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MATRIX DUALITY FOR MATRIX ORDERED SPACES

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Abstract. In this paper, we study matricially Riesz normed spaces, approximate matrix order unit spaces, matrix order unit spaces and matrix base normed spaces and develop the duality theory for these spaces generalising the duality theory of Davies, Edwards, Ellis and Ng for (real) ordered Banach spaces.

Introduction

Davies [4], Edwards [5], Ellis [6] and Ng [10] have successfully developed a duality theory for (real) ordered Banach spaces, where they studied order unit normed spaces, base normed spaces, approximate order unit normed spaces and more generally Riesz normed spaces and proved that the dual of an order unit normed space is a base normed space; the dual of a base normed space is an order unit normed space; the dual of an approximate order unit normed space is a based normed space and finally the dual of a Riesz normed space is again a Riesz normed space.

We introduced matrix ordered version of abovementioned spaces viz, matricially Riesz normed spaces (mRn spaces), approximate matrix order spaces (amou spaces), matrix order unit spaces (mou spaces) and matrix base normed spaces (mbn spaces) in [8] and [9]. In this paper, we generalise the duality theory of (real) ordered Banach spaces to these spaces in the matrix ordered context. We prove that the matrix dual of an mRn space is again an mRn space; that of an amou space is an mbn space; that of an mbn space is an mou space. Further more, we show that the reverse process also holds for dual spaces in which the cones, defining the matrix order, are norm closed. We call this structure "the A-B-O structure".

Section I. (Prerequisite)

1.1. Given a complex vector space V, the space $M_n(V)$, of $n \times n$ matrices

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with entries from V, is a complex vector space in the entry-wise operations. For $[\alpha_{ij}] \in M_{m,n}$, $[v_{ij}] \in M_n(V)$ and $[\beta_{ij}] \in M_{n,m}$, we define

$$[\alpha_{ij}][v_{ij}][\beta_{ij}] = \left[\sum_{k,l=1}^{n} \alpha_{ik} v_{kl} \beta_{lj}\right] \in M_m(V).$$

For $v \in M_n(V)$ and $w \in M_m(V)$, we write,

$$v \oplus w = \begin{bmatrix} v & 0 \\ 0 & w \end{bmatrix} \in M_{n+m}(V).$$

A **matrix norm** on V is a sequence $\{\|\cdot\|_n\}$, where $\|\cdot\|_n$ is a norm on $M_n(V)$, for all $n \in \boldsymbol{N}$. A **matricially normed space (mn space)** is a complex vector space V, together with a matrix norm $\{\|\cdot\|_n\}$, which satisfies the following properties:

I. $\|v \oplus 0\|_{n+m} = \|v\|_n$, for all $v \in M_n(V)$, $0 \in M_m(V)$,
II. $\|\alpha v \beta\|_n \leq \|\alpha\| \|v\|_n \|\beta\|$, for all $\alpha, \beta \in M_n$, and $v \in M_n(V)$.

Now consider the following L^p-conditions:
$L^p : \|v \oplus w\|_{n+m}^p = \|v\|_n^p + \|w\|_m^p (1 \leq p < \infty)$,
$L^\infty : \|v \oplus w\|_{n+m} = \max\{\|v\|_n, \|w\|_m\}$; for all $v \in M_n(V)$, $w \in M_m(V), n, m \in \boldsymbol{N}$.

An **L^p-mn space** $(1 \leq p \leq \infty)$ is an mn space $(V, \{\|\cdot\|_n\})$ which satisfies the L^p-condition [13].

1.2. Given a $\star$-vector space V, $M_n(V)$ become a $\star$-vector, if we define $[v_{ij}]^* = [v_{ji}^*]$, for all $[v_{ij}] \in M_n(V)$. In this case $M_n(V)_{sa} = \{u \in M_n(V) : u = u^*\}$ is a real vector space for all $n \in \boldsymbol{N}$.

A **matrix ordered space** is a $\star$-vector space V together with a cone $M_n(V)^+$ in $M_n(V)_{sa}$ for every $n \in \boldsymbol{N}$ and with the following property: If $v \in M_n(V)^+$ and $\gamma \in M_{n,m}$, then $\gamma^* v \gamma = M_m(V)^+$. We shall denote it by $(V, \{M_n(V)^+\})$ [4].

1.3. Given a complex vector space V and a dual pair $\langle V, V^d \rangle$, we define, for each $n \in \boldsymbol{N}$, and $[v_{ij}] \in M_n(V)$, $[f_{ij}] \in M_n(V^d)$, $\langle [v_{ij}], [f_{ij}] \rangle = \sum_{L_{ij}}^{n} \langle v_{ij}, f_{ij} \rangle$ [3]. Then $\langle M_n(V), M_n(V^d) \rangle$ is a dual pair for every $n \in \boldsymbol{N}$. We shall call this duality to be the **matrix duality** of $\langle V\ V^d \rangle$.

In particular, if $(V, \{\|\cdot\|_n\})$ is an mn space and if $(V', \|\cdot\|_1')$ is the Banach dual of $(V, \|\cdot\|_1)$, then giving $M_n(V')$ the dual norm $\|\cdot\|_n'$ of $(M_n(V), \|\cdot\|_n)$ for all $n \in \boldsymbol{N}$ we get that $(V', \{\|\cdot\|_n'\})$ is also an mn space [13]. If, in addition, $(V, \{\|\cdot\|_n\})$ satisfies L^p-condition $(1 \leq p \leq \infty)$ then $(V', \{\|\cdot\|_n'\})$ satisfies L^q-

condition $\left(\frac{1}{p}+\frac{1}{q}=1\right)$ [13]. $(V', \{\|\cdot\|'_n\})$ will be called the **matrix Banach dual** of $(V, \{\|\cdot\|_n\})$.

