

ON CERTAIN SUBALGEBRAS OF GRADED LIE ALGEBRAS

By

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Abstract. For every semi-simple graded Lie algebra $\mathfrak{g}^* = \sum_{k=-\nu}^{\nu} \mathfrak{g}_k^*$ and every $c \in \mathbb{R}$, one can determine the subalgebra $\sum_{k \in c\mathbb{Z}} \mathfrak{g}_k^*$ in terms of the restricted root systems. We also mention that these subalgebras are closely related to the orbits of the compact symmetric spaces under the actions of the isotropy subgroups.

1. Introduction

Let \mathfrak{g}^* be a real semi-simple Lie algebra. The decomposition of \mathfrak{g}^* into the direct sum of the subspaces, $\mathfrak{g}^* = \sum_{k=-\nu}^{\nu} \mathfrak{g}_k^*$, is called a *graded Lie algebra* if $[\mathfrak{g}_k^*, \mathfrak{g}_l^*] \subset \mathfrak{g}_{k+l}^*$ for every k and l . The family of the subspaces (\mathfrak{g}_k^*) is called a *gradation on \mathfrak{g}^** . A graded Lie algebra is said to be *of the ν -th kind* if $\mathfrak{g}_{\pm\nu}^* \neq 0$ and $\mathfrak{g}_k^* = 0$ for $|k| > \nu$.

Graded Lie algebras have been studied by many authors. The book [10] is a good reference. In particular, S. Kaneyuki and H. Asano ([11]) gave a bijective correspondence between isomorphism classes of all gradations on a real semi-simple Lie algebra and certain equivalence classes of partitions of a simple root system of the restricted root system. In Section 3, we restate their classification theorem in other words.

Let \mathfrak{k} be a maximal compact subalgebra of \mathfrak{g}^* . We usually assume that \mathfrak{k} is compatible with the gradation on \mathfrak{g}^* (i.e., \mathfrak{k} is the $(+1)$ -eigenspace of a grade-reversing Cartan involution). Let \mathfrak{g} be the complex dual of \mathfrak{g}^* with respect to \mathfrak{k} (i.e., \mathfrak{g} is the maximal compact subalgebra of the complexification of \mathfrak{g}^* containing \mathfrak{k}). For convenience, by a pair of Lie algebras $(\mathfrak{l}, \mathfrak{l}')$, we mean the homogeneous space L/L' , where L is a Lie group whose Lie algebra is \mathfrak{l} and L' is a subgroup of L with the Lie algebra \mathfrak{l}' . (We disregard the choice of local isomorphism classes.)

First kind semi-simple graded Lie algebras $\mathfrak{g}^* = \mathfrak{g}_{-1}^* \oplus \mathfrak{g}_0^* \oplus \mathfrak{g}_1^*$ have been classified by S. Kobayashi and T. Nagano ([12]). The homogeneous space $(\mathfrak{g}^*, \mathfrak{g}_0^* \oplus \mathfrak{g}_1^*)$

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is a symmetric R-space. Since it is compact, it coincides with $(\mathfrak{k}, \mathfrak{k} \cap \mathfrak{g}_0^*)$. It is known that every symmetric R-space arises as an orbit of the isotropy representation of the compact symmetric space $(\mathfrak{g}, \mathfrak{k})$.

Second kind semi-simple graded Lie algebras $\mathfrak{g}^* = \mathfrak{g}_{-2}^* \oplus \mathfrak{g}_{-1}^* \oplus \mathfrak{g}_0^* \oplus \mathfrak{g}_1^* \oplus \mathfrak{g}_2^*$ have been classified by S. Kaneyuki ([9]). He also determined the subalgebras \mathfrak{g}_0^* and $\mathfrak{g}_{ev}^* := \mathfrak{g}_{-2}^* \oplus \mathfrak{g}_0^* \oplus \mathfrak{g}_2^*$ for every second kind simple graded Lie algebra.

One of the aim of this paper is to investigate the subalgebras $\sum_{k \in c\mathbb{Z}} \mathfrak{g}_k^*$ for every semi-simple graded Lie algebra $\mathfrak{g}^* = \sum_{k=-\nu}^{\nu} \mathfrak{g}_k^*$ with no assumption on ν . We can determine these subalgebras in the following way. At first, for a closed subsystem Δ' , we define the subalgebra $\mathfrak{g}^*(\Delta')$ of \mathfrak{g}^* corresponding to Δ' . In Section 4, we mention that one can determine $\mathfrak{g}^*(\Delta')$ for every Δ' . We obtain the following in Section 5.

Theorem A. *Let $\mathfrak{g}^* = \sum_{k=-\nu}^{\nu} \mathfrak{g}_k^*$ be a semi-simple graded Lie algebra. For every $c \in \mathbb{R}$, $\sum_{k \in c\mathbb{Z}} \mathfrak{g}_k^*$ is the subalgebra corresponding to certain closed subsystem.*

