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BOUR'S THEOREM AND GAUSS MAP

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Abstract. A general helicoid and a rotation surface have isometric relation by Bour's theorem. In this paper, we determine pairs of surfaces of Bour's theorem with an additional condition that they have the same Gauss map.

1. Introduction

In classical differential geometry, it is well known that the right helicoid (resp. catenoid) is the only ruled (resp. rotation) surface which is minimal. Moreover, a pair of the right helicoid and the catenoid has interesting properties. That is, they are both members of a one-parameter family of isometric minimal surfaces and have the same Gauss map. This pair is a typical example of a minimal surface and its conjugate one on the Weierstrass-Enneper representation for minimal surfaces.

On the other hand, in surface theory, following Bour's theorem is well known.

Bour's theorem ([1], [3]). *A generalized helicond is isometric to a rotation surface so that helices on the helicoid correspond to parallel circles on the rotation surface.*

This theorem is a generalization of the pair of the right helicoid and the catenoid. In this generalization, however, original properties that they were minimal and preserved the Gauss map are not generally kept. Hence it is a natural question that on Bour's theorem if the Gauss map is preserved then two surfaces are determined or not.

The purpose of this paper is to answer this question.

In Section 2, we recall some formulas to study surface theory on R^3 and give an outline of the proof of Bour's theorem to make the paper self-contained. In Section 3 we determine the helicoid and the rotation surface that preserve the Gauss map on Bour's theorem.

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2. Bour's theorem

First we recall the definition of the rotation surface and the generalized helicoid and some formulas to study surface theory on R^3 [2].

For an open interval I, let $\gamma : I \longrightarrow \Pi$ be a curve in a plane Π on R^3 and let l be a straight line in Π which does not intersect the curve γ. A *rotation surface* R is defined as a surface rotating the curve γ around l (these are called the *profile curve* and the *axis*, respectively). We may suppose that the axis l is the x_3 - axis and the plane Π is the x_1x_3 - plane, without loss of generality. Then the profile curve γ is given as

$$\gamma(u) = (u,\ 0,\ \varphi(u)).$$

Hence a rotation surface R can be parametrized by

$$R(u,v) = (u\cos v,\ u\sin v,\ \varphi(u)). \tag{2.1}$$

In the rest of this paper, we shall identify a vector (a,b,c) with a transpose ${}^t(a,b,c)$ of (a,b,c).

Suppose that when a profile curve γ rotates around the axis l, it simultaneously displaces parallel to l so that the speed of displacement is proportional to the speed of rotation. Then resulting surface H is called the *generalized helicoid.* Hence this surface can be parametrized by

$$H(u,v) = (u\cos v,\ u\sin v,\ \varphi(u) + av). \tag{2.2}$$

where a is a constant. When φ is a constant function, then the generalized helicoid is called *the right helicoid.*

For a surface $X(u,v)$, the coefficients E, F and G of the first fundamental form in the base $\{X_u, X_v\}$ are defined as

$$E = \langle X_u,\ X_u\rangle,\quad F = \langle X_u,\ X_v\rangle,\quad G = \langle X_v,\ X_v\rangle. \tag{2.3}$$

The coefficients L, M and N of the second fundamental form of $X(u,v)$ are given as

$$L = \langle X_{uu},\ e\rangle,\quad M = \langle X_{uv},\ e\rangle,\quad N = \langle X_{vv},\ e\rangle, \tag{2.4}$$

by the *Gauss map*

$$e = \frac{X_u \times X_v}{\sqrt{\langle X_u \times X_v, X_u \times X_v\rangle}}. \tag{2.5}$$

By using the first fundamental form and the second fundamental form, the *mean curvature* H is defined by

$$H=\frac{EN+GL-2FM}{2(EG-F^2)}. \tag{2.6}$$

In the rest of this section, we sketch the proof of Bour's theorem to make the paper self-contained.

