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MULTIPLE SOLUTIONS FOR SEMILINEAR HEMIVARIATIONAL INEQUALITIES AT RESONANCE

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Abstract. We consider semilinear eigenvalue problems for hemivariational inequalities at resonance. First we consider problems which are at resonance in a higher eigenvalue λ_k (with $k \geq 1$) and prove two multiplicity theorems asserting the existence of at least k pairs of nontrivial solutions. Then we consider problems which are resonant at the first eigenvalue $\lambda_1 > 0$. For such problems we prove the existence of at least three nontrivial solutions. Our approach is variational and is based on the nonsmooth critical point theory of Chang, for locally Lipschitz functions.

1. Introduction

In a recent paper Goeleven-Motreanu-Panagiotopoulos [14] studied a class of eigenvalue problems for semilinear hemivariational inequalities and obtained conditions for the existence of multiple solutions. Extensions to quasilinear hemivariational inqualities were established by Gasiński-Papageorgiou [13]. The resonant case was examined by Goeleven-Motreanu-Panagiotopoulos in [15] (semilinear problems) and Gasiński-Papageorgiou [12] (quasilinear problems). In both these papers, we find results on the existence of one solution, but no multiplicity theorems. The purpose of this paper is to prove theorems on the existence of multiple solutions for semilinear hemivariational inequalities at resonance. This way we extend the work of Goeleven-Motreanu-Panagiotopoulos [14] to the resonant case (in fact at the end of [14], the resonant case was mentioned as an open problem) and also complete the other work of Goeleven-Motreanu-Panagiotopoulos [15], which deals with resonant hemivariational inequalities, but does not address the question of multiple solutions. Hemivariational inequalities are a new type of variational inequalities, where the convex subdifferential is replaced by

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the subdifferential in the sense of Clarke of a locally Lipschitz function. Such inequalities are motivated by various problems in mechanics, where the lack of convexity does not permit the use of the convex superpotential of Moreau [19]. Concrete applications to problems of theoretical mechanics and engineering can be found in the book of Panagiotopoulos [21] and Naniewicz-Panagiotopoulos [20]. Also the problems considered here incorporate the case of elliptic boundary value problems with discontinuous right hand side, which have been studied using different methods, by several researchers. We refer to the works of Ambrosetti-Badiale [2], Ambrosetti-Tuner [3], Chang [9], Massabo [18], Stuart [23] and the references therein.

Our approach is variational and is based on the critical point theory for nonsmooth locally Lipschitz functionals due to Chang [9]. For the convenience of the reader in the next section we recall some definitions and facts from the theory and also from the relevant parts of nonsmooth analysis.

2. Preliminaries

The nonsmooth critical point theory developed by Chang [9] is based on the subdifferential theory for locally Lipschitz functionals due to Clarke [10]. Let X be a Banach space and X^* its topological dual. A function $f : X \longmapsto \mathbb{R}$ is said to be locally Lipschitz, if for every $x \in X$ we can find a neighbourhood U of x and a constant $k_U > 0$, such that $|f(y) - f(z)| \leq k_U \|y - z\|$ for every $y, z \in U$. It is well-known from convex analysis that a proper, convex and lower semicontinuous function $g : X \longmapsto \overline{\mathbb{R}} = \mathbb{R} \cup \{+\infty\}$ is locally Lipschitz in the interior of its effective domain $\mathrm{dom} g = \{x \in X : \ g(x) < +\infty\}$. In analogy with the directional derivative of a convex function, for a locally Lipschitz function $f : X \longmapsto \mathbb{R}$, we can define the generalized directional derivative of f at x in the direction h by

$$f^0(x; h) \stackrel{df}{=} \limsup_{\substack{x' \to x \\ t \searrow 0}} \frac{f(x' + th) - f(x')}{t}.$$

It is easy to check that the function $X \ni h \longmapsto f^0(x; h) \in \mathbb{R}$ is sublinear and continuous (in fact $|f^0(x; h)| \leq \|h\|$, hence $f^0(x; \cdot)$ is Lipschitz). So, by a corollary to the Hahn-Banach theorem, $f^0(x; \cdot)$ is the support function of a nonempty, closed, convex and bounded (hence w^*-compact) subset of X^*, defined by

$$\partial f(x) \stackrel{df}{=} \{x^* \in X^* : \ \langle x^*, h \rangle \leq f^0(x, h) \text{ for all } h \in X\},$$

(see Clarke [10], Proposition 2.1.2, p.27). The set $\partial f(x)$ is known as the subdifferential of f at x. If $f : X \longmapsto \mathbb{R}$ is convex (so locally Lipschitz as well), then this subdifferential coincides with the subdifferential in the sense of convex analysis and we have $f^0(x;\cdot) = f'(x;\cdot)$, where $f'(x;\cdot)$ denotes the directional derivative of f at x. If f is strictly differentiable at x (in particular if f is continuously Gateaux differentiable at x), then $\partial f(x) = \{f'(x)\}$. If $f, g : X \longmapsto \mathbb{R}$ are locally Lipschitz functions, then for all $x \in X$ and all $\alpha \in \mathbb{R}$, we have $\partial(f+g)(x) \subseteq \partial f(x) + \partial g(x)$ and $\partial(\alpha f)(x) = \alpha \partial f(x)$. Finally, if f has a local extremum at $x \in X$, then $0 \in \partial f(x)$.

Now let $f : X \longmapsto \mathbb{R}$ be a locally Lipschitz function on a Banach space X. Extending the notion of a critical point theory for a smooth function to the present nonsmooth setting, we say that $x \in X$ is a critical point of f if $0 \in \partial f(x)$. We say that f satisfies the "nonsmooth Palais-Smale condition" (nonsmooth (PS)-condition), if for any sequence $\{x_n\}_{n\geq 1} \subseteq X$ such that $|f(x_n)| \leq M$ for all $n \geq 1$ and $m(x_n) \stackrel{df}{=} \inf\{\|x^*\|_* : x^* \in \partial f(x_n)\} \longrightarrow 0$ as $n \to +\infty$, we can extract a strongly converget subsequence. Since for $f \in C^1(X)$, we have that $\partial f(x) = \{f'(x)\}$ for all $x \in X$, so we see that when f is smooth, we recover the classical (PS)-condition (see e.g. Ambrosetti [1] or Rabinowitz [22]). Using this extension of the classical (PS)-condition, Chang [9] was able to obtain a deformation theorem, which led to variational minimax principles. As it was done in the smooth case by Bartolo-Benci-Fortunato [6] (Theorem 1.3), we can show using their proof (with minor modifications which involve Lemmas 3.1 up to 3.4 of Chang [9], instead of the corresponding smooth auxiliary results employed by Bartolo-Benci-Fortunato [6]), that we can still have the deformation theorem of Chang [9] (Theorem 3.1), under the following weaker compactness condition: "From any sequence $\{x_n\}_{n\geq 1} \subseteq X$ such that $|R(x_n)| \leq M$ for all $n \geq 1$ and $(1 + \|x_n\|)m(x_n) \longrightarrow 0$ as $n \to +\infty$, we can extract a strongly convergent subsequence". We call this condition, the "nonsmooth C-condition" ("C" standing for Cerami [8], who introduced it). Evidently the nonsmooth (PS)-condition implies the nonsmooth C-condition.

The following theorem is due to Chang [9] and is a nonsmooth extension of the well-known "mountain pass theorem" due to Ambrosetti-Rabinowitz [4].

THEOREM 1. **If** *X is a reflexive Banach space, $R : X \longmapsto \mathbb{R}$ is a locally Lipschitz functional which satisfies the nonsmooth C-condition and for some $r > 0$ and $y \in X$ with $\|y\| > r$ we have*

$$\max\{R(0), R(y)\} < \inf_{\|x\|=r} R(x) = \alpha,$$

then *R has a nontrivial critical point $x^* \in X$ with critical value $c = R(x^*) \geq \alpha$, which is characterized by the following minimax principle*

$$c = \inf_{\gamma\in\Gamma} \max_{\tau\in[0,1]} R(\gamma(\tau)),$$

where $\Gamma \stackrel{df}{=} \{\gamma \in \mathcal{C}([0,1],X): \ \gamma(0)=0, \ \gamma(1)=y\}$.

A slightly more general version of Theorem 1 will be needed in section 4. For this we need the following variation of the nonsmooth (PS)-condition. We say that R satisfies the nonsmooth (PS)-condition at level $c \in \mathbb{R}$, if every sequence $\{x_n\}_{n\geq 1} \subseteq X$ such that $R(x_n) \longrightarrow c$ and $m(x_n) \longrightarrow 0$ as $n \to +\infty$, has a convergent subsequence. If this condition holds at every level $c \in \mathbb{R}$, then we recover the nonsmooth (PS)-condition introduced earlier.