T. Nagano and M.S. Tanaka ([14]) pointed out that second kind graded Lie algebras $\mathfrak{g}^* = \sum_{k=-2}^2 \mathfrak{g}_k^*$ are closely related to the orbits of the compact symmetric spaces under the actions of the isotropy subgroups. Put $\mathfrak{k}_0 := \mathfrak{k} \cap \mathfrak{g}_0^*$ and $\mathfrak{k}_{ev} := \mathfrak{k} \cap \mathfrak{g}_{ev}^*$. Denote by \mathfrak{g}_0 (resp. \mathfrak{g}_{ev}) the compact dual of \mathfrak{g}_0^* (resp. \mathfrak{g}_{ev}^*) with respect to \mathfrak{k}_0 (resp. \mathfrak{k}_{ev}). The pair of the fiber bundles

$$\begin{aligned} (\mathfrak{g}_{ev}, \mathfrak{g}_0) &\longrightarrow (\mathfrak{g}, \mathfrak{g}_0) \longrightarrow (\mathfrak{g}, \mathfrak{g}_{ev}), \\ (\mathfrak{k}_{ev}, \mathfrak{k}_0) &\longrightarrow (\mathfrak{k}, \mathfrak{k}_0) \longrightarrow (\mathfrak{k}, \mathfrak{k}_{ev}). \end{aligned}$$

is called a *2-tired fibration* ([14]). The homogeneous space $(\mathfrak{g}, \mathfrak{g}_0)$ is the twistor space over the symmetric space $(\mathfrak{g}, \mathfrak{g}_{ev})$ in the sense of R. Bryant ([2]). J.H. Cheng ([3]) have mentioned that, in case $\dim \mathfrak{g}_{\pm 2}^* = 1$, $(\mathfrak{g}, \mathfrak{g}_{ev})$ is a quaternionic Kähler symmetric space and $(\mathfrak{g}, \mathfrak{g}_0)$ is the twistor space over it (refer to [18] for this twistor space).

Denote by K the isotropy subgroup of the symmetric space $M := (\mathfrak{g}, \mathfrak{k})$, and by $K(x)$ the K -orbit through $x \in M$. In [14] they proved that there exists a closed geodesic p of M such that

$$K(p(t)) = \begin{cases} (\mathfrak{k}, \mathfrak{k}) & \text{if } t \in \mathbb{Z}, \\ (\mathfrak{k}, \mathfrak{k}_{ev}) & \text{if } t \in \frac{1}{2} + \mathbb{Z}, \text{ and} \\ (\mathfrak{k}, \mathfrak{k}_0) & \text{if others.} \end{cases}$$

Remark that the geodesic p can be obtained explicitly. (In fact, $p(t) = \exp(t\pi Z) \cdot o$, where iZ is the characteristic element of the graded Lie algebra.)

In this paper, we will mention that the subalgebras $\sum_{k \in c\mathbb{Z}} \mathfrak{g}_k^*$ of a semi-simple graded Lie algebra, where ν is arbitrary, are closely related to the orbits of certain compact symmetric spaces. Let $\mathfrak{g}^* = \sum_{k=-\nu}^{\nu} \mathfrak{g}_k^*$ be a semi-simple graded Lie algebra, and $M := (\mathfrak{g}, \mathfrak{k})$ be the symmetric space, where \mathfrak{g} be the compact dual of \mathfrak{g}^* with respect to \mathfrak{k} . Denote by K the isotropy subgroup of M .

Theorem B. *There exists a closed geodesic p such that $K(p(t)) = (\mathfrak{k}, \mathfrak{k} \cap \sum_{tk \in \mathbb{Z}} \mathfrak{g}_k^*)$.*

Denote by \mathfrak{k}_p the Lie algebra of the isotropy subgroup of K at $p \in M$. We say that two orbits $K(p)$ and $K(q)$ are of the same local orbit type if \mathfrak{k}_p and \mathfrak{k}_q are conjugate under $\text{Aut } \mathfrak{g}$. The above theorem immediately leads the following.

Corollary C. *The set of the all local orbit types of K -orbits through $\{p(t) \mid t \in \mathbb{R}\}$ is bijective to $\{\sum_{k \in c\mathbb{Z}} \mathfrak{g}_k^* \mid c \in \mathbb{R}\}$.*

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2. Preliminaries

As preliminaries, we recall the restricted root systems of compact semi-simple symmetric spaces (cf. [8], [13]). They are useful to understand semi-simple graded Lie algebras.

Let \mathfrak{g} be a compact semi-simple Lie algebra, and σ be an involutive automorphism of \mathfrak{g} . Let $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{m}$ be the decomposition by σ , where $\sigma|_{\mathfrak{k}} = 1$ and $\sigma|_{\mathfrak{m}} = -1$. One knows, $[\mathfrak{k}, \mathfrak{m}] \subset \mathfrak{m}$ and $[\mathfrak{m}, \mathfrak{m}] \subset \mathfrak{k}$. We call $(\mathfrak{g}, \mathfrak{k})$ a compact semi-simple symmetric pair and $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{m}$ the symmetric decomposition of (\mathfrak{g}, σ) . Let \mathfrak{a} be a maximal abelian subspace in \mathfrak{m} , which is uniquely determined up to conjugation.

Definition 2.1. Let α be a linear form on \mathfrak{a} , and

$$\mathfrak{g}(\alpha) := \{X \in \mathfrak{g} \mid [H, [H, X]] = -\alpha(H)^2 X \text{ for any } H \in \mathfrak{a}\}.$$

A non-zero linear form α is said to be a root if $\mathfrak{a}(\alpha) \neq 0$.