1.4. Let us say, for a $\star$-vector space V and a dual pair $\langle V, V^d \rangle$, that V^d is **self adjoint** if $f^* \in V^d$ whenever $f \in V^d$ $\left[f^*(v) = \overline{f(v^*)}\right]$. Now assume that $(V, \{M_n(V)^+\})$ is a matrix ordered space then $(V^d, \{M_n(V^d)^+\})$ is also a matrix ordered space in the matrix duality, if we define for all $m \in \boldsymbol{N}$, $M_n(V^d)^+ = \{f \in M_n(V^d)_{sa} : f(v) \geq 0$ for all $v \in M_n(V)^+\}$. $(V^d, \{M_n(V^d)^+\})$ will be called a matrix ordered dual of $(V, \{M_n(V)^+\})$ [3].

In particular, the algebraic matrix dual of a matrix ordered space is a matrix ordered space.

1.5. Let $(V, \{\|\cdot\|_n\})$ be a norm complete mn space and assume that $(V, \|\cdot\|_1)$ has a predual $(V_*, \|\cdot\|_*)$. Since $(V, \|\cdot\|_1)$ is also the Banach dual of the completion of $(V_*, \|\cdot\|_*)$, we may assume, if necessary, that $(V_*, \|\cdot\|_*)$ is a Banach space. Now identifying V_* in V', we have $\|\cdot\|_* = \|\cdot\|'_1|_{V_*}$. Thus V_* together with the matrix norm $\{\|\cdot\|_{*,n}\}$ becomes a (norm complete) mn space where $\|\cdot\|_{*,n} = \|\cdot\|'_n|_{M_n(V_*)}$. $(V, \{\|\cdot\|_{*,n}\})$ will be called a **matrix (Banach) predual** of $(V, \{\|\cdot\|_n\})$. In this case, $(V, \{\|\cdot\|_n\})$ is called a **matrix dual Banach space**. In these notions, we conclude that if $(V, \{\|\cdot\|_n\})$ is a matrix dual L^p-matricially Banach space, $(1 \leq p \leq \infty)$. Then its matrix predual is an L^q-(norm complete) mn space $\left(\frac{1}{p}+\frac{1}{q}=1\right)$. Next assume that $(V, \{\|\cdot\|_m\})$ is a matrix dual Banach space with a matrix predual $(V_*, \{\|\cdot\|_{*,n}\})$, that $\{M_n(V)^+\}$ is a matrix order on V and that $\star$ is an isometry on $(M_n(V), \|\cdot\|_n)$ for every $n \in \boldsymbol{N}$. Further, assume that V_* is a self adjoint subspace when identified in V'. Then V_* is a matrix ordered space if we define $M_n(V_*)^+ = M_n(V')^+ \cap M_n(V_*)_{sa}$ for all $n \in \boldsymbol{N}$. Also then $\|f^*\|_{*,n} = \|f\|_{*,n}$ for all $f \in M_n(V_*)$ and $n \in \boldsymbol{N}$. The triple $(V_*, \{\|\cdot\|_{*,n}\}.$ $\{M_n(V_*)^+\})$ will be called **a matrix ordered Banach predual** (or simply **a matrix Banach predual**, if there is no confusion) of $(V, \{\|\cdot\|_n\}, \{M_n(V)^+\})$. Note that $M_n(V_*)^+$ is norm closed (or equivalently, weakly closed) and that its dual cone in $M_n(V)_{sa}$ is $\overline{M_n(V)^+}$ for all $n \in \boldsymbol{N}$.

1.6. In a matrix ordered space $(V, \{M_n(V)^+\})$ we define the following notions.

(a) V^+ is called **proper** if $V^+ \cap (-V^+) = \{0\}$.
(b) V^+ is called **generating** if for any $v \in V$ there are $u_1, u_2 \in V^+$ such that $\begin{bmatrix} u_1 & v \\ v^* & u_2 \end{bmatrix} \in M_2(V)^+$.

Remarks. (i) If V^+ is proper or generating, then so is $M_n(V)^+$, for all $n \in \boldsymbol{N}$ [8].

(ii) If V^+ is generating, we say that $(V, \{M_n(V)^+\})$ is a **positively generated** matrix ordered space.

1.7. Let V be positively generated. A seminorm $\|\cdot\|$ on V is called a *Riesz seminorm*, if for each $v \in V$

$$\|V\| = \inf\Big\{\max\{\|u_1\|, \|u_2\|\} : u_1, u_2 \in V^+, \begin{bmatrix} u_1 & v \\ v^* & u_2 \end{bmatrix} \in M_2(V)^+\Big\}.$$

It follows that

$$\|V\| = \inf\Big\{\|u_1\|^{1/2}\|u_2\|^{1/2} : u_1, u_2 \in V^+, \begin{bmatrix} u_1 & v \\ v^* & u_2 \end{bmatrix} \in M_2(V)\Big\}.$$

In fact, if $\begin{bmatrix} u_1 & v \\ v^* & u_2 \end{bmatrix} \in M_2(V)^+$, then $\begin{bmatrix} \lambda u_1 & v \\ v^* & \lambda^{-1}u_2 \end{bmatrix} \in M_2(V)^+$ for all $\lambda > 0$ so that

$$\inf_{\lambda>0}\Big\{\max\Big\{\|\lambda u_1\|, \|\lambda^{-1}u_2\|\Big\}\Big\} = \|u_1\|^{1/2}\|u_2\|^{1/2}.$$

Moreover,

$$\|u_1\|^{1/2}\|u_2\|^{1/2} \leq \frac{1}{2}\left(\|u_1\| + \|u_2\|\right) \leq \max\Big\{\|u_1\|, \|u_2\|\Big\}.$$

Thus, we have,

$$\|v\| = \inf\left\{\frac{1}{2}\Big(\|u_1\| + \|u_2\|\Big) : u_1, u_2 \in V^*, \begin{bmatrix} u_1 & v \\ v^* & u_2 \end{bmatrix} \in M_2(V)^+\right\}.$$

Next, it follows that for $v = v^*$,

$$\|v\| = \inf\Big\{\|u\| : u \in V^+ \text{ and } u \pm v \in V^+\Big\}.$$

In general, $\|v^*\| = \|v\|$ and $\|u_1\| \leq \|u_2\|$ whenever $0 \leq u_1 \leq u_2$.