THEOREM 2. If *X is a reflexive Banach space, $R: X \longmapsto \mathbb{R}$ is a locally Lipschitz functional, there exist $r > 0$ and $y \in X$ with $\|y\| > r$ such that*

$$\max\{R(0), R(y)\} < \inf_{\|x\|=r} R(x),$$

and

$$c \stackrel{df}{=} \inf_{\gamma\in\Gamma} \max_{\tau\in[0,1]} R(\gamma(\tau)),$$

where $\Gamma \stackrel{df}{=} \{\gamma \in \mathcal{C}([0,1],X): \ \gamma(0)=0, \ \gamma(1)=y\}$ and R satisfies the nonsmooth (PS)-condition at level c,
then

$$c \geq \inf_{\|x\|=r} R(x)$$

and there exists $x^* \in X$ such that $0 \in \partial R(x^*)$ and $R(x^*) = c$.

The next result on the existence of multiple critical points in the presence of some kind of splitting, was first proved by Szulkin (see [24], Theorem 4.4) for functions $R = \Phi + \psi$, where $\Phi \in C^1(X)$ and $\psi: X \longmapsto \overline{\mathbb{R}} = \mathbb{R} \cup \{+\infty\}$ is proper, convex and lower semicontinuous. By modifying the proof of Szulkin and using the deformation theorem of Chang [9], Goeleven-Motreanu-Panagiotopoulos extended the result of Szulkin to the case of a locally Lipschitz functionals R (see [14], Theorem 2.1). So we have the following theorem on the existence of multiple nontrivial critical points.

THEOREM 3. If *X is a reflexive Banach space and $R: X \longmapsto \mathbb{R}$ is an even, locally Lipschitz functional satisfying the nonsmooth C-condition and also*

(i) *$R(0) = 0$;*
(ii) *there exists a subspace $Y \subseteq X$ of finite codimension and numbers $\beta, r > 0$ such that $\inf\{R(x): \ x \in Y \cap \partial B_r(0)\} \geq \beta$ where $B_r = \{x \in X: \ \|x\| < r\}$ and $\partial B_r = \{x \in X: \ \|x\| = r\}$;*

(iii) *there is a finite dimensional subspace* V *of* X *with* $\dim V > \operatorname{codim} Y$ *such that* $R(y) \longrightarrow -\infty$, *as* $\|y\| \to +\infty$, *for* $y \in V$,

then R *has at least* $\dim V - \operatorname{codim} Y$ *pairs of nontrivial critical points.*

Finally let us recall the Ekeland variational principle (compare De Figueiredo [11], Hu-Papageorgiou [16], p.519 or Clarke [10], Chapter 7.5).

THEOREM 4. **If** (Y, d) *is a complete metric space and* $R : Y \longmapsto \overline{\mathbb{R}} = \mathbb{R} \cup \{+\infty\}$ *is lower semicontinuous and bounded from below,*
then *for any* $\varepsilon > 0$ *there exists* $y_\varepsilon \in Y$ *such that*

$$\begin{cases} R(y_\varepsilon) & \leq \inf\limits_{y \in Y} R(y) + \varepsilon, \\ R(y_\varepsilon) & < R(y) + \varepsilon d(y, y_\varepsilon) \qquad \forall y \in Y,\ y \neq y_\varepsilon. \end{cases}$$

Using the eigenfunction expansion theory for self-adjoint compact operators, we know that $(-\Delta, H_0^1(Z))$ has a sequence of eigenvalues $\{\lambda_k\}_{k \geq 1}$ such that $0 < \lambda_1 < \lambda_2 \leq \lambda_3 \leq \ldots$, $\lambda_k \longrightarrow +\infty$ as $k \to +\infty$ and the corresponding eigenfunctions $\{w_k\}_{k \geq 1}$ form an ortonormal basis of $L^2(Z)$. In what follows, we will denote $V_k \stackrel{df}{=} \operatorname{span}\{w_1, w_2, \ldots, w_k\}$ for $k \geq 1$.

3. Resonant problems at λ_k

Let $Z \subseteq \mathbb{R}^N$ $(N \geq 2)$ be a bounded domain with a C^1-boundary Γ. In this section we study the following resonant at λ_k hemivariational inequality:

$$(RHI_k) \qquad \begin{cases} -\Delta x(z) - \lambda_k x(z) \in \partial j(z, x(z)) \text{ a.e. on } Z, \\ x|_\Gamma = 0. \end{cases}$$

Our hypotheses on the function j are the following:

$H(j)_1$ $j : Z \times \mathbb{R} \longmapsto \mathbb{R}$ is an even locally Lipschitz integrand (which means that for all $\zeta \in \mathbb{R}$: $Z \ni z \longmapsto j(z, \zeta) \in \mathbb{R}$ is measurable and for almost all $z \in Z$: $\mathbb{R} \ni \zeta \longmapsto j(z, \zeta) \in \mathbb{R}$ is even and locally Lipschitz), such that:

(i) for almost all $z \in Z$, all $\zeta \in \mathbb{R}$ and all $v(z, \zeta) \in \partial j(z, \zeta)$, we have $|v(z, \zeta)| \leq a_1(z) + c_1|\zeta|$ with $a_1 \in L^\infty(Z)$ and $c_1 > 0$;

(ii) $j(\cdot, 0) \in L^\infty(Z)$;

(iii) $\liminf\limits_{|\zeta| \to +\infty} \dfrac{2j(z, \zeta)}{\zeta^2} > 0$ uniformly for almost all $z \in Z$;

(iv) $\limsup\limits_{\zeta \to 0} \dfrac{2j(z, \zeta)}{\zeta^2} \leq -\lambda_k$ uniformly for almost all $z \in Z$;

(v) there exists $0 < \mu < 2$ such that $\limsup\limits_{|\zeta| \to +\infty} \dfrac{v(z, \zeta)\zeta - 2j(z, \zeta)}{|\zeta|^\mu} < 0$ uniformly for almost all $z \in Z$ and all $v(z, \zeta) \in \partial j(z, \zeta)$, with $v(\cdot, \zeta) \in L^2(Z)$.

Now we can prove the following multiplicity result for (RHI_k).

THEOREM 5. If *hypotheses* $H(j)_1$ *hold and* $k \geq 1$,
then *problem* (RHI_k) *has at least k-pairs* $\{\pm x_i\}_{i=1}^k$ *of nontrivial solutions.*

Proof. Let $R_k : H_0^1(Z) \longmapsto \mathbb{R}$ be the energy function defined by

$$R_k(x) \stackrel{df}{=} \frac{1}{2}\|\nabla x\|_2^2 - \frac{\lambda_k}{2}\|x\|_2^2 - \int_Z j(z, x(z))\, dz.$$

From theorem 2.7.5, p.83 of Clarke [10], we know that R_k is a locally Lipschitz functional.

Claim #1 R_k satisfies C-condition.

Let $\{x_n\}_{n\geq 1} \subseteq H_0^1(Z)$ be a sequence such that $|R_k(x_n)| \leq M_1$ for $n \geq 1$ and $(1 + \|x_n\|)m(x_n) \longrightarrow 0$ as $n \to +\infty$. We have to produce a strongly convergent subsequence. To this end let $x_n^* \in \partial R_k(x_n)$, for $n \geq 1$, be such that $m(x_n) = \|x_n^*\|_*$. Its existence follows from the fact that $\partial R_k(x_n)$ is weakly compact and the norm functional is weakly lower semicontinuous (so we can apply the theorem of Weierstrass and obtain such x_n^*). Let $A \in \mathcal{L}(H_0^1(Z), H^{-1}(Z))$ be the self-adjoint, monotone operator defined by

$$\langle Ax, y\rangle = \int_Z (\nabla x, \nabla y)_{\mathbb{R}^N}\, dz \qquad \text{for all } x, y \in H_0^1(Z).$$

Here by $\langle \cdot, \cdot \rangle$ we denote the duality brackets for the pair $(H_0^1(Z), H^{-1}(Z))$. For every $n \geq 1$, we have $x_n^* = Ax_n - \lambda_k x_n - u_n^*$, with $u_n^* \in \partial\psi(x_n)$, where $\psi : H_0^1(Z) \longmapsto \mathbb{R}$ is defined by $\psi(x) \stackrel{df}{=} \int_Z j(z, x(z))\, dz$. It is well known (see e.g. Clarke [10] Theorem 2.7.3, p.80 or Aubin-Clarke [5] Theorem 2) that $u_n^*(z) \in \partial j(z, x_n(z))$ for almost all $z \in Z$ and that $u_n^* \in L^2(Z)$ (see e.g. Chang [9], Theorem 2.2). From the choice of the sequence $\{x_n\}_{n\geq 1}$, we have $-2R_k(x_n) \leq 2M_1$, for $n \geq 1$, and so

$$-\|\nabla x_n\|_2^2 + \lambda_k\|x_n\|_2^2 + 2\int_Z j(z, x(z))\, dz \leq 2M_1. \tag{1}$$

Because $(1 + \|x_n\|)\|x_n^*\|_* = (1 + \|x_n\|)m(x_n) \longrightarrow 0$ as $n \to +\infty$, so also

$$\langle x_n^*, x_n\rangle \longrightarrow 0 \qquad \text{as } n \to +\infty, \tag{2}$$

and in particular the sequence $\{\langle x_n^*, x_n\rangle\}_{n\geq 1}$ is bounded. This implies that there exists $M_2 > 0$, such that

$$\langle Ax_n, x_n\rangle - \lambda_k\|x_n\|_2^2 - \int_Z u_n^*(z)x_n(z)\, dz \leq M_2,$$

and so

$$\|\nabla x_n\|_2^2 - \lambda_k \|x_n\|_2^2 - \int_Z u_n^*(z) x_n(z)\, dz \le M_2. \tag{3}$$