We remark that $\mathfrak{g}(\alpha) = \mathfrak{g}(-\alpha)$. Let

$$\mathfrak{k}(\alpha) := \mathfrak{k} \cap \mathfrak{g}(\alpha) \quad \text{and} \quad \mathfrak{m}(\alpha) := \mathfrak{m} \cap \mathfrak{g}(\alpha).$$

It is easily seen that $\mathfrak{m}(0) = \mathfrak{a}$. We call $\dim \mathfrak{k}(\alpha)$ the multiplicity of a root α . The

set of all roots with the multiplicities is called the *restricted root system* of the symmetric space. Denote by Δ the restricted root system.

Proposition 2.2. ([8] pp. 335–336) *Let $\alpha, \beta \in \Delta \cup \{0\}$. Then*

- (i) $[\mathfrak{k}(\alpha), \mathfrak{k}(\beta)] \subset \mathfrak{k}(\alpha + \beta) + \mathfrak{k}(\alpha - \beta),$
- (ii) $[\mathfrak{m}(\alpha), \mathfrak{m}(\beta)] \subset \mathfrak{k}(\alpha + \beta) + \mathfrak{k}(\alpha - \beta),$ and
- (iii) $[\mathfrak{k}(\alpha), \mathfrak{m}(\beta)] \subset \mathfrak{m}(\alpha + \beta) + \mathfrak{m}(\alpha - \beta).$

From this proposition, $\mathfrak{k}(0)$ is a subalgebra. The subalgebra $\mathfrak{k}(0)$ is called the *principal isotropy subalgebra*, since it coincides with the Lie algebra of the isotropy group at certain element in the principal orbit in \mathfrak{m} . All the principal isotropy subalgebras have been determined (see e.g. [16]).

Proposition 2.3. *Let $\alpha \in \Delta$. For every $X_\alpha \in \mathfrak{k}(\alpha)$, there exists $Y_\alpha \in \mathfrak{m}(\alpha)$ such that*

$$(1) \quad [H, X_\alpha] = \alpha(H)Y_\alpha \quad \text{and} \quad [H, Y_\alpha] = -\alpha(H)X_\alpha \quad \text{for all } H \in \mathfrak{a}.$$

Proof. Let $\alpha \in \Delta$ and $X_\alpha \in \mathfrak{k}(\alpha)$. Take $H_0 \in \mathfrak{a}$ satisfying $\alpha(H_0) = 1$, and put $Y_\alpha := [H_0, X_\alpha]$. We agree $Y_\alpha \in \mathfrak{m}(\alpha)$ from Proposition 2.2, (iii). Polarization of the definition of $\mathfrak{g}(\alpha)$ gives that, $X \in \mathfrak{g}(\alpha)$ if and only if $[H_1, [H_2, X]] = -\alpha(H_1)\alpha(H_2)X$ for every $H_1, H_2 \in \mathfrak{a}$. This leads that Y_α satisfies (1). Q.E.D.

Let $\mathfrak{g}^{\mathbb{C}}$ be the complexification of \mathfrak{g} . One can easily see that $\mathfrak{g}^* := \mathfrak{k} \oplus i\mathfrak{m}$ is a real subalgebra of $\mathfrak{g}^{\mathbb{C}}$. The Lie algebra \mathfrak{g}^* is called the *non-compact dual of $(\mathfrak{g}, \mathfrak{k})$* . Let $\mathfrak{g}^\pm(\alpha)$ be the subspaces in $\mathfrak{k} \oplus i\mathfrak{m}$ spanned by $X_\alpha \pm iY_\alpha$ respectively, where $X_\alpha \in \mathfrak{k}(\alpha)$ and $Y_\alpha \in \mathfrak{m}(\alpha)$ satisfy the relation (1). Put $\mathfrak{g}^*(\alpha) := \mathfrak{g}^+(\alpha) \oplus \mathfrak{g}^-(\alpha)$. One can easily see that $\mathfrak{g}(\alpha) = \mathfrak{g}^+(\alpha) \oplus \mathfrak{g}^-(\alpha) = \mathfrak{k}(\alpha) \oplus i\mathfrak{m}(\alpha)$.

Proposition 2.4. *For every $H \in \mathfrak{a}$, $X^+ \in \mathfrak{g}^+(\alpha)$ and $X^- \in \mathfrak{g}^-(\alpha)$,*

$$[iH, X^+] = \alpha(H)X^+ \quad \text{and} \quad [iH, X^-] = -\alpha(H)X^-.$$

Proof. We can prove the proposition by direct calculations. Q.E.D.

Let $\Pi := \{\alpha_1, \dots, \alpha_r\}$ be a simple root system of Δ , and

$$C := \{H \in \mathfrak{a} \mid \alpha_1(H), \dots, \alpha_r(H) \in \mathbb{Z}_{\geq 0}\},$$

where $\mathbb{Z}_{\geq 0}$ denotes the set of all non-negative integers. The set C is the intersection of the center lattice $\{H \in \mathfrak{a} \mid \alpha_1(H), \dots, \alpha_r(H) \in \mathbb{Z}\}$ and the closed fundamental Weyl chamber $\{H \in \mathfrak{a} \mid \alpha_1(H), \dots, \alpha_r(H) \geq 0\}$. We call the set C the *fundamental center lattice in \mathfrak{a} with respect to Π* . Let $\{H^1, \dots, H^r\}$ be the

dual basis of Π . Then it is obvious that

$$C = \left\{ \sum_{j=1}^r c_j H^j \in \mathfrak{a} \mid c_1, \dots, c_r \in \mathbb{Z}_{\geq 0} \right\}.$$

3. Graded Lie algebras

In this section, we mention the classification of semi-simple graded Lie algebras in terms of partitions of simple root systems of the restricted root systems, mentioned by S. Kaneyuki and H. Asano ([11]). On the final of the section, we restate their theorem in terms of the fundamental center lattices.