1.8. Let $A \subset V$ and put

$$S(A) = \Big\{v \in V : \begin{bmatrix} u_1 & v \\ v^* & u_2 \end{bmatrix} \in M_2(V)^+ \text{ for some } u_1, u_2 \in A^+\Big\}.$$

If $A^+ \neq \phi$, then $S(A)$ is circled and self-adjoint. If, in addition, A is convex, then $S(A)$ is also convex. We say, A is solid, if $A = S(A)$. In this terminology, we have the following characterization of Riesz seminorms.

1.9. Lemma. *Let $(V, \{M_n(V)^+\})$ be a matrix ordered space and $\|\cdot\|$ a seminorm on V. Then V^+ is generating and $\|\cdot\|$ is a Riesz seminorm if and only if the open unit ball of $(V, \|\cdot\|)$ is solid* [8].

1.10. An **(L^p-) matricially Riesz normed space ((L^p-) mRn space)** $(1 \leq p \leq \infty)$ is an (L^p-) mn space $(V, \{\|\cdot\|_n\})$ together with a matrix order $\{M_n(V)^+\}$ such that $(V, \{M_n(V)^+\})$ is positively generated and $\{\|\cdot\|_n\}$ is a matrix Riesz norm on V.

1.11. Now we consider some special types of Riesz norms. Let $(V, \{M_n(V)^+\})$ be a matrix ordered space. An increasing net $\{e_\lambda\}_{\lambda \in D}$ in V^+ is called an **approximate order unit** for V if given $v \in V$ there are $\lambda \in D$ and $\alpha > 0$ such that $\begin{bmatrix} \alpha e_\lambda & v \\ v^* & \alpha e_\lambda \end{bmatrix} \in M_2(V)^+$.

For each $v \in V$, we define

$$\|v\|^a = \inf\left\{\alpha > 0 : \begin{bmatrix} \alpha e_\lambda & v \\ v^* & \alpha e_\lambda \end{bmatrix} \in M_2(V)^+ \text{ for some } \lambda \in D\right\}.$$

Then $\| \ \|^a$ is a Riesz semi norm on V [8]. It is called the approximate order unit semi norm on V determined by $\{e_\lambda\}$. In the following lemma we characterized approximate order unit norms among Riesz norms.

1.12. Lemma. *Let $(V, \{M_n(V)^+\})$ be a matrix ordered space and suppose $\|\cdot\|$ be a Riesz norm on V. Then $\|\cdot\|$ is an approximate order unit norm on V if and only if the positive part of U^+ the open unit ball of $(V, \|\cdot\|)$ is directed upwards. In this case U^+ is an approximate order for V* [8].

An approximate matrix order unit space (amou space) is an L^∞-mRn space $(V, \{\| \ \|_n\}, \{M_n(V)^+\})$ in which $\| \ \|_n$ is an approximate order unit norm on $M_n(V)$ for every $n \in \boldsymbol{N}$.

1.13. Let $(V, \{M_n(V)^+\})$ be a matrix ordered space. An element $e \in V^+$ is called an **order unit** for V if for every $v \in V$ there is $\alpha > 0$ such that $\begin{bmatrix} \alpha e_\lambda & v \\ v^* & \alpha e \end{bmatrix} \in M_2(V)^+$.

Thus an order unit is a special kind of approximate order unit, so all the properties of approximate order units also hold for order units. For $v \in V$,

$$\|v\|^0 = \inf\left\{\alpha > 0 : \begin{bmatrix} \alpha e & v \\ v^* & \alpha e \end{bmatrix} \in M_2(V)^+\right\}$$

defines a Riesz semi norm on V called the **order unit semi norm** on V determined by e. We have the following charactrization for an order unit norm.

1.14. Lemma. *Let $(V, \{M_n(V)^+\})$ be a matrix ordered space and suppose $\|\cdot\|$ is a Riesz norm on V. Then $\|\cdot\|$ is an order unit norm on V if and only if the positive part of the open unit ball of $(V, \|\cdot\|)$ is dominated by a positive element $e \in V^+$ with $\|e\| \leq 1$* [9].

1.15. A **matrix order unit space (mou space)** is an L^∞-mRn space $(V, \{\| \ \|_n\}, \{M_n(V)^+\})$ in which $\| \ \|_n$ is an order unit norms on $M_n(V)$ for all $n \in \boldsymbol{N}$.

Remark. For the sake of characterization, our definition of an mou space looks different from that of an mou space given by Choi and Effros in [3]. However, in a subsequent paper we show that the two defintions are equivalent in a certain sense.

1.16. Now we discuss another type of mRn space. Let $(V, \{M_n(V)^+\})$ be a matrix ordered space. A convex subset B of $V^+ \backslash \{0\}$ is called a **base** for V^+ if B has the following property: if $u \in V^+ \backslash \{0\}$ then there are unique $b \in B$ and $\alpha > 0$ such that $u = \alpha b$. Uniqueness of α and convexity of B determines a strictly positive linear functional f on V (*a linear functional on V will be called strictly positive if $f^* = f$, and $f(V^+ \backslash \{0\}) = (0, \infty)$*.)

It follows from [14] that the family of bases of V^+ and the set of strictly positive linear functionals are in a correspondence, which is one to one if V is positively generated.