Adding (1) and (3), we obtain

$$\int_Z (2j(z, x_n(z)) - u_n^*(z) x_n(z))\, dz \le 2M_1 + M_2. \tag{4}$$

By virtue of hypothesis $H(j)_1$ (v), we know that there exists $c_2 > 0$, such that $\limsup_{|\zeta| \to +\infty} \frac{v(z,\zeta)\zeta - 2j(z,\zeta)}{|\zeta|^\mu} \le -2c_2$ (with $0 < \mu < 2$) uniformly for almost all $z \in Z$ and all $v(z,\zeta) \in \partial j(z,\zeta)$, with $v(\cdot,\zeta) \in L^2(Z)$. So we can find $M_3 = M_3(c_2) > 0$ such that for almost all $z \in Z$, all ζ such that $|\zeta| \ge M_3$ and all $v(z,\zeta) \in \partial j(z,\zeta)$, we have

$$\frac{v(z,\zeta)\zeta - 2j(z,\zeta)}{|\zeta|^\mu} \le -c_2 < 0,$$

and so

$$v(z,\zeta)\zeta - 2j(z,\zeta) \le -c_2 |\zeta|^\mu.$$

On the other hand, from the Lebourg mean value theorem (see e.g. Clarke [10], Theorem 2.3.7, p.41), for almost all $z \in Z$ and all $\zeta \in \mathbb{R}$, we have

$$|j(z,\zeta) - j(z,0)| \le |v_1(z,\xi)||\zeta|,$$

for some $v_1 \in \partial j(z,\xi)$ with $\xi = t\zeta$, $0 < t < 1$, and so using $H(j)_1$ (i), we get

$$|j(z,\zeta)| \le |j(z,0)| + a_1(z)|\zeta| + c_1|\zeta|^2. \tag{5}$$

Then, for almost all $z \in Z$ and all ζ such that $|\zeta| < M_3$, we have

$$|j(z,\zeta)| \le c_3,$$

where $c_3 \stackrel{df}{=} \|j(\cdot,0)\|_{L^\infty(Z)} + M_3 \|a_1\|_{L^\infty(Z)} + c_1 M_3^2$, and again from $H(j)_1$ (i), we have

$$|v(z,\zeta)\zeta| \le c_4,$$

where $c_4 \stackrel{df}{=} M_3 \|a_1\|_{L^\infty(Z)} + c_1 M_3^2$ (see hypotheses $H(j)_1$ (i) and $H(j)_1$ (ii)). Therefore, it follows that for almost all $z \in Z$, all $\zeta \in \mathbb{R}$ and all $v(z,\zeta) \in \partial j(z,\zeta)$, we have

$$v(z,\zeta)\zeta - 2j(z,\zeta) \le -c_2|\zeta|^\mu + c_5,$$

with $c_5 \stackrel{df}{=} c_4 + 2c_3 + c_2 M_3^{\mu}$. Using this inequality in (4), we obtain

$$\int_Z c_2 |x_n(z)|^{\mu}\, dz - c_5|Z| \leq 2M_1 + M_2,$$

so, for all $n \geq 1$, we also have

$$\|x_n\|_{\mu} \leq c_6, \tag{6}$$

with $c_6 \stackrel{df}{=} \left(\dfrac{2M_1 + M_2 + c_5|Z|}{c_2}\right)^{\frac{1}{\mu}}$. Let us choose q such that $2 < q < \min \left\{2^*, 2\dfrac{N+\mu}{N}\right\}$, where

$$2^* \stackrel{df}{=} \begin{cases} \dfrac{2N}{N-2} & \text{if } N > 2, \\ +\infty & \text{if } N = 2. \end{cases}$$

From (5) it follows, that for almost all $z \in Z$ and all $\zeta \in \mathbb{R}$, we have

$$j(z, \zeta) \leq c_7 + c_8 |\zeta|^q, \tag{7}$$

with $c_7 \stackrel{df}{=} \|j(\cdot, 0)\|_{L^{\infty}(Z)} + \|a_1\|_{L^{\infty}(Z)} + c_1$ and $c_8 \stackrel{df}{=} \|a_1\|_{L^{\infty}(Z)} + c_1$. Let

$$\vartheta \stackrel{df}{=} \begin{cases} \dfrac{2^*(q-\mu)}{q(2^*-\mu)} & \text{if } N > 2, \\ 1 - \dfrac{\mu}{q} & \text{if } N = 2. \end{cases}$$

Using the interpolation inequality (see e.g. Brezis [7], Remarque 2, p.57, and note that $0 < \vartheta < 1$ is chosen such that $\frac{1}{q}$ is the "convex combination" of $\frac{1}{\mu}$ and $\frac{1}{2^*}$, namely $\frac{1}{q} = \frac{1-\vartheta}{\mu} + \frac{\vartheta}{2^*}$), inequality (6) and the Sobolev embedding theorem, for $n \geq 1$, we have

$$\|x_n\|_q \leq \|x_n\|_{\mu}^{1-\vartheta} \|x_n\|_{2^*}^{\vartheta} \leq c_6^{1-\vartheta} \|x_n\|_{2^*}^{\vartheta} \leq c_9 \|x_n\|^{\vartheta}, \tag{8}$$

with some $c_9 > 0$. Then, as for all $n \geq 1$ we have $R_k(x_n) \leq M_1$, so using also (7) and (8), we have that

$$\begin{aligned} \frac{1}{2}\|\nabla x_n\|_2^2 &\leq \frac{\lambda_k}{2}\|x_n\|_2^2 + \int_Z j(z, x_n(z))\, dz + M_1 \\ &\leq \frac{\lambda_k}{2}\|x_n\|_2^2 + c_7|Z| + c_8\|x_n\|_q^q + M_1 \\ &\leq \frac{\lambda_k}{2}|Z|^{\frac{q-2}{q}}\|x_n\|_q^2 + c_8\|x_n\|_q^q + c_7|Z| + M_1 \\ &\leq c_{10}\|x_n\|_q^q + c_{11} \leq c_{10}c_9^q\|x_n\|^{\vartheta q} + c_{11}, \end{aligned}$$

where $c_{10} \stackrel{df}{=} \frac{\lambda_k}{2}|Z|^{\frac{q-2}{q}} + c_8$ and $c_{11} \stackrel{df}{=} \frac{\lambda_k}{2}|Z|^{\frac{q-2}{2}} + c_7|Z| + M_1$. Using the Poincaré inequality, we obtain

$$\frac{1}{2}\|\nabla x_n\|_2^2 \leq c_{12}\|\nabla x_n\|_2^{\vartheta q} + c_{11}, \tag{9}$$

with some $c_{12} > 0$ depending on λ_k. Let us calculate ϑq. In case $N > 2$, from the choice of q, we have $2\mu + 2N > Nq$, so

$$\vartheta q = 2^* \frac{q-\mu}{2^*-\mu} = \frac{2N}{N-2} \cdot \frac{(q-\mu)(N-2)}{2N+2\mu-\mu N} < \frac{2N}{N-2} \cdot \frac{(q-\mu)(N-2)}{Nq-N\mu} = 2.$$

In case $N = 2$, we have $2^* = +\infty$ and as $\mu < 2 < q$, so

$$\vartheta q = \left(1 - \frac{\mu}{q}\right) q = q - \mu < 2.$$

Thus, we always have $\vartheta q < 2$. Therefore from (9), we infer that $\{x_n\}_{n\geq 1} \subseteq H_0^1(Z)$ is bounded. By passing to a subsequence if necessary and using compactness of the embedding $H_0^1(Z) \subseteq L^2(Z)$, we may assume that

$$\begin{aligned} x_n &\longrightarrow x \text{ weakly in } H_0^1(Z), \\ x_n &\longrightarrow x \text{ in } L^2(Z), \\ x_n(z) &\longrightarrow x(z) \quad \text{a.e on } Z \text{ as } n \to +\infty, \end{aligned}$$

and $|x_n(z)| \leq h(z)$ a.e. on Z for all $n \geq 1$ with $h \in L^2(Z)$ (see e.g. Brezis [7], Theorem IV.9, p.58). Then we have $\lambda_k x_n \longrightarrow \lambda_k x$ in $L^2(Z)$. Also $u_n^* \in \partial\psi(x_n)$, for $n \geq 1$, and from Theorem 2.2 of Chang [9], we know that $\partial\psi(x_n) \subseteq L^2(Z)$. Moreover, by virtue of hypothesis $H(j)_1$ (i), we have that $\{u_n^*\}_{n\geq 1} \subseteq L^2(Z)$ is bounded. Then, if by $(\cdot,\cdot)_2$ we denote the inner product in $L^2(Z)$, we have

$$\langle x_n^*, x_n - x\rangle = \langle Ax_n, x_n - x\rangle - \lambda_k(x_n, x_n - x)_2 - (u_n^*, x_n - x)_2$$

so, using also (2), we get

$$\limsup_{n\to+\infty} \langle Ax_n, x_n - x\rangle \leq 0.$$

From the monotonicity of $A \in \mathcal{L}(H_0^1(Z), H^{-1}(Z))$, we have

$$\langle Ax_n, x_n\rangle \longrightarrow \langle Ax, x\rangle \qquad \text{as } n \to +\infty,$$

so

$$\|\nabla x_n\|_2 \longrightarrow \|\nabla x\|_2 \qquad \text{as } n \to +\infty.$$

But we also have $\nabla x_n \longrightarrow \nabla x_n$ weakly in $L^2(Z, \mathbb{R}^N)$ as $n \to +\infty$. Thus, from Kadeck-Klee property, we infer that $\nabla x_n \longrightarrow \nabla x$ in $L^2(Z, \mathbb{R}^N)$ as $n \to +\infty$. So finally $x_n \longrightarrow x$ in $H_0^1(Z)$ as $n \to +\infty$. This proves Claim #1.

