $h : G \to H$ is a map from $V(G)$ to $V(H)$ such that if $xy \in E(G)$, then $h(x)h(y) \in E(H)$. This induces a map from $E(G)$ to $E(H)$ naturally. It is easy to see that a complete N-coloring of a graph G corresponds to a homomorphism $h : G \to K_N$, which induces a surjection from $E(G)$ to $E(K_N)$. Such a homomorphism is said to be *complete*. Thus, a graph G is complete N-colorable if and only if there is a complete homomorphism $h : G \to K_N$.

Let P_n denote the path with n vertices. It is clear that a homomorphism $h : P_n \to K_N$ is complete if and only if it induces a walk W_h in K_N which traces each edge at least once. We define $O(d)$ and $E(d)$ to express $\psi(P_n)$, as follows:

$$O(d) = \max\left\{N : N \text{ is odd}, \tfrac{N(N-1)}{2} \leq d\right\} = 2\left\lfloor\sqrt{\tfrac{d}{2}+\tfrac{1}{16}}-\tfrac{1}{4}\right\rfloor+1$$

$$E(d) = \max\left\{N : N \text{ is even}, \tfrac{N^2}{2}-1 \leq d\right\} = 2\left\lfloor\sqrt{\tfrac{d+1}{2}}\right\rfloor$$

LEMMA 3. $\psi(P_n) = \max\{O(n-1), E(n-1)\}$.

Proof. Let N be an odd number. Then K_N has an euler tour. We can construct a complete homomorphism $h : P_n \to K_N$ so that W_h covers the euler tour if $N(N-1)/2 \leq n-1$. Thus, if $\psi(P_n)$ is odd, then it must be the value of N which maximizes $N(N-1)/2$ under this inequality.

On the other hand, if N is an even number, then consider an independent set $H \subset E(K_N)$ consisting of exactly $(N-2)/2$ edges. Let K'_N be the graph obtained from K_N by replacing each edge of H with parallel edges. Then, K'_N has exactly two vertices x, y of odd degree and K'_N has an euler tour from x to y. Tracing this euler tour, we can construct a complete homomorphism $h : P_n \to K_N$ if $N(N-1)/2 + (N-2)/2 = N^2/2 - 1 \leq n-1$. Conversely, it is easy to see that any walk covering all edges of K_N must pass through at least $(N-2)/2$ edges twice or more. Thus, if $\psi(P_n)$ is even, it must be the maximum value of N with $N^2/2 - 1 \leq n-1$. ■

Lemma 4. *If H is an induced subgraph of a graph G, then $\psi(G) \geq \psi(H)$.* ■

Lemma 5. *Let G be a graph with vertices $v_1, \ldots, v_n$ and let G_i be the subgraph of G induced by $\{v_1, \ldots, v_i\}$, for $i = 1, \ldots, n$. Suppose that for each i, the neighborhood of v_i in G_i induces a complete graph. Then $\psi(G) \geq \psi(P_{n-m+1})$, where*

$$m = |\{i \in \{1, \ldots, n-1\} : v_i v_{i+1} \notin E(G)\}| + 1.$$

Proof. Let P be the spanning subgraph in G whose edges are those of the form $v_i v_{i-1}$. Then P is a disjoint union of m paths and there is a natural surjective homomorphism $q : P \to P_{n-m+1}$ such that $q(v_i) = q(v_{i+1})$ whenever $v_i v_{i+1} \notin E(G)$. Let $\bar{c} : P_{n-m+1} \to \{1, \ldots, N\}$ be a complete coloring of P_{n-m+1} which attains $\psi(P_{n-m+1})$ $(= N)$. First, we define a color-assignment $c_1 : V(G) \to \{0, 1, \ldots, N\}$ by $c_1(v_i) = \bar{c}(q(v_i))$. This induces a complete coloring of P, but it might not be a coloring of G yet.

Suppose that $c_i : V(G) \to \{0, 1, \ldots, N\}$ has been defined and that c_i induces a complete coloring of $G_i \cup P - c_i^{-1}(0)$ with exactly N colors. It is clear that c_1 and c_2 satisfy this condition with $c_i^{-1}(0) = \emptyset$. Thus, we suppose that $i \geq 3$ and shall construct color-assignments $c_3, c_4, \ldots$ with this condition inductively as follows.

Let $x_1, \ldots, x_s$ be the neighbors of v_{i+1} in G_{i+1}. Since they induce a complete graph in G_{i+1} by the assumption of the lemma, we have $c_i(x_j) \neq c_i(x_k)$ for any distinct $j, k \in \{1, \ldots, s\}$, unless $c_i(x_j) = c_i(x_k) = 0$. If $c_i(v_{i+1}) \neq c_i(x_j)$ for each $j \in \{1, \ldots, s\}$, or if $c_i(v_{i+1}) = 0$, then c_i induces a coloring of $G_{i+1} - c_i^{-1}(0)$. In this case, we can put $c_{i+1} := c_i$.

Suppose that $c_i(v_{i+1}) \neq 0$ and that $c_i(v_{i+1}) = c_i(x_j)$ for some j. Let l denote the minimum index k such that $v_k v_{k+1} \notin E(G)$ with $k \geq i$. Thus v_l is one of ends of the component of P including v_{i+1}. In this case, we make a color-assignment $c_i' : V(G) \to \{0, 1, \dots, N\}$ temporarily by:

$$\begin{cases} c_i'(v_k) = c_i(v_k) & (1 \le k \le i,\ l+1 \le k \le n) \\ c_i'(v_k) = c_i(v_{k+1}) & (i+1 \le k \le l-1) \\ c_i'(v_k) = 0 & (k = l) \end{cases}$$

Call this deformation of c_i a *color shift* here.