1.17. Let V be positively generated and assume that B is a base for V^+. Let f be the corresponding strictly positive linear functional on V. For $v \in V$, define

$$\begin{aligned} \|v\|^b &= \inf\{\alpha > 0 : v \in \alpha S(B)\} \\ &= \inf\{\alpha > 0 : \begin{bmatrix} \alpha b & v \\ v^* & \alpha c \end{bmatrix} \in M_2(V)^+ \text{ for some } b, c \in B\} \\ &= \inf\{\max\{f(u_1), f(u_2)\} : u_1, u_2 \in V^+, \begin{bmatrix} u_1 & v \\ v^* & u_2 \end{bmatrix} \in M_2(V)^+\}. \end{aligned}$$

Then $\|\cdot\|^b$ is a Riesz seminorm on V called the *base seminorm* on V determined by the base B. Also for $u \in V^+$, we have $\|u\|^b = f(u)$. In fact, $\begin{bmatrix} u & u \\ u & u \end{bmatrix} \in M_2(V)^+$ implies that $\|u\|^b \leq f(u)$. If $\|u\| \leq f(u)$, we can find $u_1, u_2 \in V^+$ with $\begin{bmatrix} u_1 & u \\ u & u_2 \end{bmatrix} \in M_2(V)^+$ such that $\|u\|^b < \max\{f(u_1), f(u_2)\} < f(u)$. Now $u_1 + u_2 - 2u \in V^+$ so that $2f(u) \leq f(u_1) + f(u_2) \leq 2\max\{f(u_1), f(u_2)\}$, a contradiction. Thus $\|u\|^b = f(u)$. In particular, we have that $\|\cdot\|^b$ is additive on V^+ and $\|u\|^b \neq 0$ if $u \in V^+\setminus\{0\}$. These properties of base seminorms characterize them among Riesz seminorms.

1.18. Lemma. *Let $(V, \{M_n(V)^+\})$ be a positively generated matrix ordered space and suppose that $\|\cdot\|$ is a Riesz seminorm on V. Then $\|\cdot\|$ is a base seminorm if and only if $\|\cdot\|$ is additive on V^+ and $\|u\| \neq 0$ if $u \in V^+\setminus\{0\}$.*

1.19. A matricially base normed space (mbn space) is an L^1-mRn space $(V, \{\|\cdot\|_n\}, \{M_n(V)^+\})$ in which $\|\cdot\|_n$ is a base norm on $M_n(V)$ for every $n \in \boldsymbol{N}$ [9].

We also require the following result .

1.20. Proposition. *Let $(V, \{M_n(V)^+\})$ be a positively generated matrix ordered space and suppose that $\|\cdot\|$ is a base norm on V. Let B be the corresponding base of V^+. If U and Σ denote the open and the closed unit balls of $(V, \|\cdot\|)$, respectively, then*

$$U_{sa} \subset co(B \cup -B) \subset \Sigma_{sa}.$$

In particular, if B is closed, then

$$co(B \cup -B) = \Sigma_{sa}.$$

Proof. Let $v \in U_{sa}$, then there is $b \in B$ such that $b \pm v \in V^+$. If $v = u$ or $v = -u$, then $v \in co(B \cup -B)$. Assume, therefore, that $v \neq u$, $v \neq -u$. Put $b_1 = \|u+v\|^{-1}(u+v)$, $b_2 = \|u+v\|^{-1}(u-v)$. then $b_1, b_2 \in B$. Also $\|u+v\| + \|u-v\| = \|2u\| = 2$, so that

$$v = \frac{1}{2}\|u+v\|b_1 - \frac{1}{2}\|u-v\|b_2 \in co(B \cup -B).$$

Next, let $b_1, b_2 \in B$, $\alpha \in [0,1]$. Then

$$\|\alpha b_1 - (1-\alpha)b_2\| \leq \alpha + (1-\alpha) = 1$$

so that $\alpha b_1 - (1-\alpha)b_2 \in \Sigma_{sa}$. Hence, $U_{sa} \subset co(B \cup -B) \subset \Sigma_{sa}$. ■

Section II.

2.1. Let (V, V^+) be a real ordered vector space (i.e. a real vector space V, together with a cone V^+).

(a) A functional $p : V \longrightarrow \boldsymbol{R}$ is called a *sublinear functional*, if $p(v+w) \leq p(v)+p(w)$, and $p(\alpha v) = \alpha p(v)$, whenever $v, w \in V$, $\alpha \geq 0$.
(b) A functional $q : V \longrightarrow \boldsymbol{R}$ is called *superlinear* if $-q$ is sublinear.

The following theorem is obtained from [7].

2.2. Theorem. *Let (V, V^+) be a real ordered vector space and let U be a convex, absorbent subset of V. Suppose that f is a (real) linear functional on V such that for some $\alpha > 0$, $\sup\{f(x) : u \pm x \in V^+\} \leq \alpha$ for all $u \in U^+$. Then there exists $g \in V^*$ such that $-g \leq f \leq g$ and $\sup\limits_{x \in U} g(x) \leq \alpha$.*

2.3. Lemma. *Let $(V, \{M_n(V)^+\})$ be a matrix ordered space and let A be a solid, convex, absorbent subset of V. Then A^0 is solid and convex.*

Proof. Clearly, A^0 is self-adjoint, circled and convex. We show that A^0 is solid.

Let $f \in S(A^0)$. Then there exist $g_1, g_2 \in (A^0)^+$ such that $\begin{bmatrix} g_1 & f \\ f^* & g_2 \end{bmatrix} \in M_2(V^*)^+$. If $v \in A = S(A)$, there are $u_1, u_2 \in A^+$ such that $\begin{bmatrix} u_1 & v \\ v^* & u_2 \end{bmatrix} \in M_2(V)^+$. Choose $\theta \in [0, 2\pi]$ such that $|f(v)| = -e^{i\theta} f(v)$. Then $\begin{bmatrix} u_1 & e^{i\theta} v \\ e^{-i\theta} v^* & u_2 \end{bmatrix} \in M_2(V)^+$, so that

$$0 \leq \left\langle \begin{bmatrix} g_1 & f \\ f^* & g_2 \end{bmatrix}, \begin{bmatrix} u_1 & e^{i\theta} v \\ e^{-i\theta} v^* & u_2 \end{bmatrix} \right\rangle = g_1(u_1) + g_2(u_2) - 2|f(v)|.$$

Thus $2|f(v)| \leq g_1(u_1) + g_2(u_2) \leq 2$ and $f \in A^0$.