Recall that $V_k \stackrel{df}{=} \operatorname{span}\{w_i\}_{i=1}^k$, where $\{w_i\}_{i\geq 1}$ are the eigenfunctions corresponding to the eigenvalues $\{\lambda_i\}_{i\geq 0}$ of $(-\Delta, H_0^1(Z))$. So $\dim V_k = k$.

Claim #2 $R_k(x) \longrightarrow -\infty$ as $\|x\| \to +\infty$ and $x \in V_k$.

From hypothesis $H(j)_1$ (iii) it follows that there exists $c_{13} > 0$, such that $\liminf\limits_{|\zeta|\to+\infty} \dfrac{2j(z,\zeta)}{\zeta^2} > 4c_{13}$, uniformly for almost all $z \in Z$. So we can find $M_4 > 0$ such that, for almost all $z \in Z$ and all ζ such that $|\zeta| \geq M_4$, we have

$$j(z,\zeta) \geq c_{13}\zeta^2.$$

On the other hand, from (5), we see that for almost all $z \in Z$ and all ζ such that $|\zeta| \leq M_4$, we have

$$j(z,\zeta) \geq -c_{14},$$

with $c_{14} \stackrel{df}{=} \|j(\cdot,0)\|_{L^\infty(Z)} + M_4\|a_1\|_{L^\infty(Z)} + c_1 M_4^2$. So for almost all $z \in Z$ and all $\zeta \in \mathbb{R}$, we can write that

$$j(z,\zeta) \geq c_{13}|\zeta|^2 - c_{15}, \tag{10}$$

with $c_{15} \stackrel{df}{=} c_{14} + c_{13}M_4^2$. For $x \in V_k$, we have that $\|\nabla x\|_2^2 \leq \lambda_k \|x\|_2^2$ (see e.g. Kesavan [17], Theorem 3.6.2, p.149). So using also (10), for $x \in V_k$, we have

$$R_k(x) = \frac{1}{2}\|\nabla x\|_2^2 - \frac{\lambda_k}{2}\|x\|_2^2 - \int_Z j(z,x(z))\,dz \leq -c_{13}\|x\|_2^2 + c_{15}|Z|.$$

Thus finally $R_k(x) \longrightarrow -\infty$ as $\|x\| \to +\infty$ for $x \in V_k$ (recall that V_k is finite dimensional, so all norms of V_k are equivalent). This proves Claim #2.

Claim #3 There exists $r > 0$ such that $\inf\{R_k(x) : x \in \partial B_r(0)\} > 0$.

By virtue of hypothesis $H(j)_1$ (iv), we can find $\delta > 0$ such that for almost all $z \in Z$ and all $|\zeta| \leq \delta$, we have $j(z,\zeta) \leq \left(-\lambda_k + \dfrac{\lambda_1}{2}\right)\dfrac{\zeta^2}{2}$. On the other hand, form (5) we see that for almost all $z \in Z$ and all $|\zeta| \geq \delta$, we have

$$j(z,\zeta) \leq c_{16}|\zeta|^\eta$$

with $c_{16} = \left(\|j(\cdot,0)\|_{L^\infty(Z)} + \frac{1}{4}\|a_1\|_{L^\infty(Z)} + (\|a_1\|_{L^\infty(Z)} + c_1)\delta^2\right)\delta^{-\eta}$ and $2 < \eta \leq 2^*$. Thus finally for almost all $z \in Z$ and all $\zeta \in \mathbb{R}$, we have

$$j(z,\zeta) \leq \left(-\lambda_k + \frac{\lambda_1}{2}\right)\frac{\zeta^2}{2} + c_{17}|\zeta|^\eta,$$

with $c_{17} = c_{16} + \left(\lambda_k - \frac{\lambda_1}{2}\right)\frac{\delta^{2-\eta}}{2}$. Using the fact that $\|\nabla x\|_2^2 \geq \lambda_1\|x\|_2^2$ for all $x \in H_0^1(Z)$ (see e.g. Kesavan [17], Theorem 3.6.2, p.149) and the Sobolev embedding theorem, for all $x \in H_0^1(Z)$, we have

$$\begin{aligned} R_k(x) &= \frac{1}{2}\|\nabla x\|_2^2 - \frac{\lambda_k}{2}\|x\|_2^2 - \int_Z j(z,x(z))\,dz \\ &\geq \frac{1}{2}\|\nabla x\|_2^2 - \frac{\lambda_k}{2}\|x\|_2^2 + \left(\frac{\lambda_k}{2} - \frac{\lambda_1}{4}\right)\|x\|_2^2 - c_{17}\|x\|_\eta^\eta \\ &\geq \frac{1}{4}\|\nabla x\|_2^2 - c_{17}\|x\|_\eta^\eta \geq \frac{1}{4}\|\nabla x\|_2^2 - c_{17}\|x\|^\eta. \end{aligned}$$

Since $\|\nabla x\|_2$ is an equivalent norm on $H_0^1(Z)$, we see that for all $x \in H_0^1(Z)$, we have

$$R_k(x) \geq c_{18}\|x\|^2 - c_{17}\|x\|^\eta$$

with some $c_{18} > 0$. Since $2 < \eta$, from the last inequality, we see that choosing $0 < r < \left(\frac{c_{18}}{c_{17}}\right)^{\frac{1}{\eta-2}}$, we will have $\inf\{R_k(x) : x \in \partial B_r(0)\} > 0$. This proves Claim #3.

Now since R_k is even and because of Claims #1, #2 and #3, we can apply Theorem 3, with $V = V_k$ ($\dim V_k = k$) and $Y = H_0^1(Z)$ (codim$Y = 0$) and deduce that R_k has k pairs $\{\pm x_i\}_{i=1}^k$ of nontrivial critical points. It is easy to see that these are solutions of (RHI_k). So problem (RHI_k) has k pairs of nontrivial solutions. Q.E.D.

We can have another such a multiplicity result, under a new set of hypotheses that involve a Landesman-Lazer type condition (see hypothesis $H(j)_2$ (iv)). Our new set of hypotheses on the integrand j is the following

$H(j)_2$ $j : Z \times \mathbb{R} \longmapsto \mathbb{R}$ is an even locally Lipschitz integrand (see $H(j)_1$), such that:

(i) for almost all $z \in Z$, all $\zeta \in \mathbb{R}$ and all $v(z,\zeta) \in \partial j(z,\zeta)$, we have $|v(z,\zeta)| \leq a(z)$ with $a \in L^\infty(Z)$;

(ii) $j(\cdot,0) \in L^\infty$ and $\partial j(z,0) = \{0\}$ for almost all $z \in Z$;

(iii) there exist $v_-, v_+ : Z \longmapsto \mathbb{R}$ measurable functions, such that for almost all $z \in Z$, we have $v(z,\zeta) \longrightarrow v_-(z)$ as $\zeta \to -\infty$ and $v(z,\zeta) \longrightarrow v_+(z)$ as $\zeta \to +\infty$, uniformly for all $v(z,\zeta) \in \partial j(z,\zeta)$ with $v(\cdot,\zeta) \in L^2(Z)$;

(iv) $\displaystyle\int_Z \left(v_-(z)u_k^-(z) - v_+(z)u_k^+(z)\right) dz \neq 0$ for all eigenfunctions u_k corresponding to the eigenvalue λ_k;

(v) $\displaystyle\limsup_{\zeta\to 0} \frac{2j(z,\zeta)}{\zeta^2} \leq -\lambda_k$ uniformly for almost all $z \in Z$.

THEOREM 6. **If** *hypotheses* $H(j)_2$ *hold and* $k \geq 2$,
then *problem* (RHI_k) *has at least* $k-1$ *pairs* $\{\pm x_i\}_{i=1}^{k-1}$ *of nontrivial solutions.*

Proof. As before let $R_k : H_0^1(Z) \longmapsto \mathbb{R}$ be the energy functional defined by

$$R_k(x) \overset{df}{=} \frac{1}{2}\|\nabla x\|_2^2 - \frac{\lambda_k}{2}\|x\|_2^2 - \psi(x),$$

where $\psi : H_0^1(Z) \ni x \longmapsto \displaystyle\int_Z j(z,x(z))\, dz \in \mathbb{R}$. We know that R_k is locally Lipschitz.