Let $\mathfrak{g}^* = \sum_{k=-\nu}^{\nu} \mathfrak{g}_k^*$ be a semi-simple graded Lie algebra. An element $Z \in \mathfrak{g}^*$ is called a *characteristic element* if each \mathfrak{g}_k^* is the k -eigenspace of $\text{ad}(Z)$. It is well-known that every semi-simple graded Lie algebra has the unique characteristic element.

Let (\mathfrak{g}_k^*) and $(\mathfrak{g}_k^{*'})$ be two gradations on \mathfrak{g}^* . We say that they are *isomorphic* if there exists $a \in \text{Aut } \mathfrak{g}^*$ such that a sends \mathfrak{g}_k^* into $\mathfrak{g}_k^{*'}$ for every k . Denote by Z and Z' the characteristic elements of (\mathfrak{g}_k^*) and $(\mathfrak{g}_k^{*'})$ respectively. One can easily see that the gradations are isomorphic if and only if $a(Z) = Z'$ for some $a \in \text{Aut } \mathfrak{g}^*$.

Before we mention the classification theorem by S. Kaneyuki and H. Asano, we give the notations. Let $(\mathfrak{g}, \mathfrak{k}; \sigma)$ be a compact semi-simple symmetric pair, $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{m}$ be the symmetric decomposition of (\mathfrak{g}, σ) , and $\mathfrak{g}^* = \mathfrak{k} \oplus i\mathfrak{m}$ be the non-compact dual. Fix a maximal abelian subspace \mathfrak{a} of \mathfrak{m} , and a simple root system Π of the restricted root system Δ with respect to \mathfrak{a} . By a partition of Π we mean a disjoint union $\Pi = \bigcup_{k=0}^n \Pi_k$ such that $\Pi_n \neq \emptyset$. (In [11], they assume that $\Pi_1 \neq \emptyset$. But it is not essential.)

Theorem 3.1. ([11]) *There exists a bijective correspondence between the set of isomorphism classes of gradations on \mathfrak{g}^* , and the set of equivalent classes of partitions of Π under the automorphism group of the Dynkin diagram of Π .*

We mention how to construct the gradation from a partition of $\Pi = \{\alpha_1, \dots, \alpha_r\}$. Let $\Pi = \bigcup_{k=0}^n \Pi_k$ be a partition. Every root α can be expressed as $\alpha = \sum_{j=1}^r c_j \alpha_j$. Define the integer $w(\alpha)$ by

$$(2) \quad w(\alpha_j) := k \quad \text{if } \alpha_j \in \Pi_k, \quad \text{and} \quad w(\alpha) := \sum_{j=1}^r c_j \cdot w(\alpha_j).$$

Put

$$\mathfrak{g}_0^* := \mathfrak{g}^*(0) \oplus \sum_{w(\alpha)=0} \mathfrak{g}^*(\alpha), \quad \text{and} \quad \mathfrak{g}_k^* := \sum_{w(\alpha)=k} \mathfrak{g}^*(\alpha).$$

Thus we obtain the gradation (\mathfrak{g}_k^*) .

Keep the above notations (that is, fix \mathfrak{m} , \mathfrak{a} , and Π). Let C be the fundamental center lattice in \mathfrak{a} with respect to Π .

Theorem 3.2. *For every element $Z \in C$, the eigenspace decomposition of \mathfrak{g}^* with respect to $\text{ad}(iZ)$ gives the gradation of \mathfrak{g}^* . Conversely, every gradation on \mathfrak{g}^* can be obtained in this way up to isomorphisms.*

Proof. Let $Z \in C$. From Proposition 2.4, one has

$$\text{ad}(iZ)|_{\mathfrak{g}^{\pm(\alpha)}} = \alpha(Z) \cdot \text{id}.$$

for every $\alpha \in \Delta \cup \{0\}$. Thus every eigenvalue of $\text{ad}(iZ)$ is an integer. Let \mathfrak{g}_k^* be the k -eigenspace of \mathfrak{g}^* with respect to $\text{ad}(iZ)$. Then $\mathfrak{g}^* = \sum \mathfrak{g}_k^*$. Jacobi identity leads that $[\mathfrak{g}_k^*, \mathfrak{g}_l^*] \subset \mathfrak{g}_{k+l}^*$ for every k and l . Thus every $Z \in C$ defines the gradation (\mathfrak{g}_k^*) on \mathfrak{g}^* whose characteristic element is iZ .

Next, we show the latter part. Let $\mathfrak{g}^* = \sum_{k=-\nu}^{\nu} \mathfrak{g}_k^*$ be a semi-simple graded Lie algebra. Take the characteristic element E . We have only to prove that there exists $Z \in C$ such that iZ is conjugate to E under the action of $\text{Aut } \mathfrak{g}^*$. Let \mathfrak{k}' be a maximal compact subalgebra which is orthogonal to E . There exists $a \in \text{Aut } \mathfrak{g}^*$ such that $a(\mathfrak{k}') = \mathfrak{k}$. Thus $a(E) \in \mathfrak{im}$. Since all the maximal abelian subspaces are conjugate, there exists $b \in \text{Aut } \mathfrak{g}^*$ such that $-i \cdot b \circ a(E) \in \mathfrak{a}$. Furthermore, from the conjugacy of Weyl chambers, we have $c \in \text{Aut } \mathfrak{g}^*$ such that

$$\alpha_j(-i \cdot c \circ b \circ a(E)) \geq 0 \quad \text{for } j = 1, \dots, r.$$

Put $Z := -i \cdot c \circ b \circ a(E)$. Since every eigenvalue of $\text{ad}(iZ)$ is an integer, we have $Z \in C$. We have found $Z \in C$ such that $iZ = c \circ b \circ a(E)$. Q.E.D.