Conversely, let $f \in A^0$. If $f = 0$, then $f \in S(A^0)$. Assume that $f \neq 0$. Let $\sup\limits_{v \in A} |f(v)| = \alpha \leq 1$. Consider the real ordered vector space $(M_2(V)_{sa}, M_2(V)^+)$ and put $U = \left\{ \begin{bmatrix} u_1 & v \\ v^* & u_2 \end{bmatrix} : u_1, u_2 \in A_{sa}, v \in A \right\}$. Then U is a convex, absorbent subset of $M_2(V)_{sa}$ such that $S(U)_{sa} \subset U$. Put $\overline{f} = \begin{bmatrix} 0 & f \\ f^* & 0 \end{bmatrix} \in M_2(V^*)_{sa}$, so that $\sup\limits_{u \in U} \overline{f}(u) = 2\alpha$. Applying 2.2, there exists $\overline{g} \in M_2(V^*)_{sa}$ such that $-\overline{g} \leq \overline{f} \leq \overline{g}$ and $\sup\limits_{u \in U} \overline{g}(u) \leq 2\alpha$. Let $\overline{g} = \begin{bmatrix} g_1 & h \\ h^* & g_2 \end{bmatrix}$. Then $\begin{bmatrix} g_1 & h \pm f \\ h^* \pm f^* & g_2 \end{bmatrix} \in M_2(V^*)^+$ so that

$$\begin{bmatrix} g_1 & f \\ f^* & g_2 \end{bmatrix} = \frac{1}{2} \begin{bmatrix} g_1 & h+f \\ h^* + f^* & g_2 \end{bmatrix} + \frac{1}{2} \begin{bmatrix} g_1 & -h+f \\ -h^* + f^* & g_2 \end{bmatrix} \in M_2(V^*)^+.$$

Given $v \in A$, there are $u_1, u_2 \in A^+$ such that $\begin{bmatrix} u_1 & v \\ v^* & u_2 \end{bmatrix} \in M_2(V)^+$. Thus, as above, we have

$$\begin{aligned} 2|f(v)| &\leq g_1(u_1) + g_2(u_2) \\ &\leq \sup_{u_1 \in A_{sa}} g_1(u_1) + \sup_{u_2 \in A_{sa}} g_2(u_2) + 2 \sup_{v \in A} |h(v)| \\ &= \sup_{u \in U} \overline{g}(u) \leq 2\alpha. \end{aligned}$$

Since $\sup_{v \in A} f(v) = 2\alpha$, we conclude that $h = 0$, and that

$$\sup_{v \in A} |g_k(v)| = \sup_{u \in A_{sa}} g_k(u) = \sup_{u \in A^+} g_k(u) = \alpha_k \text{ (say)}$$

$k = 1, 2$ with $\alpha_1 + \alpha_2 = 2\alpha$. Now $\begin{bmatrix} g_1 & f \\ f^* & g_2 \end{bmatrix} \in M_2(V^*)^+$ implies $\begin{bmatrix} \lambda g_1 & f \\ f^* & \lambda^{-1} g_2 \end{bmatrix} \in M_2(V^*)^+$ for all $\lambda > 0$. We may calculate, as above, that $2\alpha \leq \lambda\alpha_1 + \lambda^{-1}\alpha_2$ for all $\lambda > 0$. Thus $\alpha_1 + \alpha_2 = 2\alpha \leq 2\alpha_1^{1/2}\alpha_2^{1/2} \leq \alpha_1 + \alpha_2$ and $\alpha_1 = \alpha_2 = \alpha \leq 1$. Hence $g_1, g_2 \in (A^0)^+$ so that $f \in S(A^0)$. Therefore, A^0 is solid. ■

2.4. Theorem. *Let $(V, \{M_n(V)^+\})$ be a positively generated matrix ordered space and let $\|\cdot\|$ be a Riesz norm on V. Consider the matrix Banach dual $(V', \{M_n(V')^+\})$. Then the closed unit ball of $(V', \|\cdot\|')$ is solid. In particular, $\|\cdot\|'$ is a Riesz norm on V'.*

Proof. Let U be the open unit ball of $(V, \|\cdot\|)$ so that U is solid, convex and absorbent in V. Thus, by 2.3, $\Sigma' = U^0$ is also solid, where Σ' denotes the closed unit ball of $(V', \|\cdot\|')$. Let U' be the open unit ball of $(V', \|\cdot\|')$. We show that U' is solid. $f \in U$, $f = 0$ implies $f \in S(U')$. Let $f \in U'$, $0 < \|f\|' = \alpha < 1$. Then $\alpha^{-1} f \in \Sigma'$. Since Σ' is solid, there are $g_1', g_2' \in \Sigma'^+$ such that $\begin{bmatrix} g_1' & \alpha^{-1} f \\ \alpha^{-1} f^* & g_2' \end{bmatrix} \in M_2(V')^+$. Put $g_1 = \alpha g_1', g_2 = \alpha g_2'$. Then $g_1, g_2 \in U'^+$ and $\begin{bmatrix} g_1 & f \\ f^* & g_2 \end{bmatrix} \in M_2(V)^+$. Thus $f \in S(U')$. Conversely, let $f \in S(U')$. Then there are $g_1, g_2 \in U'^+$ such that $\begin{bmatrix} g_1 & f \\ f^* & g_2 \end{bmatrix} \in M_2(V')^+$. Put $\alpha = \max\left\{\|g_1\|, \|g_2\|\right\} < 1$. Then $\begin{bmatrix} \alpha^{-1} g_1 & \alpha^{-1} f \\ \alpha^{-1} f^* & \alpha^{-1} g_2 \end{bmatrix} \in M_2(V')^+$ (assuming $\alpha \neq 0$) and $\max\left\{\|\alpha^{-1} g_1\|', \|\alpha^{-1} g_2\|'\right\} = 1$. Thus $\alpha^{-1} g_1, \alpha^{-1} g_2 \in \Sigma'^+$ and $\alpha^{-1} f \in S(\Sigma') = \Sigma'$. Hence $f \in U'$ and U' is solid. ■

2.5. Theorem. *Let $(V, \{M_n(V^*)^+\})$ be a positively generated matrix ordered space, $\|\cdot\|$ a Riesz norm on V and assume that $(V_*, \|\cdot\|_*)$ is a predual of*

$(V, \|\cdot\|)$ such that V_* is a self-adjoint subspace of V'. If $\|\cdot\|$ is a Riesz norm on V and if V^+ is norm-closed, then $\|\cdot\|_*$ is also a Riesz norm on V_*.