The line element of a generalized helicoid (2.2) is

$$ds^2=(1+\varphi'^2)du^2+2a\varphi' du dv+(u^2+a^2)dv^2. \tag{2.7}$$

A helix on the generalized helicoid is a curve of u =(const.). To give a curve orthogonal to the helix, we consider the orthogonal condition

$$a\varphi'(u)du+(u^2+a^2)dv=0.$$

From this equation, we obtain

$$v=-\int\frac{a\varphi'}{u^2+a^2}du+c,$$

where c is a constant. Hence if we put

$$\overline{v}=v+\int\frac{a\varphi'}{u^2+a^2}du$$

then the orthogonal curve is given by $\overline{v}=$ (const.). So, it follows that

$$ds^2=\left(1+\frac{u^2\varphi'^2}{u^2+a^2}\right)du^2+(u^2+a^2)d\overline{v}^2.$$

If we put

$$\overline{u}=\int\sqrt{1+\frac{u^2\varphi'^2}{u^2+a^2}}du,\quad \sqrt{u^2+a^2}=f(\overline{u}),$$

then the line element of the generalized helicoid is given as

$$ds^2=d\overline{u}^2+f^2(\overline{u})d\overline{v}^2. \tag{2.8}$$

On the other hand, the line element of a rotation surface

$$(u_R\cos v_R,\ u_R\sin v_R,\ \varphi_R(u_R)) \tag{2.9}$$

is

$$ds_R^2 = (1 + {\varphi'_R}^2)du_R^2 + u_R^2 dv_R^2.$$

So, if we put

$$\overline{u}_R = \int \sqrt{1 + {\varphi'_R}^2} du_R, \quad u_R = f_R(\overline{u}_R), \quad \overline{v}_R = v_R,$$

then the line element of the rotation surface is rewritten as

$$ds_R^2 = d\overline{u}_R^2 + f_R^2(\overline{u}_R) d\overline{v}_R^2. \tag{2.10}$$

From (2.8) and (2.10), if we put

$$\overline{u} = \overline{u}_R, \quad \overline{v} = \overline{v}_R, \quad u_R = f_R(\overline{u}_R) = f(\overline{u}) = \sqrt{u^2 + a^2}, \tag{2.11}$$

then we have an isometry. Therefore, a generalized helicoid

$$H(u, v) = \begin{bmatrix} u \cos v \\ u \sin v \\ \varphi(u) + av \end{bmatrix} \tag{2.12}$$

is isometric to the rotation surface

$$R(u, v) = \begin{bmatrix} \sqrt{u^2 + a^2} \cos(v + \int \frac{a\varphi'}{u^2 + a^2} du) \\ \sqrt{u^2 + a^2} \sin(v + \int \frac{a\varphi'}{u^2 + a^2} du) \\ \int \sqrt{\frac{a^2 + u^2 {\varphi'}^2}{u^2 + a^2}} du \end{bmatrix}. \tag{2.13}$$

3. Gauss map

In this section, we prove the following theorem.

THEOREM. *Let a generalized helicoid and a rotation surface be isometrically related by Bour's theorem. If these two surfaces have the same Gauss map, then the pair of two surfaces is*

$$\begin{bmatrix} u \cos v \\ u \sin v \\ \varphi(u) + av \end{bmatrix} \quad \textit{and} \quad \begin{bmatrix} \sqrt{u^2 + a^2} \cos\left(v + \int \frac{a\varphi'}{u^2 + a^2} du\right) \\ \sqrt{u^2 + a^2} \sin\left(v + \int \frac{a\varphi'}{u^2 + a^2} du\right) \\ b \cosh^{-1}\left(\frac{\sqrt{u^2 + a^2}}{b}\right) \end{bmatrix}$$

where

$$\varphi(u) = a\tan^{-1}\left(\frac{\sqrt{b^2-a^2}}{a}\sqrt{\frac{u^2+a^2}{u^2+a^2-b^2}}\right) + \sqrt{b^2-a^2}\log\sqrt{\frac{\sqrt{u^2+a^2}+\sqrt{u^2+a^2-b^2}}{\sqrt{u^2+a^2}-\sqrt{u^2+a^2-b^2}}},$$

and a and b are constants of $b^2-a^2 \geq 0$.