Claim #1 R_k satisfies the nonsmooth (PS)-condition.

Let $\{x_n\}_{n\geq 1} \subseteq H_0^1(Z)$ be such that $|R_k(x_n)| \leq M_1$ for all $n \geq 1$ and let $m(x_n) \longrightarrow 0$ as $n \to +\infty$. We will show that $\{x_n\}_{n\geq 1} \subseteq H_0^1(Z)$ is bounded. Suppose this is not true. Then, by passing to a subsequence if necessary, we may assume that $\|x_n\| \longrightarrow +\infty$ as $n \to +\infty$. Let us set $y_n = \dfrac{x_n}{\|x_n\|}$ for $n \geq 1$. Then $\|y_n\| = 1$ and so we may assume that

$$\begin{aligned} y_n &\longrightarrow y \text{ weakly in } H_0^1(Z),\\ y_n &\longrightarrow y \text{ in } L^2(Z),\\ y_n(z) &\longrightarrow y(z) \quad \text{a.e on } Z \text{ as } n \to +\infty, \end{aligned}$$

and $|y_n(z)| \leq h(z)$ a.e. on Z with $h \in L^2(Z)$. We know that we can find $x_n^* \in \partial R_k(x_n)$ such that $\|x_n^*\|_* = m(x_n)$, for $n \geq 1$, and

$$x_n^* = Ax_n - \lambda_k x_n - u_n^*,$$

where $A \in \mathcal{L}\left(H_0^1(Z), H^{-1}(Z)\right)$ is defined by $\langle Ax, y\rangle = \displaystyle\int_Z (\nabla x, \nabla y)_{\mathbb{R}^N}\, dz$ for all $x, y \in H_0^1(Z)$ and $u_n^* \in \partial\psi(x_n)$. Dividing the last equality by $\|x_n\|$, we obtain

$$Ay_n - \lambda_k y_n - \frac{u_n^*}{\|x_n\|} = \frac{x_n^*}{\|x_n\|}. \tag{11}$$

Since $u_n^* \in \partial\psi(x_n)$, we have $u_n^*(z) \in \partial j(z, x_n(z))$ for almost all $z \in Z$ (see Clarke [10], Theorem 2.7.5, p.83) and $u_n^* \in L^2(Z)$ (see Chang [9], Theorem 2.2). From hypothesis $H(j)_2$ (i), we get $\|u_n^*\|_2 \le \|a\|_2$, so the sequence $\{u_n^*\}_{n\ge1}$ is also bounded in $H^{-1}(Z)$. Also from the choice of the sequence $\{x_n\}_{n\ge1}$, we have that $x_n^* \longrightarrow 0$ in $H^{-1}(Z)$ as $n \to +\infty$. Hence, by passing to the limit in (11), as $n \to +\infty$, we obtain

$$Ay = \lambda_k y,$$

and so

$$\begin{cases} -\Delta x(z) - \lambda_k y(z) = 0 \text{ a.e. on } Z \\ y|_\Gamma = 0. \end{cases} \tag{12}$$

Now, we will show that $y \neq 0$. Suppose this is not true. Then, using the Poincaré inequality, we have

$$\begin{aligned} \frac{R_k(x_n)}{\|x_n\|^2} &= \frac{1}{2}\|\nabla y_n\|_2^2 - \frac{\lambda_k}{2}\|y_n\|_2^2 - \int_Z \frac{j(z, x_n(z))}{\|x_n\|^2}\, dz \\ &\ge c_1\|y_n\|^2 - \frac{\lambda_k}{2}\|y_n\|_2^2 - \int_Z \frac{j(z, x_n(z))}{\|x_n\|^2}\, dz \\ &= c_1 - \frac{\lambda_k}{2}\|y_n\|_2^2 - \int_Z \frac{j(z, x_n(z))}{\|x_n\|^2}\, dz, \end{aligned}$$

with some $c_1 > 0$. Note that $\dfrac{R_k(x_n)}{\|x_n\|^2} \longrightarrow 0$ as $n \to +\infty$. Also, as before, via the Lebourg mean value theorem and using hypothesis $H(j)_2$ (i), we can check that $\displaystyle\int_Z \frac{j(z, x_n(z))}{\|x_n\|^2}\, dz \longrightarrow 0$ as $n \to +\infty$. But we also have $\|y_n\|_2^2 \longrightarrow \|y\|_2^2 = 0$ as $n \to +\infty$. So, passing to the limit in last inequality, we obtain $c_1 \le 0$ and we reach a contradiction to the fact that $c_1 > 0$. Therefore $y \neq 0$ and this combined with (12) implies that y is an eigenfunction corresponding to the eigenvalue λ_k. We have

$$\begin{aligned} x_n &\longrightarrow +\infty \text{a.e. on } \{y > 0\} \text{ and} \\ x_n &\longrightarrow -\infty \text{a.e. on } \{y < 0\}. \end{aligned}$$

From the choice of the sequence $\{x_n\}_{n\ge1}$, for $n \ge 1$, we have

$$|R_k(x_n)| \le M_1$$

and, by passing to a subsequence if necessary, we have

$$|\langle x_n^*, u\rangle| \le \varepsilon_n\|u\|,$$

for all $u \in H_0^1(Z)$ with $\varepsilon_n \searrow 0$. Putting $u = x_n \in H_0^1(Z)$ in the last two inequalities we have

$$-2M_1 \leq \|\nabla x_n\|_2^2 - \lambda_k \|x_n\|_2^2 - 2\int_Z j(z, x_n(z))\, dz \leq 2M_1$$

and

$$-\varepsilon_n \|x_n\| \leq -\|\nabla x_n\|_2^2 + \lambda_k \|x_n\|_2^2 + \int_Z u_n^*(z) x_n(z)\, dz \leq \varepsilon_n \|x_n\|.$$

Adding the last two inequalities, we obtain

$$-2M_1 - \varepsilon \|x_n\| \leq \int_Z (u_n^*(z) x_n(z) - 2j(z, x_n(z)))\, dz \leq 2M_1 + \varepsilon_n \|x_n\|.$$

Dividing by $\|x_n\|$, we have

$$-\frac{2M_1}{\|x_n\|} - \varepsilon_n \leq \int_Z \left(u_n^*(z) y_n(z) - \frac{2j(z, x_n(z))}{\|x_n\|} \right) dz \leq \frac{2M_1}{\|x_n\|} + \varepsilon_n. \tag{13}$$

By virtue of hypothesis $H(j)_2$ (iii) and (i), we have

$$\begin{array}{lcll} u_n^*(z) y_n(z) & \longrightarrow & v_+(z) y(z) & \text{a.e. on } \{y > 0\}, \\ u_n^*(z) y_n(z) & \longrightarrow & v_-(z) y(z) & \text{a.e. on } \{y < 0\}, \\ u_n^*(z) y_n(z) & \longrightarrow & 0 & \text{a.e. on } \{y = 0\}. \end{array}$$

Next, let N be the Lebesgue-null subset of $Z_1 \stackrel{df}{=} \{y \neq 0\}$, outside of which we have $x_n \longrightarrow \pm\infty$ and $u_n^* \longrightarrow v_\pm$ as $n \to +\infty$. Fix $z \in Z_1 \setminus N$ and assume $x_n(z) \longrightarrow +\infty$, $u_n^*(z) \longrightarrow v_+(z)$ as $n \to +\infty$ (the analysis of the other case is similar). For a given $0 < \varepsilon < 1$, via the Lebourg mean value theorem, we have

$$j(z, x_n(z)) = j(z, \varepsilon x_n(z)) + v_n(z)(1 - \varepsilon) x_n(z),$$

where $v_n(z) \in \partial j(z, w_n(z))$ and $w_n(z) = (1 - t_n) x_n(z) + t_n \varepsilon x_n(z)$, for $0 < t_n < 1$ and $n \geq 1$. Note that for $n \geq 1$ large enough, we have $x_n(z) > 0$ and so $w_n(z) = x_n(z) - t_n(1 - \varepsilon) x_n(z) \geq x_n(z) - (1 - \varepsilon) x_n(z) = \varepsilon x_n(z)$. Therefore $w_n(z) \longrightarrow +\infty$ as $n \to +\infty$ and so, by virtue of hypothesis $H(j)_2$ (iii), we have $v_n(z) \longrightarrow v_+(z)$ as $n \to +\infty$. Let $n_0 = n_0(\varepsilon, z) \geq 1$ be such that, for $n \geq n_0$, we have $x_n(z) > 0$ and $|v_n(z) - v_+(z)| < \varepsilon$. So for $n \geq n_0$, we have

$$\frac{2j(z, x_n(z))}{x_n(z)} = \frac{2j(z, \varepsilon x_n(z))}{x_n(z)} + \frac{2v_n(z)(1 - \varepsilon) x_n(z)}{x_n(z)}.$$