Proof. Consider $(V', \|\cdot\|')$ and $(V', \{M_n(V')^+\})$. Then, by 2.4, $\|\cdot\|'$ is a Riesz norm on V'. Identifying $(V_*, \|\cdot\|_*)$ in $(V', \|\cdot\|')$, we have $\|\cdot\|' = \|\cdot\|_*$ on V_*. Let U_* be the open unit ball of $(V_*, \|\cdot\|_*)$. We show that U_* is solid. Let $f \in S(U_*)$. Then $\begin{bmatrix} g_1 & f \\ f^* & g_2 \end{bmatrix} \in M_2(V_*)^+ \subset M_2(V')^+$, for some $g_1, g_2 \in U^+$. Since $\|\cdot\|'$ is a Riesz norm on V', we have, $\|f\|_* = \|f\|' \leq \max\{\|g_1\|', \|g_2\|'\}$ $= \max\{\|g_1\|_*, \|g_2\|_*\} < 1$. Thus $S(U_*) \subset U_*$. Conversely, let $f_0 \in U_*$. Put $\|f_0\|_* = \alpha = 1-\varepsilon$, for some $\varepsilon > 0$. We may assume that $f_0 \neq 0$. Consider the real ordered vector space $(M_2(V_*)_{sa}, M_2(V_*)^+)$ with the dual $(M_2(V)_{sa}, M_2(V)^+)$. For $f \in V_*$, $g_1, g_2 \in (V_*)_{sa}$, define $\left\| \begin{bmatrix} g_1 & f \\ f^* & g_2 \end{bmatrix} \right\|_2^* = \|g_1\|_* + \|g_2\|_* + 2\|f\|_*$. Then $\|\cdot\|_2^*$ is $\star$-norm on $M_2(V_*)_{sa}$ and its dual norm $\|\cdot\|_2$ on $M_2(V)_{sa}$ is given by

$$\left\| \begin{bmatrix} u_1 & v \\ v^* & u_2 \end{bmatrix} \right\|_2 = \max\{\|u_1\|, \|u_2\|, \|f\|\}.$$

Let A be the closed unit ball of $(M_2(V)_{sa}, \|\cdot\|_2)$. Then A is circled, convex and absorbent. We show that $S(A)_{sa} \subset A$. Let $\begin{bmatrix} u_1 & v \\ v^* & u_2 \end{bmatrix} \in S(A)_{sa}$. Then there is $\begin{bmatrix} u_1' & v' \\ v'^* & u_2' \end{bmatrix} \in A^+$ such that $\begin{bmatrix} u_1' & v' \\ v'^* & u_2' \end{bmatrix} \pm \begin{bmatrix} u_1 & v \\ v^* & u_2 \end{bmatrix} \in M_2(V)^+$. It follows that $\begin{bmatrix} u_1' & v \\ v^* & u_2' \end{bmatrix} \in M_2(V)^+$. In fact, $\begin{bmatrix} u_1' & v \\ v^* & u_2' \end{bmatrix} = \frac{1}{2} \begin{bmatrix} u_1' + u_1 & v' + v \\ v'^* + v^* & u_2' + u_2 \end{bmatrix} +$ $\frac{1}{2} \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \begin{bmatrix} u_1' - u_1 & v' - v \\ v'^* - v^* & u_2' - u_2 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$. Thus $\|v\| \leq \max\{\|u_1'\|, \|u_2'\|\} \leq 1$. Also, $u_k' \pm u_k \in V^+$ so that $\|u_k\| \leq \|u_k'\| \leq 1$, $k = 1, 2$. Hence $\begin{bmatrix} u_1 & v \\ v^* & u_2 \end{bmatrix} \in A$. Next, define $q_* : M_2(V)^+ \longrightarrow \boldsymbol{R}$ by

$q_*\left(\begin{bmatrix} u_1 & v \\ v^* & u_2 \end{bmatrix}\right) = \sup \left\langle \begin{bmatrix} u_1' & v' \\ v'^* & u_2' \end{bmatrix}, \begin{bmatrix} 0 & f_0 \\ f_0^* & 0 \end{bmatrix} \right\rangle$, where supremum is taken over $\begin{bmatrix} u_1' & v' \\ v'^* & u_2' \end{bmatrix} \in M_2(V)_{sa}$ with $\begin{bmatrix} u_1 & v \\ v^* & u_2 \end{bmatrix} \pm \begin{bmatrix} u_1' & v' \\ v'^* & u_2' \end{bmatrix} \in M_2(V)^+$

for every $\begin{bmatrix} u_1 & v \\ v^* & u_2 \end{bmatrix} \in M_2(V)^+$. If $\begin{bmatrix} u_1 & v \\ v^* & u_2 \end{bmatrix} \in A^*$ and $\begin{bmatrix} u_1' & v' \\ v'^* & u_2 \end{bmatrix} \in M_2(V)_{sa}$ is such that $\begin{bmatrix} u_1 & v \\ v^* & u_2 \end{bmatrix} \pm \begin{bmatrix} u_1' & v' \\ v'^* & u_2' \end{bmatrix} \in M_2(V)^+$. Then $\begin{bmatrix} u_1' & v' \\ v'^* & u_2' \end{bmatrix} \in S(A)_{sa} \subset A$. Thus