Let $\mathfrak{g}^* = \sum_{k=-\nu}^{\nu} \mathfrak{g}_k^*$ be the graded Lie algebra defined by $Z \in C$. Then one can easily see that $\nu = \tilde{\alpha}(Z)$, where $\tilde{\alpha}$ denotes the highest root.

This theorem says that there exists a surjection from C onto \mathcal{G} , the set of isomorphism classes of gradations on \mathfrak{g}^* . We call $Z, Z' \in C$ *equivalent* if there exists $a \in \text{Aut } \mathfrak{g}^*$ such that $a(Z) = Z'$. Then it can be easily seen that the set of equivalent classes of C is bijective to \mathcal{G} .

Here we mention the relationship between C and partitions of Π . For $Z \in C$, we can define the partition $\Pi = \bigcup_{k=0}^n \Pi_k$ by

$$\Pi_k := \{\alpha \in \Pi \mid \alpha(Z) = k\}.$$

Conversely, from a partition $\Pi = \bigcup_{k=0}^n \Pi_k$, we can obtain $Z \in C$ by

$$Z := \sum_{j=1}^r w(\alpha_j) \cdot H^j,$$

where $w(\alpha_j)$ is the integer defined in (2), and $\{H^1, \dots, H^r\}$ is the dual basis of Π . These arguments give a bijective correspondence between the set of equivalence classes of C and the set of partitions of Π .

4. The corresponding subalgebras

In this section, we define the subalgebras of \mathfrak{g}^* which correspond to closed subsystems of the restricted root systems. In [16], the corresponding subalgebras of \mathfrak{k} were considered. Every corresponding subalgebra of \mathfrak{g}^* can be determined easily by the closed subsystem, as in [16].

Definition 4.1. A subset Δ' in a root system Δ is called a *closed subsystem* if the following two property holds: (i) if $\alpha, \beta \in \Delta'$ and $\alpha + \beta \in \Delta$ then $\alpha + \beta \in \Delta'$, and (ii) $\Delta' = -\Delta'$.

For an element $H \in \mathfrak{a}$ and $c \in \mathbf{R}$, we set $\Delta_H^c := \{\alpha \in \Delta \mid \alpha(H) \in c\mathbf{Z}\}$. This is a typical example of closed subsystems.

Let Δ be the restricted root system of a compact semi-simple symmetric pair $(\mathfrak{g}, \mathfrak{k})$. Every closed subsystem in Δ becomes the root system with the multiplicities in the natural way.

Proposition 4.2. ([16]) *For every closed subsystem Δ' in Δ , there exists the compact semi-simple symmetric pair $(\mathfrak{g}', \mathfrak{k}')$ whose restricted root system is Δ' . Furthermore, the principal isotropy subalgebra $\mathfrak{k}'(0)$ is an ideal of $\mathfrak{k}(0)$.*

We recall the proof briefly. One can construct \mathfrak{g}' and \mathfrak{k}' as follows: let $\mathfrak{k}'(0) := \sum_{\alpha \in \Delta'} [\mathfrak{k}(\alpha), \mathfrak{k}(\alpha)]_{\mathfrak{k}(0)}$ (the subscript $\mathfrak{k}(0)$ denotes the $\mathfrak{k}(0)$ -component), \mathfrak{a}' be the dual space of the \mathbf{R} -span of Δ' , $\mathfrak{k}' := \mathfrak{k}'(0) \oplus \sum_{\alpha \in \Delta'} \mathfrak{k}(\alpha)$, $\mathfrak{m}' := \mathfrak{a}' \oplus \sum_{\alpha \in \Delta'} \mathfrak{m}(\alpha)$, and $\mathfrak{g}' := \mathfrak{k}' \oplus \mathfrak{m}'$.

Definition 4.3. Let Δ' be a closed subsystem in Δ . We call

$$\mathfrak{g}^*(\Delta') := \mathfrak{k}(0) \oplus i\mathfrak{a} \oplus \sum_{\alpha \in \Delta'} (\mathfrak{k}(\alpha) \oplus i\mathfrak{m}(\alpha))$$

the subalgebra of \mathfrak{g}^* which corresponds to Δ' .

Let $(\mathfrak{g}', \mathfrak{k}')$ be the symmetric pair with the restricted root system Δ' , and $\mathfrak{k}'(0)$ be the principal isotropy subalgebra of this pair. Since $\mathfrak{k}'(0)$ is an ideal of $\mathfrak{k}(0)$, the orthogonal complement of $\mathfrak{k}'(0)$ (with respect to the Killing form of \mathfrak{g}) is also an ideal of $\mathfrak{k}(0)$. We denote it by $\mathfrak{k}(0)/\mathfrak{k}'(0)$. Let $(\mathfrak{g}')^*$ denote the non-compact dual of $(\mathfrak{g}', \mathfrak{k}')$.