From the Lebourg mean value theorem, we also get $|j(z, \varepsilon x_n(z))| \leq c_2 + c_3|x_n(z)|$ for some $c_2, c_3 > 0$ (see hypotheses $H(j)_2$ (i) and (ii)). Since for $n \geq n_0$ we have $-\varepsilon + v_+(z) \leq v_n(z) \leq \varepsilon + v_+(z)$ and $x_n(z) > 0$, so we can write

$$\frac{-2c_2 - 2c_3\varepsilon x_n(z)}{x_n(z)} + \frac{2(-\varepsilon + v_+(z))(1-\varepsilon)x_n(z)}{x_n(z)} \leq \frac{2j(z, x_n(z))}{x_n(z)}$$
$$\leq \frac{2c_2 + 2c_3\varepsilon x_n(z)}{x_n(z)} + \frac{2(\varepsilon + v_+(z))(1-\varepsilon)x_n(z)}{x_n(z)}.$$

Since $x_n(z) \longrightarrow +\infty$ as $n \to +\infty$ and $0 < \varepsilon < 1$ was arbitrary, we infer that

$$\frac{2j(z, x_n(z))}{x_n(z)} \longrightarrow 2v_+(z) \qquad \text{a.e. on } \{y > 0\}.$$

As we already mentioned, in a similar way we can show that

$$\frac{2j(z, x_n(z))}{x_n(z)} \longrightarrow 2v_-(z) \qquad \text{a.e. on } \{y < 0\}.$$

Finally for almost all $z \in \{y = 0\}$ we have

$$\left|\frac{2j(z, x_n(z))}{\|x_n\|}\right| \leq \frac{2c_2 + c_3|x_n(z)|}{\|x_n\|} \longrightarrow 0, \qquad \text{as } n \to +\infty.$$

Therefore, by passing to the limit in (13), we obtain

$$\int_Z \big[v_+(z)\chi_{\{y>0\}}(z) + v_-(z)\chi_{\{y<0\}}(z)$$
$$-2v_+(z)\chi_{\{y>0\}}(z) - 2v_-(z)\chi_{\{y<0\}}(z)\big]\, y(z)\, dz = 0$$

so

$$\int_Z \big(v_+(z)\chi_{\{y>0\}}(z) + v_-(z)\chi_{\{y<0\}}(z)\big)\, y(z)\, dz = 0$$

and

$$\int_Z \big(v_+(z)y^+(z) + v_-(z)y^-(z)\big)\, dz = 0,$$

so we get a contradiction to hypothesis $H(j)_2$ (iv). Thus $\{x_n\}_{n\geq 1}$ is bounded in $H_0^1(Z)$ and so we may assume that $x_n \longrightarrow x$ weakly in $H_0^1(Z)$ as $n \to +\infty$. Using the Kadec-Klee property and proceeding as in the proof of Theorem 5 (see Claim #1), we can have that $x_n \longrightarrow x$ in $H_0^1(Z)$. This proves Claim #1.

Claim #2 $R_k(x) \longrightarrow -\infty$ as $\|x\| \to +\infty$ and $x \in V_{k-1}$.

For every $x \in V_{k-1}$, we know that $\|\nabla x\|_2^2 \leq \lambda_{k-1}\|x\|_2^2$ and so we have

$$\begin{aligned} R_k(x) &= \frac{1}{2}\|\nabla x\|_2^2 - \frac{\lambda_k}{2}\|x\|_2^2 - \int_Z j(z,x(z))\,dz \\ &\leq \frac{1}{2}\left(1 - \frac{\lambda_k}{\lambda_{k-1}}\right)\|\nabla x\|_2^2 + c_4\|\nabla x\|_2 + c_5, \end{aligned}$$

for some $c_4, c_5 > 0$ (see hypothesis $H(j)_2$ (i)). Since $1 - \dfrac{\lambda_k}{\lambda_{k-1}} < 0$, we deduce that $R_k(x) \longrightarrow -\infty$ as $\|x\| \to +\infty$ for $x \in V_{k-1}$. This proves the Claim #2.

Claim #3 There exists $r > 0$ such that $\inf\{R_k(x) : x \in \partial B_r(0)\} > 0$.

This follows from hypothesis $H(j)_2$ (v) as in the proof of Theorem 4 (see Claim #3).

Since R_k is even, claims #1, #2 and #3 permit the use of Theorem 3 with $V = V_{k-1}$ ($\dim V = k-1$) and $Y = H_0^1(Z)$ ($\operatorname{codim} Y = 0$), which gives us $k-1$ pairs $\{\pm x_i\}_{i=1}^{k-1}$ of nontrivial critical points of R_k. We can easily check that these functions solve (RHI_k). Q.E.D.

4. Resonant problems at λ_1

In this section we consider semilinear hemivariational inequalities at resonance at $\lambda_1 > 0$. So we deal with the following problem

$$(RHI_1) \qquad \begin{cases} -\Delta x(z) - \lambda_1 x(z) \in \partial j(z, x(z)) \text{ a.e. on } Z, \\ x|_\Gamma = 0. \end{cases}$$

We will show that problem (RHI_1) has at least three nontrivial solutions, when we assume that the potential $j(z,\zeta)$ has a finite limit for almost all $z \in Z$ as $\zeta \to \pm\infty$. Such problems were called by Bartolo-Benci-Fortunato "strongly resonant" (see [6]). Besides Bartolo-Benci-Fortunato, such "smooth" strongly resonant problems were also studied by Thews [25] and Ward [26]. In all these papers we have existence but no multiplicity results.

Our hypotheses on j are the following

$H(j)_3$ $j : Z \times \mathbb{R} \longmapsto \mathbb{R}$ is a locally Lipschitz integrand, such that:

(i) for almost all $z \in Z$, all $\zeta \in \mathbb{R}$ and all $v(z,\zeta) \in \partial j(z,\zeta)$, we have $|v(z,\zeta)| \leq a(z)$ with $a \in L^\infty(Z)$;

(ii) $j(\cdot, 0) \in L^\infty(Z)$ and $\displaystyle\int_Z j(z,0)\,dz \geq 0$;

(iii) $j(z,\zeta) \longrightarrow j_\pm(z)$ as $\zeta \to \pm\infty$ uniformly for almost all $z \in Z$, $j_\pm \in L^\infty(Z)$, $\int_Z j_\pm(z)\,dz > 0$ and $v(z,\zeta) \longrightarrow 0$ as $|\zeta| \to +\infty$, for almost all $z \in Z$ and all $v(z,\zeta) \in \partial j(z,\zeta)$, with $v(\cdot,\zeta) \in L^2(Z)$;

(iv) there exist $\vartheta_- < 0 < \vartheta_+$ such that $\int_Z j(z,\vartheta_\pm w_1(z))\,dz > \int_Z j_\pm(z)\,dz$ (here w_1 is the first eigenfunction corresponding to the first eigenvalue $\lambda_1 > 0$; recall $w_1(z) > 0$ for all $z \in Z$), and there exists $\vartheta \neq 0$, such that $\int_Z j(z,\vartheta w_1(z))\,dz \leq 0$;

(v) there exists $\mu > \lambda_1$ such that $\limsup\limits_{\zeta\to 0} \dfrac{2j(z,\zeta)}{\zeta^2} < -\mu$ uniformly for almost all $z \in Z$;

(vi) for almost all $z \in Z$ and all $\zeta \in \mathbb{R}$, we have $2j(z,\zeta) \leq (\lambda_2 - \lambda_1)\zeta^2$.

THEOREM 7. If *hypotheses $H(j)_3$ hold,*
then *problem (RHI_1) has at least three nontrivial solutions.*

Proof. We introduce the energy functional $R_1 : H_0^1(Z) \longmapsto \mathbb{R}$ defined by

$$R_1(x) = \frac{1}{2}\|\nabla x\|_2^2 - \frac{\lambda_1}{2}\|x\|_2^2 - \int_Z j(z,x(z))\,dz.$$

Claim #1 R_1 is bounded below.

By virtue of hypothesis $H(j)_3$ (iii), we can find $M_1 > 0$ such that for almost all $z \in Z$, we have

$$|j(z,\zeta) - j_-(z)| \leq 1 \text{for all } \zeta \leq -M_1 \text{ and}$$
$$|j(z,\zeta) - j_+(z)| \leq 1 \text{for all } \zeta \geq M_1.$$

Also from hypotheses $H(j)_3$ (i) and (ii) and the Lebourg mean value theorem, for almost all $z \in Z$ and all ζ such that $|\zeta| < M_1$, we get that $|j(z,\zeta)| \leq a_1(z)$, where $a_1 \in L^\infty(Z)$, namely $a_1(z) = M_1 a(z) + \|j(\cdot,0)\|_{L^\infty(Z)}$. Then for all $x \in H_0^1(Z)$, we have

$$\begin{aligned} R_1(x) &= \frac{1}{2}\|\nabla x\|_2^2 - \frac{\lambda_1}{2}\|x\|_2^2 - \int_{\{|x|<M_1\}} j(z,x(z))\,dz \\ &\quad - \int_{\{x\leq -M_1\}} j(z,x(z))\,dz - \int_{\{x\geq M_1\}} j(z,x(z))\,dz \\ &\geq -\|a_1\|_1 - \|j_+\|_1 - \|j_-\|_1 - 2|Z| \end{aligned}$$

(recall $\|\nabla x\|_2^2 \geq \lambda_1\|x\|_2^2$ for all $x \in H_0^1(Z)$). This proves Claim #1.