$$\left\langle \begin{bmatrix} 0 & f_0 \\ f_0^* & 0 \end{bmatrix}, \begin{bmatrix} u_1' & v' \\ v'^* & u_2' \end{bmatrix} \right\rangle = 2\left|\mathrm{Re}\ f_0(v)\right| \leq 2\|f_0\| = 2\alpha,$$

so that $q_*\left(\begin{bmatrix} u_1 & v \\ v^* & u_2 \end{bmatrix}\right) \le 2\alpha$ for every $\begin{bmatrix} u_1 & v \\ v^* & u_2 \end{bmatrix} \in A^+$. Hence q_* is well defined. Also,
$q_*\left(\alpha\begin{bmatrix} u_1 & v \\ v^* & u_2 \end{bmatrix}\right) = \alpha q_*\left(\begin{bmatrix} u_1 & v \\ v^* & u_2 \end{bmatrix}\right)$ for all $\begin{bmatrix} u_1 & v \\ v^* & u_2 \end{bmatrix} \in M_2(V)^+$ and $\alpha \ge 0$.
We show that q_* is super additive. Let $\begin{bmatrix} u_1 & v \\ v^* & u_2 \end{bmatrix}, \begin{bmatrix} u_1' & v' \\ v'^* & u_2' \end{bmatrix} \in M_2(V)^+$ and $\varepsilon > 0$. Then there are $\begin{bmatrix} x_1 & y \\ y^* & x_2 \end{bmatrix}, \begin{bmatrix} x_1' & y' \\ y' & x_2' \end{bmatrix} \in M_2(V)_{sa}$ such that

$$\begin{bmatrix} u_1 & v \\ v^* & u_2 \end{bmatrix} \pm \begin{bmatrix} x_1 & y \\ y^* & x_2 \end{bmatrix} \in M_2(V)^+, \begin{bmatrix} u_1' & v' \\ v'^* & u_2' \end{bmatrix} \pm \begin{bmatrix} x_1' & y' \\ y'^* & x_2' \end{bmatrix} \in M_2(V)^+;$$
$$q_*\left(\begin{bmatrix} u_1 & v \\ v^* & u_2 \end{bmatrix}\right) < \left\langle \begin{bmatrix} 0 & f_0 \\ f_0^* & 0 \end{bmatrix}, \begin{bmatrix} x_1 & y \\ y^* & x_2 \end{bmatrix} \right\rangle + \varepsilon/2,$$
$$q_*\left(\begin{bmatrix} u_1' & v' \\ v'^* & u_2' \end{bmatrix}\right) < \left\langle \begin{bmatrix} 0 & f_0 \\ f_0^* & 0 \end{bmatrix}, \begin{bmatrix} x_1' & y' \\ y'^* & x_2' \end{bmatrix} \right\rangle + \varepsilon/2.$$

Thus

$$\begin{aligned} q\left(\begin{bmatrix} u_1 & v \\ v^* & u_2 \end{bmatrix} + \begin{bmatrix} u_1' & v' \\ v'^* & u_2' \end{bmatrix}\right) &\ge \left\langle \begin{bmatrix} 0 & f_0 \\ f_0^* & 0 \end{bmatrix}, \begin{bmatrix} x_1 + x_1' & y + y' \\ y^* + y'^* & x_2 + x_2' \end{bmatrix} \right\rangle \\ &= \left\langle \begin{bmatrix} 0 & f_0 \\ f_0^* & 0 \end{bmatrix}, \begin{bmatrix} x_1 & y \\ y^* & x_2 \end{bmatrix} \right\rangle + \left\langle \begin{bmatrix} 0 & f_0 \\ f_0^* & 0 \end{bmatrix}, \begin{bmatrix} x_1' & y' \\ y'^* & x_2' \end{bmatrix} \right\rangle \\ &> q_*\left(\begin{bmatrix} u_1 & v \\ v^* & u_2 \end{bmatrix}\right) + q_*\left(\begin{bmatrix} u_1' & v' \\ v'^* & u_2' \end{bmatrix}\right) - \varepsilon. \end{aligned}$$

It follows that q_* is a super linear functional on $M_2(V)^+$. Next, we show that q_* is weak*-upper semi continuous on A^+. Assume to the contrary and let $\left\{\begin{bmatrix} u_1^\lambda & v^\lambda \\ v^{\lambda *} & u_2^\lambda \end{bmatrix}\right\}_\lambda$ be a net in A^+ converging to $\begin{bmatrix} u_1 & v \\ v^* & u_2 \end{bmatrix}$ in the weak*-topology and there is $k > 0$ such that $q_*\left(\begin{bmatrix} u_1^\lambda & v^\lambda \\ v^{\lambda *} & u_2^\lambda \end{bmatrix}\right) > k > q_*\left(\begin{bmatrix} u_1 & v \\ v^* & u_2 \end{bmatrix}\right)$ for every λ. Then, for each λ, we can find $\begin{bmatrix} x_1^\lambda & y^\lambda \\ y^{\lambda *} & x_2^\lambda \end{bmatrix} \in M_2(V)_{sa}$ with $\begin{bmatrix} u_1^\lambda & v^\lambda \\ v^{\lambda *} & u_2^\lambda \end{bmatrix} \pm \begin{bmatrix} x_1^\lambda & y^\lambda \\ y^{\lambda *} & x_2^\lambda \end{bmatrix} \in M_2(V)^+$ such that $\left\langle \begin{bmatrix} 0 & f_0 \\ f_0^* & 0 \end{bmatrix}, \begin{bmatrix} x_1^\lambda & y^\lambda \\ y^{\lambda *} & x_2^\lambda \end{bmatrix} \right\rangle > k$. Now, for each λ, $\begin{bmatrix} x_1^\lambda & y^\lambda \\ y^{\lambda *} & x_2^\lambda \end{bmatrix} \in S(A)_{sa} \subset A$ and A is weak*-compact, so that, on passing to a subnet, we may assume that $\begin{bmatrix} x_1^\lambda & y^\lambda \\ y^{\lambda *} & x_2^\lambda \end{bmatrix} \longrightarrow \begin{bmatrix} x_1 & y \\ y^* & x_2 \end{bmatrix}$ (weak*) in A. Thus

$$\begin{bmatrix} u_1 & v \\ v^* & u_2 \end{bmatrix} \pm \begin{bmatrix} x_1 & y \\ y^* & x_2 \end{bmatrix} = \text{weak}^* - \lim\left(\begin{bmatrix} u_1^\lambda & v^\lambda \\ v^{\lambda *} & u_2^\lambda \end{bmatrix} \pm \begin{bmatrix} x_1^\lambda & y^\lambda \\ y^{\lambda *} & x_2^\lambda \end{bmatrix}\right) \in M_2(V)^+$$