Theorem 4.4. *The subalgebra $\mathfrak{g}^*(\Delta')$ corresponds to a closed subsystem Δ' in Δ can be decomposed as the following direct sum of Lie algebras:*

$$\mathfrak{g}^*(\Delta') = (\mathfrak{g}(0)/\mathfrak{k}'(0)) \oplus \mathbf{R}^{\text{rank}\Delta - \text{rank}\Delta'} \oplus (\mathfrak{g}')^*,$$

where $\mathbf{R}^{\text{rank}\Delta - \text{rank}\Delta'}$ is abelian.

Proof. We can obtain following decompositions as Lie algebras,

$$\begin{aligned} \mathfrak{k}(0) &= (\mathfrak{k}(0)/\mathfrak{k}'(0)) \oplus \mathfrak{k}'(0), \quad \text{and} \\ i\mathfrak{a} &= \mathbf{R}^{\text{rank}\Delta - \text{rank}\Delta'} \oplus i\mathfrak{a}'. \end{aligned}$$

From the proof of Proposition 4.2, we have the decomposition

$$\begin{aligned} \mathfrak{g}^*(\Delta') &= \mathfrak{k}(0) \oplus \sum_{\alpha \in \Delta'} \mathfrak{k}(\alpha) \oplus i\mathfrak{a} \oplus \sum_{\alpha \in \Delta'} i\mathfrak{m}(\alpha) \\ (3) \quad &= (\mathfrak{k}(0)/\mathfrak{k}'(0)) \oplus \mathbf{R}^{\text{rank}\Delta - \text{rank}\Delta'} \oplus (\mathfrak{g}')^* \end{aligned}$$

as a vector space.

To complete the proof, we prove that each component of (3) is an ideal. From the definition of $\mathfrak{k}'(0)$, $\mathfrak{k}(0)/\mathfrak{k}'(0)$ centralizes $\mathfrak{k}' \oplus i\mathfrak{m}'$. Since $\mathfrak{k}(0)$ acts on \mathfrak{a} trivially, $\mathfrak{k}(0)/\mathfrak{k}'(0)$ is an ideal of $\mathfrak{g}^*(\Delta')$. Furthermore, one can see that

$$\mathbf{R}^{\text{rank}\Delta - \text{rank}\Delta'} = \{H \in \mathfrak{a} \mid \alpha(H) = 0 \text{ for every } \alpha \in \Delta'\}.$$

Thus $\mathbf{R}^{\text{rank}\Delta - \text{rank}\Delta'}$ is an ideal of $\mathfrak{g}^*(\Delta')$, and so is $(\mathfrak{g}')^*$.

Q.E.D.

From the table of the principal isotropy subalgebras (see e.g. [16]) and Theorem 4.4, every corresponding subalgebra $\mathfrak{g}^*(\Delta')$ can be determined easily, if we know the type and the multiplicities of Δ' .

5. The subalgebras of graded Lie algebras

In Section 3, we showed that every element $Z \in C$ defines the graded Lie algebra $\mathfrak{g}^* = \sum_{k=-\nu}^{\nu} \mathfrak{g}_k^*$, where $\nu = \tilde{\alpha}(Z)$. In this section we prove that every subalgebra $\sum_{k \in c\mathbf{Z}} \mathfrak{g}_k^*$ corresponds to certain closed subsystems.

Lemma 5.1. *For a graded Lie algebra $\mathfrak{g}^* = \sum_{k=-\nu}^{\nu} \mathfrak{g}_k^*$, defined by Z ,*

- (i) $\mathfrak{g}_0 = \mathfrak{k}(0) \oplus i\mathfrak{a} \sum_{\alpha(Z)=0} \mathfrak{g}(\alpha)$, and
- (ii) $\mathfrak{g}_{\pm k} = \sum_{\alpha(Z)=k} \mathfrak{g}^{\pm}(\alpha)$ for every $k > 0$.

Proof. By Proposition 2.4, the linear map $\text{ad}(iZ)$ acts on $\mathfrak{g}^{\pm}(\alpha)$ as $\pm\alpha(Z) \cdot \text{id}$. The subspace \mathfrak{g}_k is the k -eigenspace with respect to $\text{ad}(iZ)$. Then the lemma follows (we remark that $\text{ad}(iZ) = 0$ on $\mathfrak{k}(0) \oplus i\mathfrak{a}$).

Q.E.D.

Theorem 5.2. For every real number c , $\sum_{k \in c\mathbb{Z}} \mathfrak{g}_k^*$ is the subalgebra which corresponds to $\Delta_Z^c := \{\alpha \in \Delta \mid \alpha(Z) \in c\mathbb{Z}\}$.

Proof. From Lemma 5.1, we have

$$\sum_{k \in c\mathbb{Z}} \mathfrak{g}_k^* = \mathfrak{k}(0) \oplus i\mathfrak{a} \oplus \sum_{\alpha(Z) \in c\mathbb{Z}} \mathfrak{g}(\alpha).$$

This coincides with the subalgebra which corresponds to Δ_Z^c .

Q.E.D.

If we understand Δ_Z^c , then the subalgebra $\sum_{k \in c\mathbb{Z}} \mathfrak{g}_k^*$ can be easily determined by Theorem 4.4. In general, it is not easy to determine Δ_Z^c . (We have to calculate $\alpha(Z)$ for every $\alpha \in \Delta$.) But one can see the Dynkin diagrams of the closed subsystems Δ_Z^c in case $c = 0$ or ν . Let $\{H^1, \dots, H^r\}$ be the dual basis of a simple root system $\{\alpha_1, \dots, \alpha_r\}$.