Proof. First we consider a helicoid (2.12). Since

$$H_u = \begin{bmatrix}\cos v\\ \sin v\\ \varphi'\end{bmatrix}, \quad H_v = \begin{bmatrix}-u\sin v\\ u\cos v\\ a\end{bmatrix},$$

the Gauss map e_H of the helicoid is given as

$$e_H = \frac{1}{\sqrt{a^2+u^2+u^2\varphi'^2}}\begin{bmatrix}a\sin v - u\varphi'\cos v\\ -u\varphi'\sin v - a\cos v\\ u\end{bmatrix}, \tag{3.1}$$

by virtue of (2.5). Differentiating H_u and H_v, we have

$$H_{uu} = \begin{bmatrix}0\\ 0\\ \varphi''\end{bmatrix}, \quad H_{vv} = \begin{bmatrix}-u\cos v\\ -u\sin v\\ 0\end{bmatrix}, \quad H_{uv} = \begin{bmatrix}-\sin v\\ \cos v\\ 0\end{bmatrix}.$$

Hence the mean curvature H_H is given as

$$H_H = \frac{(1+\varphi'^2)u^2\varphi' + u\varphi''(u^2+a^2) + 2a^2\varphi'}{2(u^2(1+\varphi'^2)+a^2)^{3/2}}, \tag{3.2}$$

by virtue of the first and second fundamental forms

$$E_H = 1+\varphi'^2, \quad F_H = a\varphi', \quad G_H = u^2+a^2,$$

$$L_H = \frac{u\varphi''}{\sqrt{u^2(1+\varphi'^2)+a^2}}, \quad M_H = \frac{-a}{\sqrt{u^2(1+\varphi'^2)+a^2}},$$
$$N_H = \frac{u^2\varphi'}{\sqrt{u^2(1+\varphi'^2)+a^2}}.$$

Next we calculate the Gauss map e_R and the mean curvature H_R of the rotation surface (2.13). Since

$$R_u = \begin{bmatrix} \frac{u}{\sqrt{u^2+a^2}} \cos\left(v + \int \frac{a\varphi'}{u^2+a^2} du\right) - \frac{a\varphi'}{\sqrt{u^2+a^2}} \sin\left(v + \int \frac{a\varphi'}{u^2+a^2} du\right) \\ \frac{u}{\sqrt{u^2+a^2}} \sin\left(v + \int \frac{a\varphi'}{u^2+a^2} du\right) + \frac{a\varphi'}{\sqrt{u^2+a^2}} \cos\left(v + \int \frac{a\varphi'}{u^2+a^2} du\right) \\ \sqrt{\frac{a^2+u^2\varphi'^2}{u^2+a^2}} \end{bmatrix},$$

$$R_v = \begin{bmatrix} -\sqrt{u^2+a^2} \sin\left(v + \int \frac{a\varphi'}{u^2+a^2} du\right) \\ \sqrt{u^2+a^2} \cos\left(v + \int \frac{a\varphi'}{u^2+a^2} du\right) \\ 0 \end{bmatrix},$$

the Gauss map e_R of the rotation surface is given as

$$(3.3) \qquad e_R = \frac{1}{\sqrt{a^2+u^2+u^2\varphi'}} \begin{bmatrix} -\sqrt{a^2+u^2\varphi'} \cos\left(v + \int \frac{a\varphi'}{u^2+a^2} du\right) \\ -\sqrt{a^2+u^2\varphi'} \sin\left(v + \int \frac{a\varphi'}{u^2+a^2} du\right) \\ u \end{bmatrix}.$$

By a straight calculation, we have the coefficients of the second fundamental form as follows

$$L_R = \frac{1}{\sqrt{a^2+u^2+u^2\varphi'^2}} \left(-\frac{(a^2 - a^2\varphi'^2)\sqrt{a^2+u^2\varphi'^2}}{(u^2+a^2)^{3/2}} + \frac{u^2(-a^2 + a^2\varphi'^2 + (a^2+u^2)u\varphi'\varphi'')}{(u^2+a^2)^{3/2}\sqrt{a^2+u^2\varphi'^2}} \right),$$

$$N_R = \frac{\sqrt{u^2+a^2}\sqrt{a^2+u^2\varphi'^2}}{\sqrt{a^2+u^2+u^2\varphi'^2}},$$

$$M_R = \frac{a\varphi'\sqrt{a^2+u^2\varphi'^2}}{\sqrt{u^2+a^2}\sqrt{a^2+u^2+u^2\varphi'^2}}.$$

So, the mean curvature H_R is

$$(3.4) \qquad H_R = \frac{u^2\varphi'(2a^2\varphi' + u^2\varphi' + u^2\varphi'^3 + a^2u\varphi'' + u^3\varphi'')}{2\sqrt{u^2+a^2}\sqrt{a^2+u^2\varphi'^2}(a^2+u^2+u^2\varphi'^2)^{3/2}}.$$

Now suppose that the Gauss map e_H is identically equal to e_R.