Next consider the following splitting for $H_0^1(Z)$. Let $H_0^1(Z) = V_{w_1} \oplus Y_{w_1}$, with $V_{w_1} = \operatorname{span}\{w_1\}$ and $Y_{w_1} = V_{w_1}^\perp$.

Claim #2 $R_1(v) \geq 0$ for all $v \in Y_{w_1}$.

Let $v \in Y_{w_1}$. Using hypothesis $H(j)_3$ (vi), we obtain

$$\begin{aligned} R_1(v) &= \frac{1}{2}\|\nabla v\|_2^2 - \frac{\lambda_1}{2}\|v\|_2^2 - \int_Z j(z, v(z))\, dz \\ &\geq \frac{1}{2}\|\nabla v\|_2^2 - \frac{\lambda_1}{2}\|v\|_2^2 - \frac{1}{2}(\lambda_2 - \lambda_1)\|v\|_2^2 \\ &\geq \frac{1}{2}\|\nabla v\|_2^2 - \frac{\lambda_1}{2}\|v\|_2^2 - \frac{1}{2}\|\nabla v\|_2^2 + \frac{\lambda_1}{2}\|v\|_2^2 = 0, \end{aligned}$$

(recall that $\|\nabla v\|_2^2 \geq \lambda_2 \|v\|_2^2$ for all $v \in Y_{w_1}$), which proves the Claim #2.

Claim #3 R_1 satisfies the nonsmooth (PS)-condition at level $c \neq -\displaystyle\int_Z j_{\pm}(z)\, dz$.

Let $\{x_n\}_{n\geq 1} \subseteq H_0^1(Z)$ be a sequence such that $R_1(x_n) \longrightarrow c$ with $c \neq -\displaystyle\int_Z j_{\pm}(z)\, dz$ and $m(x_n) \longrightarrow 0$ as $n \to +\infty$. We will show that $\{x_n\}_{n\geq 1}$ is bounded in $H_0^1(Z)$. Suppose that it is not true. Then, passing to a subsequence if necessary, we may assume that $\|x_n\| \longrightarrow +\infty$ as $n \to +\infty$. Let us set $y_n = \dfrac{x_n}{\|x_n\|}$ for $n \geq 1$. By passing to a subsequence if necessary, we may assume that

$$\begin{aligned} y_n &\longrightarrow y && \text{weakly in } H_0^1(Z), \\ y_n &\longrightarrow y && \text{in } L^2(Z), \\ y_n(z) &\longrightarrow y(z) && \text{a.e on } Z \text{ as } n \to +\infty, \end{aligned}$$

and $|y_n(z)| \leq h(z)$ a.e. on Z with $h \in L^2(Z)$. From the choice of the sequence $\{x_n\}_{n\geq 1}$, we know that there exists $M_2 > 0$, such that

$$|R_1(x_n)| \leq M_2,$$

so

$$-M_2 \leq \frac{1}{2}\|\nabla x_n\|_2^2 - \frac{\lambda_1}{2}\|x_n\|_2^2 - \int_Z j(z, x_n(z))\, dz \leq M_2$$

and so

$$-\frac{M_2}{\|x_n\|^2} \leq \frac{1}{2}\|\nabla y_n\|_2^2 - \frac{\lambda_1}{2}\|y_n\|_2^2 - \int_Z \frac{j(z, x_n(z))}{\|x_n\|^2}\, dz \leq \frac{M_2}{\|x_n\|^2}. \tag{14}$$

Using hypothesis $H(j)_3$ (i) and the Lebourg mean value theorem, it follows that $\displaystyle\int_Z \frac{j(z, x_n(z))}{\|x_n\|^2}\, dz \longrightarrow 0$ as $n \to +\infty$. Passing to the limit in (14) as $n \to +\infty$, we obtain

$$\lim_{n\to+\infty} \|\nabla y_n\|_2^2 = \lambda_1 \|y\|_2^2, \tag{15}$$

so from the weak lower semicontinuity of the norm functional, we get

$$\|\nabla y\|_2^2 \leq \liminf_{n\to+\infty} \|\nabla y_n\|_2^2 = \lambda_1 \|y\|_2^2.$$

But from the Rayleigh quotient, we know that $\|\nabla y\|_2^2 \geq \lambda_1\|y\|_2^2$, so finally, we have that

$$\|\nabla y\|_2^2 = \lambda_1 \|y\|_2^2. \tag{16}$$

Moreover, from (15) and (16), we get that $\|\nabla y_n\|_2 \longrightarrow \|\nabla y\|_2$, so, using Kadec-Klee property, we also have $y_n \longrightarrow y$ in $H_0^1(Z)$ as $n \to +\infty$, and so $y \neq 0$. Thus, from (16), we deduce that $y = \pm w_1$. Without any loss of generality we may assume that $y = w_1$ (the analysis is similar if $y = -w_1$). Since $w_1(z) > 0$ for all $z \in Z$, we have that $x_n(z) \longrightarrow +\infty$, for all $z \in Z$, as $n \to +\infty$. For $n \geq 1$, let $x_n^* \in \partial R_1(x_n)$ be such that $m(x_n) = \|x_n^*\|_*$. We know that $x_n^* = Ax_n - \lambda_1 x_n - u_n^*$ with $A \in \mathcal{L}(H_0^1(Z), H^{-1}(Z))$ being defined by $\langle Ax, y\rangle = \int_Z (\nabla x(z), \nabla y(z))_{\mathbb{R}^N}\, dz$ for $x, y \in H_0^1(Z)$ and $u_n^* \in \partial\psi(x_n)$, where in this case $\psi : H_0^1(Z) \longmapsto \mathbb{R}$ is defined by $\psi(x) = \int_Z j(z, x(x))\, dz$. So $u_n^*(z) \in \partial j(z, x_n(z))$ a.e. on Z. In particular we have that

$$|\langle x_n^*, v\rangle| \leq \varepsilon_n\|v\| \quad \forall v \in H_0^1(Z) \text{ with } \varepsilon_n \searrow 0,$$

so

$$\left|\langle Ax_n, v\rangle - \lambda_1(x_n, v) - \int_Z u_n^*(z)v(z)\, dz\right| \leq \varepsilon_n\|v\|$$

for all $v \in H_0^1(Z)$, with $\varepsilon_n \searrow 0$. Let $x_n = t_n w_1 + v_n$ with $t_n \in \mathbb{R}$ (i.e. $t_n w_1 \in V_{w_1}$) and $v_n \in V_{w_1}^{\perp} = Y_{w_1}$. Taking $v = v_n$, we have

$$\|\nabla v_n\|_2^2 - \lambda_1\|v_n\|_2^2 - \int_Z u_n^*(z)v_n(z)\, dz \leq \varepsilon_n\|v_n\|,$$

so

$$\left(1 - \frac{\lambda_1}{\lambda_2}\right)\|\nabla v_n\|_2^2 - c_3\|u_n^*\|_\infty\|\nabla v_n\|_2 \leq \varepsilon_n'\|\nabla v_n\|_2$$

for some $c_3 > 0$ and with $\varepsilon_n' \searrow 0$ (recall that $\|\nabla v\|_2^2 \geq \lambda_2\|v\|_2^2$ for all $v \in Y_{w_1}$).

Note that by virtue of hypothesis $H(j)_3$ (iii) and the fact that $x_n(z) \longrightarrow +\infty$ as $n \to +\infty$ for all $z \in Z$, we have $u_n^*(z) \longrightarrow 0$ for almost all $z \in Z$ as

$n \to +\infty$. This, together with hypothesis $H(j)_3$ (i) and the Lebesgue dominated convergence theorem, implies that $\|u_n^*\|_\infty \longrightarrow 0$ as $n \to +\infty$. Thus we have

$$\left(1 - \frac{\lambda_1}{\lambda_2}\right) \|\nabla v_n\|_2 \leq \varepsilon_n'',$$

with $\varepsilon_n'' \searrow 0$ (namely $\varepsilon_n'' = \varepsilon_n' + c_3\|u_n^*\|_\infty$). So we have that $v_n \longrightarrow 0$ in $H_0^1(Z)$ as $n \to +\infty$. Using this convergence, we see that for a given $\varepsilon > 0$, we can find $n_0 = n_0(\varepsilon)$ such that for $n \geq n_0$ we have

$$\begin{aligned} R_1(x_n) &= \frac{1}{2}\|\nabla x_n\|_2^2 - \frac{\lambda_1}{2}\|x_n\|_2^2 - \int_Z j(z, x_n(z))\, dz \\ &= \frac{1}{2}t_n^2\|\nabla w_1\|_2^2 + \frac{1}{2}\|\nabla v_n\|_2^2 - \frac{\lambda_1}{2}t_n^2\|w_1\|_2^2 - \frac{\lambda_1}{2}\|v_n\|_2^2 \\ &\quad - \int_Z j(z, x_n(z))\, dz \\ &= \frac{1}{2}\|\nabla v_n\|_2^2 - \frac{\lambda_1}{2}\|v_n\|_2^2 - \int_Z j(z, x_n(z))\, dz \\ &\leq \varepsilon - \int_Z j(z, x_n(z))\, dz, \end{aligned}$$

so from Fatou lemma, we have

$$\limsup_{n\to+\infty} R_1(x_n) \leq \varepsilon - \int_Z j_+(z)\, dz. \tag{17}$$

On the other hand, since $\|\nabla x_n\|_2^2 \geq \lambda_1\|x_n\|_2^2$, we have

$$\limsup_{n\to+\infty} R_1(x_n) \geq - \int_Z j_+(z)\, dz. \tag{18}$$

From (17), (18) and since $\varepsilon > 0$ was arbitrary, we infer that

$$\limsup_{n\to+\infty} R_1(x_n) = - \int_Z j_+(z)\, dz,$$

which is a contradiction to our assumption. This proves that the sequence $\{x_n\}_{n\geq 1} \subseteq H_0^1(Z)$ is bounded. Then, as in the proof of Theorem 5, Claim #1, via the Kadec-Klee property, we can show that $x_n \longrightarrow x$ in $H_0^1(Z)$, which proves Claim #3.