It is clear that a color shift preserves the adjacency of colors $1, \dots, N$ lying on $G_i \cup P - v_{i+1}$, but the adjacency of two colors $c_i(v_{i+1})$ and $c_i(x_k)$ for some k with $k \ne j$ and that of $c_i(v_{i+1})$ and $c_i(v_{i+2})$ might be lost. However, these colors are still adjacent on $x_j x_k$ and $x_j v_{i+1}$, and hence c_i' satisfies the same condition as c_i.

Repeat color shifts until $c_i'(v_{i+1}) = 0$ or until $c_i'(v_{i+1}) \ne c_i'(x_j)$ for all $j \in \{1, \dots, s\}$. Then we can put $c_{i+1} := c_i'$ with the desired condition. Finally, we obtain a color-assignment $c_n : V(G) \to \{0, 1, \dots, N\}$ such that c_n induces a complete coloring of $G - c_n^{-1}(0)$ with exactly N colors. Since $G - c_n^{-1}(0)$ is an induced subgraph of G, we have $\psi(G) \ge \psi(G - c_n^{-1}(0)) \ge N = \psi(P_{n-m+1})$ by Lemma 4. ■

Now we shall prove Theorem 1.

Proof of Theorem 1. It suffices to make a labeling of vertices satisfying the assumptions in Lemma 5 and to evaluate the value of m for a maximal outerplanar graph G with n vertices.

It is easy to see that any maximal outerplanar graph with at least three vertices has a vertex of degree 2, which forms a triangle together with its two neighbors. Let v_n be one of vertices of degree 2 in $G_n = G$ and put $G_{n-1} = G_n - v_n$. For $i = n-1, \dots, 4$, let v_i be a vertex of degree 2 in G_i chosen in such a way that if there is a vertex of degree 2 in G_i which is adjacent to v_{i+1} in G_{i+1}, then we take it as v_i, and otherwise, take any vertex of degree 2 in G_i as v_i. Then, we put $G_{i-1} = G_i - v_i$. Finally, let v_2 and v_3 be the two neighbors of v_4 in G_4 and v_1 the other. This labeling satisfies the condition in Lemma 5 clearly.

Let $Q_n = P$ be the same subgraph in $G_n = G$ as in Lemma 5, which is a disjoint union of m paths, and let $V_2 = V_2(G_n)$ denote the number of vertices of degree 2 in G_n. We shall show that Q_n has exactly $V_2 - 1$ components, using induction on n. This implies that $m = V_2 - 1$. If $n = 4$, then this holds obviously with $V_2 = 2$. Suppose that $n \ge 5$.

Let x and y be two neighbors of v_n in $G = G_n$. If v_n is isolated in Q_n, then $\deg_{G_{n-1}}(x) \ge 3$ and $\deg_{G_{n-1}}(y) \ge 3$, and hence $V_2(G_{n-1}) = V_2(G_n) - 1$. By induction hypothesis, Q_{n-1} has $V_2(G_n) - 2$ components and hence Q_n has

$V_2(G_n) - 1$ components.

If v_n is not isolated in Q_n, then one of x and y, say x, has degree 2 in G_{n-1} and $v_{n-1} = x$ by the definition of labeling of vertices. (If y also had degree 2 in G_{n-1}, then G_{n-1} would be isomorphic to K_3 and hence $n = 4$, contrary to the assumption $n \geq 5$.) Thus, Q_{n-1} has the same number of components as Q_n and also has the same number of vertices of degree 2 as Q_n. By induction hypothesis, the number of components of Q_n is equal to $V_2(G_{n-1}) - 1 = V_2(G_n) - 1$. The induction completes.

Therefore, we have $V_2 = m + 1$ and $\psi(G) \geq \psi(P_{n-m+1})$ by Lemma 5. The theorem follows from Lemma 3 with this. ■

Now we discuss the sharpness of estimation in Theorem 1. Let $G = P_{n-1} + K_1$. It is clear that $\psi(G) = \psi(P_{n-1}) + 1$ since the vertex corresponding to K_1 is adjacent to all other vertices. Assume that

$$|E(P_{n-1})| = n - 2 = N(N-1)/2 - 1$$

for some odd number $N \geq 3$. Since G has exactly two vertices of degree 2, we have $m = 1$ and $\psi(G) \geq \psi(P_n)$ by Theorem 1. On the other hand, by the assumption of n, we have $\psi(P_{n-1}) = N - 1$ while $\psi(P_n) = N$. Hence, $\psi(G) = \psi(P_{n-1}) + 1 = \psi(P_n) = N = O(n-1)$.

The same arguments as in the proof of Theorem 1 can be used to show a lower bound of $\psi(G)$ for a more general family of graphs, defined as follows. First, put $G_1 = K_{t+1}$. To obtain a graph G_{i+1} with exactly $t + i + 1$ vertices, choose any clique in G_i consisting of t vertices and join those t vertices to a new vertex so that they form K_{t+1}. These graphs G_i's are called *t-trees*. It is easy to see that a t-tree G has minimum degree exactly t, and the vertices of G of degree t are independent if $|V(G)| \geq t + 2$. Obviously, an ordinary tree is a 1-tree, and a maximal outerplanar graph is a 2-tree. (Note that there is a 2-tree which is not maximal outerplanar.)

THEOREM 6. *Let t be a positive integer and let G be a t-tree with n vertices and with exactly $m+1$ vertices of degree t. Then, $\psi(G) \geq \max\{O(n-m), E(n-m)\}$.* ■

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