If $\varphi' = 0$, then the helicoid reduces to right helicoid and the mean curvature H_R of the rotation surface is identically zero. So the rotation surface is the catenoid and the function $\varphi_R(u_R)$ of (2.9) is $\varphi_R(u_R) = b\cosh^{-1}\left(\frac{u_R}{b}\right)$, where b is a constant. Comparing this equation and the third element of (2.13), we have

$$b\cosh^{-1}\left(\frac{\sqrt{u^2+a^2}}{b}\right) = \int \frac{a}{\sqrt{u^2+a^2}}du.$$

By differentiating this equation, it follows that

$$\frac{bu}{\sqrt{u^2+a^2-b^2}} = a.$$

Hence we have $a = b$.

Next we suppose $\varphi' \neq 0$. Then compairing (3.1) and (3.3), we have

$$\tan^{-1}\left(\frac{a}{u\varphi'}\right) = \int \frac{a\varphi'}{u^2+a^2}du.$$

Differentiating this equation, we obtain

$$a^2u\varphi'' + u^3\varphi'' + u^2\varphi'^3 + u^2\varphi' + 2a^2\varphi' = 0. \tag{3.5}$$

From (3.2) and (3.4), this equation means that the generalized helicoid and the rotation surface have zero mean curvature. Hence, again, the rotation surface reduces to the catenoid. So, it follows that

$$b\cosh^{-1}\left(\frac{\sqrt{u^2+a^2}}{b}\right) = \int \sqrt{\frac{a^2+u^2\varphi'^2}{u^2+a^2}}du.$$

From this equation, we can give the profile curve φ of the generalized helicoid. In fact, differentiating this equatin, we have

$$\varphi'^2 = \frac{(b^2-a^2)u^2 - a^4 + a^2b^2}{u^2(u^2+a^2-b^2)}. \tag{3.6}$$

By substituting this equation into (3.5), it follows that

$$(ua^2+u^3)\varphi'' + \left(a^2+u^2+\frac{b^2u^2}{u^2+a^2-b^2}\right)\varphi' = 0,$$

so that

$$-\frac{\varphi''}{\varphi'} = \frac{a^2+u^2+\frac{b^2u^2}{u^2+a^2-b^2}}{a^2u+u^3}.$$

From this, we obtain

$$-\log \varphi' = \log c \frac{u\sqrt{u^2 + a^2 - b^2}}{\sqrt{u^2 + a^2}},$$

where c is a constant, so

$$\varphi' = \frac{\sqrt{u^2 + a^2}}{cu\sqrt{u^2 + a^2 - b^2}}.$$

Comparing this equation with (3. 6), we have

$$(b^2 - a^2)c^2 = 1,$$

so

$$\varphi' = \frac{\sqrt{b^2 - a^2}\sqrt{u^2 + a^2}}{u\sqrt{u^2 + a^2 - b^2}}. \tag{3.7}$$

To solve this differential equation, we put

$$\sqrt{\frac{u^2 + a^2}{u^2 + a^2 - b^2}} = t.$$

Then it follows that

$$\begin{aligned}\varphi &= \sqrt{b^2 - a^2} \int \frac{\sqrt{u^2 + a^2}}{u\sqrt{u^2 + a^2 - b^2}} du = \sqrt{b^2 - a^2} \int \frac{b^2 t^2}{((a^2 - b^2)t^2 - a^2)(t^2 - 1)} dt \\ &= \sqrt{b^2 - a^2} \left(\frac{a}{\sqrt{b^2 - a^2}} \tan^{-1} \left(\frac{\sqrt{b^2 - a^2}}{a} t \right) + \log \sqrt{\frac{t+1}{t-1}} \right) + d,\end{aligned}$$

where d is a constant. Hence we can give the profile curve of the generalized helicoid and this completes the proof.

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