Proposition 5.3. ([16]) Let $Z := c_1 H^{i_1} + \dots + c_k H^{i_k}$, where $c_1, \dots, c_k > 0$, and $\nu := \tilde{\alpha}(Z)$. Then

- (i) $\Pi - \{\alpha_{i_1}, \dots, \alpha_{i_k}\}$ is a simple root system of Δ_Z^0 , and
- (ii) $(\Pi - \{\alpha_{i_1}, \dots, \alpha_{i_k}\}) \cup \{-\tilde{\alpha}\}$ is a simple root system of Δ_Z^ν .

For example, we consider the Lie algebra $e_{6(6)}$, which is the non-compact dual of $(e_6, sp(4))$. We write the extended Dynkin diagram of the restricted root system of the pair in Figure 1. The highest root $\tilde{\alpha}$ can be expressed as

$$\tilde{\alpha} = \alpha_1 + 2\alpha_2 + 2\alpha_3 + 3\alpha_4 + 2\alpha_5 + \alpha_6.$$

Since $\tilde{\alpha}(H^4) = 3$, H^4 defines the third kind graded Lie algebra.

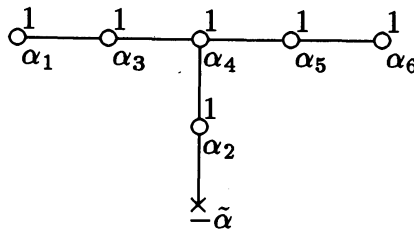


Figure 1 The extended Dynkin diagram of $(e_6, sp(4))$

Example 5.4. Let $e_{6(6)} = \sum_{k=-3}^3 \mathfrak{g}_k$ be the third kind graded Lie algebra defined by H^4 . Then

$$\mathfrak{g}_0 = \mathbf{R} \oplus sl(3, \mathbf{R}) \oplus sl(3, \mathbf{R}) \oplus sl(2, \mathbf{R}),$$

$$\mathfrak{g}_{-2} \oplus \mathfrak{g}_0 \oplus \mathfrak{g}_2 = \mathfrak{sl}(6, \mathbf{R}) \oplus \mathfrak{sl}(2, \mathbf{R}), \quad \text{and}$$

$$\mathfrak{g}_{-3} \oplus \mathfrak{g}_0 \oplus \mathfrak{g}_3 = \mathfrak{sl}(3, \mathbf{R}) \oplus \mathfrak{sl}(3, \mathbf{R}) \oplus \mathfrak{sl}(3, \mathbf{R}).$$

Proof. By Proposition 5.3, we obtain that $\Delta_{H^4}^0$ and $\Delta_{H^4}^3$ are the root systems of types $A_2 \oplus A_2 \oplus A_1$ and $A_2 \oplus A_2 \oplus A_2$ respectively, and all the multiplicities are 1. Furthermore, direct calculations show that $\Delta_{H^4}^2$ is of type $A_5 \oplus A_1$ with multiplicities 1. The restricted root system of $(\mathfrak{su}(n+1), \mathfrak{so}(n+1))$ is A_n with the multiplicities 1. Its non-compact dual is $\mathfrak{sl}(n+1, \mathbf{R})$. The corresponding subalgebras can be determined by Theorem 4.4. Q.E.D.

6. Relations to orbits

In this section, we mention relationships between semi-simple graded Lie algebras and the orbits of semi-simple symmetric spaces under the actions of the isotropy subgroups investigated in [16].

Let $(\mathfrak{g}, \mathfrak{k})$ be a compact semi-simple symmetric pair and C be the fundamental center lattice defined in Section 3. Let G be the connected Lie group with the Lie algebra \mathfrak{g} , and K be the connected subgroup of G with the Lie algebra \mathfrak{k} . Then the homogeneous space $M := G/K$ is Riemannian symmetric. We remark that the local orbit types of M only depends on the locally isomorphism class of M . Thus the choice of the locally isomorphism class of M is not important.

Let $\mathfrak{g}^* = \sum_{k=-\nu}^{\nu} \mathfrak{g}_k^*$ be the graded Lie algebra defined by $Z \in C$ (i.e., iZ is the characteristic element). Let us consider the closed geodesic $p(t) := \exp(t\pi Z)$, where \exp denotes the exponential map of M at the origin. We denote by $\mathfrak{k}_{p(t)}$ the Lie algebra of the isotropy subgroup of K at $p(t)$.

Theorem 6.1. $\mathfrak{k}_{p(t)} = \mathfrak{k} \cap \sum_{tk \in \mathbf{Z}} \mathfrak{g}_k^*$ for every $t \in \mathbf{R}$.

Proof. From Theorem 5.2, we have

$$\begin{aligned} \mathfrak{k} \cap \sum_{tk \in \mathbf{Z}} \mathfrak{g}_k^* &= \mathfrak{k} \cap \left(\mathfrak{k}(0) \oplus i\alpha \oplus \sum_{t\alpha(Z) \in \mathbf{Z}} \mathfrak{k}(\alpha) \oplus im(\alpha) \right) \\ &= \mathfrak{k}(0) \oplus \sum_{t\alpha(Z) \in \mathbf{Z}} \mathfrak{k}(\alpha) \\ &= \mathfrak{k}(0) \oplus \sum_{\alpha(t\pi Z) \in \pi\mathbf{Z}} \mathfrak{k}(\alpha). \end{aligned}$$

According to [17], this coincides with $\mathfrak{k}_{\exp(t\pi Z)}$.