Claim #4 There exist $r_0 > 0$ such that $\inf\{R_1(x) : x \in \partial B_r(0)\} > 0$ for all $r \in (0, r_0)$.

From hypothesis $H(j)$ (v), we can find $\delta > 0$ such that for almost all $z \in Z$ and all ζ such that $|\zeta| \le \delta$, we have

$$j(z,\zeta) \le \frac{1}{2}\left(-\frac{\mu+\lambda_1}{2}\right)|\zeta|^2$$

(recall that $\mu > \lambda_1$ and so $-\dfrac{\mu+\lambda_1}{2} > -\mu$). On the other hand, from the hypothesis $H(j)_3$ (i) and the Lebourg mean value theorem, for almost all $z \in Z$ and all ζ such that $|\zeta| > \delta$, we obtain

$$|j(z,\zeta)| \le c_4 + c_5|\zeta|,$$

with some $c_4, c_5 > 0$. Thus for almost all $z \in Z$ and all $\zeta \in \mathbb{R}$, we have

$$j(z,\zeta) \le \frac{1}{2}\left(-\frac{\mu+\lambda_1}{2}\right)|\zeta|^2 + c_6|\zeta|^{\vartheta},$$

with $c_6 = (c_4 + c_5\delta)\delta^{-\vartheta} + \dfrac{1}{2}\left(-\dfrac{\mu+\lambda_1}{2}\right)\delta^{2-\vartheta}$ and $2 < \vartheta \le 2^* = \dfrac{2N}{N-2}$. Using this we obtain that

$$\begin{aligned} R_1(x) &= \frac{1}{2}\|\nabla x\|_2^2 - \frac{\lambda_1}{p}\|x\|_2^2 - \int_Z j(z,x(z))\,dz \\ &\ge \frac{1}{2}\|\nabla x\|_2^2 - \frac{\lambda_1}{2}\|x\|_2^2 + \frac{1}{2}\left(\frac{\mu+\lambda_1}{2}\right)\|x\|_2^2 - c_6\|x\|_{\vartheta}^{\vartheta} \\ &= \frac{1}{2}\|\nabla x\|_2^2 + \frac{\mu-\lambda_1}{4}\|x\|_2^2 - c_6\|x\|_{\vartheta}^{\vartheta} \;\ge\; \frac{1}{2}\|\nabla x\|_2^2 - c_6\|x\|_{\vartheta}^{\vartheta}. \end{aligned}$$

From the Sobolev embedding theorem, we have that $H_0^1(Z)$ is embedded continuously in $L^{\vartheta}(Z)$. So using the Poincaré inequality, it follows that

$$R_1(x) \ge c_7\|x\|^2 - c_8\|x\|^{\vartheta},$$

with some $c_7, c_8 > 0$ and all $x \in H_0^1(Z)$, and so

$$R_1(x) \ge c_7\|x\|^2\left(1 - \frac{c_8}{c_7}\|x\|^{\vartheta-2}\right).$$

Let $r_0 \stackrel{df}{=} \left(\dfrac{c_8}{c_7}\right)^{\frac{1}{\vartheta-2}}$. Now, for $r \in (0, r_0)$, we have

$$\inf_{\|x\|=r} R_1(x) > 0,$$

and this proves Claim #4.

Now, let $U^{\pm} \stackrel{df}{=} \{x \in H_0^1(Z) : \ x = \pm t w_1 + v, \ t > 0, \ v \in Y_{w_1}\}$. We will show that R_1 attains its infimum on both open sets U^+ and U^-. To this end let $m_+ \stackrel{df}{=} \inf\{R_1(x) : \ x \in U^+\}$. Since R_1 is locally Lipschitz, we have that $m_+ = \inf\{R_1(x) : \ x \in \overline{U^+}\}$. Let

$$\overline{R_1^+} \stackrel{df}{=} \begin{cases} R_1(x) & \text{if } x \in \overline{U^+}, \\ +\infty & \text{otherwise.} \end{cases}$$

Evidently $\overline{R_1}$ is lower semicontinuous and bounded below (see Claim #1). Thus we can apply the Ekeland variational principle (see Theorem 4) and obtain a sequence $\{x_n\}_{n\geq 1} \subseteq U^+$ such that $R_1(x_n) \searrow m_+$ as $n \longrightarrow +\infty$ and

$$\overline{R_1^+}(x_n) < \overline{R_1^+}(y) + \varepsilon_n \|x_n - y\|,$$

for all $y \in H_0^1(Z)$, $y \neq x_n$, with $\varepsilon_n \searrow 0$. So

$$R_1(x_n) < R_1(y) + \varepsilon_n \|x_n - y\|$$

for all $y \in \overline{U^+}$, $y \neq x_n$. This means that $x_n \in U^+$ minimizes the functional $y \longrightarrow R_1(y) + \varepsilon_n \|y - x_n\|$ on $\overline{U^+}$. Since U^+ is open, we have $0 \in \partial R_1(x_n) + \varepsilon_n \overline{B}_1^*$, where $\overline{B}_1^* \stackrel{df}{=} \{x^* \in H^{-1}(Z) : \ \|x^*\|_* \leq 1\}$ (recall that $\partial \| \cdot \| = \overline{B}_1^*$; see e.g. Hu-Papageorgiou [16]). Hence, we can find $x_n^* \in \partial R_1(x_n)$ such that $\|x_n^*\|_* \leq \varepsilon_n$ for $n \geq 1$. If follows that $m(x_n) \leq \|x_n^*\|_* \leq \varepsilon_n \longrightarrow 0$. Using hypothesis $H(j)_3$ (iv), we obtain

$$m_+ = \inf_{x \in U^+} R_1(x) \leq R_1(\vartheta_+ w_1) = -\int_Z j(z, \vartheta_+ w_1(z))\, dz < -\int_Z j_+(z)\, dz.$$

Since $R_1(x_n) \longrightarrow m_+ \neq -\int_Z j_+(z)\, dz$, so from Claim #3, we infer that, by passing to a subsequence if necessary, we may assume that $x_n \longrightarrow y_1$ in $H_0^1(Z)$ with $y_1 \in \overline{U^+}$. We will show that $y_1 \in U^+$. Let us assume that $y_1 \in \partial \overline{U^+} = Y_{w_1}$. Then from Claim #2, we have $0 \leq R_1(y_1)$. On the other hand, from hypotheses $H(j)_3$ (iv) and (iii), we get $m_+ < 0$. But $R_1(y_1) = m_+$, so we get a contradiction. Hence $y_1 \in U^+$, $y_1 \neq 0$ and $0 \in \partial R_1(y_1)$. Similarly we obtain $y_2 \in U^-$, $y_2 \neq 0$ such that $0 \in \partial R_1(y_2)$. Clearly $y_1 \neq y_2$.

Finally from Claim #4, we know that we can find $0 < r < \min\{r_0, |\vartheta|\}$ (compare hypothesis $H(j)_3$ (iv)) such that $\inf\{R_1(x) : \ x \in \partial B_r(0)\} > 0$. From hypothesis $H(j)_3$ (iv), we also have that $\max\{R_1(0), R_1(\vartheta w_1)\} \leq 0$. These facts combined with Claim #3, permit the use of Theorem 2, with $y = w$ (note that $c \neq -\int_Z j_{\pm}(z)\, dz$, because $c = \inf_{\gamma \in \Gamma} \max_{\tau \in [0,1]} R_1(\gamma(\tau)) \geq R_1(0) = 0 > -\int_Z j_{\pm}(z)\, dz$), which gives us $y_3 \in H_0^1(Z)$ such that $0 \in \partial R_1(y_3)$ and

$$R_1(y_3) = c > -\int_Z j_{\pm}(z)\, dz > m_{\pm},$$

so clearly $y_3 \neq 0$ and $y_3 \neq y_1$, $y_3 \neq y_2$. Finally we can easily check that y_1, y_2, y_3 satisfy (RHI_1) and so are three different, nontrivial solutions. Q.E.D.

REMARK 8. It will be interesting to have such mutliplicity result for quasilinear hemivariational inequalities, like the one studied by Gasiński-Papageorgiou [12].

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