Q.E.D.

From this theorem one can see that $\mathfrak{k}_{p(t)} = \mathfrak{k}_{p(t+1)}$ for every $t \in \mathbf{R}$. (We remark that $\mathfrak{k}_{p(0)} = \mathfrak{k}_{p(1)}$ does not imply $p(0) = p(1)$.)

In case $t = 0$, $\sum_{0 \cdot k \in \mathbf{Z}} \mathfrak{g}_k^* = \mathfrak{g}^* = \sum_{k \in \mathbf{Z}} \mathfrak{g}_k^*$. If $t \neq 0$, put $c := 1/t$. Then we have $\sum_{tk \in \mathbf{Z}} \mathfrak{g}_k^* = \sum_{k \in c\mathbf{Z}} \mathfrak{g}_k^*$, which was discussed in Section 5. Then the next corollary immediately follows from Theorem 6.1.

Corollary 6.2. *Let $Z \in C$. There exists a bijective correspondence between the set of the local orbit types of K -orbits through $\{\exp(tZ) \in M \mid t \in \mathbf{R}\}$, and $\{\sum_{k \in c\mathbf{Z}} \mathfrak{g}_k^* \mid c \in \mathbf{R}\}$.*

For example let us consider the case that Z defines the third kind graded Lie algebra $\mathfrak{g}^* = \sum_{k=-3}^3 \mathfrak{g}_k^*$. Corollary 6.2 leads that there are four local orbit types among the K -orbits through $\{\exp(tZ) \mid t \in \mathbf{R}\}$. The isotropy subalgebras $\mathfrak{k}_{p(t)}$ at $p(t) = \exp(t\pi Z)$ can be described from Theorem 6.1 as follows:

$$\mathfrak{k}_{p(t)} = \begin{cases} \mathfrak{k} & \text{if } t \in \mathbf{Z}, \\ \mathfrak{k} \cap (\mathfrak{g}_{-3} \oplus \mathfrak{g}_0^* \oplus \mathfrak{g}_3^*) & \text{if } t \in \frac{1}{3} + \mathbf{Z} \text{ or } \frac{2}{3} + \mathbf{Z}, \\ \mathfrak{k} \cap (\mathfrak{g}_{-2}^* \oplus \mathfrak{g}_0^* \oplus \mathfrak{g}_2^*) & \text{if } t \in \frac{1}{2} + \mathbf{Z}, \\ \mathfrak{k} \cap \mathfrak{g}_0^* & \text{if others.} \end{cases}$$

Example 6.3. Let $e_{6(6)} = \sum_{k=-3}^3 \mathfrak{g}_k$ be the third kind graded Lie algebra investigated in Example 5.4. Let $p(t) := \exp(t\pi H^4)$ be the geodesic in $E_6/Sp(4)$. Then

$$sp(4)_{p(t)} = \begin{cases} sp(4) & \text{if } t \in \mathbf{Z}, \\ sp(1) \oplus sp(1) \oplus sp(1) & \text{if } t \in \frac{1}{3} + \mathbf{Z} \text{ or } \frac{2}{3} + \mathbf{Z}, \\ u(4) & \text{if } t \in \frac{1}{2} + \mathbf{Z}, \\ sp(1) \oplus sp(1) \oplus so(2) & \text{if others.} \end{cases}$$

This is direct consequence of Example 5.4. Among them, the orbit through $p(\frac{1}{2})$ is locally isomorphic to the symmetric space $Sp(4)/U(4)$.

Corollary 6.4. *Let $p(t) := \exp(t\pi Z)$ be the geodesic with $Z \in C$. Then the K -orbit through $p(\frac{1}{2})$ is symmetric.*

Proof. Let $\mathfrak{g}^* = \sum_{k=-\nu}^{\nu} \mathfrak{g}_k^*$ be the graded Lie algebra defined by Z . Let $\mathfrak{g}_{ev} := \sum_{k: \text{even}} \mathfrak{g}_k$ and $\mathfrak{g}_{odd} := \sum_{k: \text{odd}} \mathfrak{g}_k$. Then we have $\mathfrak{k} = (\mathfrak{k} \cap \mathfrak{g}_{ev}) \oplus (\mathfrak{k} \cap \mathfrak{g}_{odd})$. Theorem 6.1 says that $\mathfrak{k}_{p(\frac{1}{2})} = \mathfrak{k} \cap \mathfrak{g}_{ev}$. Since $[\mathfrak{g}_{odd}, \mathfrak{g}_{odd}] \subset \mathfrak{g}_{ev}$, the pair $(\mathfrak{k}, \mathfrak{k}_{p(\frac{1}{2})})$ is symmetric. Q.E.D.

B.Y. Chen and T. Nagano ([4]) have mentioned the following theorem: *the K -orbit through p is symmetric, if p is the antipodal of the closed geodesic starting from the origin*. Corollary 6.4 gives the easy proof of their theorem. In fact, the isotropy subalgebra at the antipodal point p coincides with \mathfrak{k} or $\mathfrak{k} \cap \mathfrak{g}_{ev}$.

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