(being a dual cone, $M_2(V)^+$ is weak*-closed). Hence, by definition,

$$q_*\left(\begin{bmatrix} u_1 & v \\ v^* & u_2 \end{bmatrix}\right) \geq \left\langle \begin{bmatrix} 0 & f_0 \\ f_0^* & 0 \end{bmatrix}, \begin{bmatrix} x_1 & y \\ y^* & x_2 \end{bmatrix} \right\rangle$$
$$= \lim_\lambda \left(\begin{bmatrix} 0 & f_0 \\ f_0^* & 0 \end{bmatrix}, \begin{bmatrix} x_1^\lambda & y^\lambda \\ y^{\lambda *} & x_2^\lambda \end{bmatrix} \right) \geq k,$$

which contradicts the initial assumption. Hence q_* must be weak*-upper semi continuous on A^+. Consider

$$Q = \left\{ \begin{bmatrix} u_1 & v \\ v^* & u_2 \end{bmatrix} \in M_2(V)^+ : q_*\left(\begin{bmatrix} u_1 & v \\ v^* & u_2 \end{bmatrix}\right) \geq 2\alpha + \varepsilon \right\}.$$

Then Q is convex and weak*-closed. Also A is convex and weak*-compact (Banach-Alaoglu Theorem) with $A \cap Q = \emptyset$. Hence, by a geometric form of Hahn-Banach Theorem, we can find a weak*-continuous linear functional on $M_2(V)_{sa}$ (i.e. an element $\begin{bmatrix} g_1 & f \\ f^* & g_2 \end{bmatrix} \in M_2(V_*)_{sa}$) such that

$$\left\langle \begin{bmatrix} g_1 & f \\ f^* & g_2 \end{bmatrix}, \begin{bmatrix} u_1 & v \\ v^* & u_2 \end{bmatrix} \right\rangle < 2\alpha + \varepsilon, \text{ if } \begin{bmatrix} u_1 & v \\ v^* & u_2 \end{bmatrix} \in A$$

and

$$\left\langle \begin{bmatrix} g_1 & f \\ f^* & g_2 \end{bmatrix}, \begin{bmatrix} u_1 & v \\ v^* & u_2 \end{bmatrix} \right\rangle > 2\alpha + \varepsilon, \text{ if } \begin{bmatrix} u_1 & v \\ v^* & u_2 \end{bmatrix} \in Q.$$

Claim: $\left\langle \begin{bmatrix} g_1 & f \\ f^* & g_2 \end{bmatrix}, \begin{bmatrix} u_1 & v \\ v^* & u_2 \end{bmatrix} \right\rangle \geq q_*\left(\begin{bmatrix} u_1 & v \\ v^* & u_2 \end{bmatrix}\right) \geq 0$

for every $\begin{bmatrix} u_1 & v \\ v^* & u_2 \end{bmatrix} \in M_2(V)^+$.

Clearly, $q_*\left(\begin{bmatrix} u_1 & v \\ v^* & u_2 \end{bmatrix}\right) \geq 0$, for all $\begin{bmatrix} u_1 & v \\ v^* & u_2 \end{bmatrix} \in M_2(V)^+$. If $q_*\left(\begin{bmatrix} u_1 & v \\ v^* & u_2 \end{bmatrix}\right) > 0$, letting $\begin{bmatrix} u_1 & v \\ v^* & u_2 \end{bmatrix}_\varepsilon = (2\alpha + \varepsilon) q_*\left(\begin{bmatrix} u_1 & v \\ v^* & u_2 \end{bmatrix}\right)^{-1} \begin{bmatrix} u_1 & v \\ v^* & u_2 \end{bmatrix}$, we have,

$$q_*\left(\begin{bmatrix} u_1 & v \\ v^* & u_2 \end{bmatrix}\right) = 2\alpha + \varepsilon.$$

Thus $\begin{bmatrix} u_1 & v \\ v^* & u_2 \end{bmatrix} \in Q$ so that $\left\langle \begin{bmatrix} g_1 & f \\ f^* & g_2 \end{bmatrix}, \begin{bmatrix} u_1 & v \\ v^* & u_2 \end{bmatrix}_\varepsilon \right\rangle < 2\alpha + \varepsilon$. Therefore, $\left\langle \begin{bmatrix} g_1 & f \\ f^* & g_2 \end{bmatrix}, \begin{bmatrix} u_1 & v \\ v^* & u_2 \end{bmatrix} \right\rangle < q_*\left(\begin{bmatrix} u_1 & v \\ v^* & u_2 \end{bmatrix}\right)$.

Next, let $q_*\left(\begin{bmatrix} u_1 & v \\ v^* & u_2 \end{bmatrix}\right) = 0$. Fix $\begin{bmatrix} u_1' & v' \\ v'^* & u_2' \end{bmatrix} \in Q$. Then for any $n \in \boldsymbol{N}$,

$n\begin{bmatrix} u_1 & v \\ v^* & u_2 \end{bmatrix} + \begin{bmatrix} u_1' & v' \\ v'^* & u_2' \end{bmatrix} \in Q$, for q_* is super additive. Thus,
$\left\langle \begin{bmatrix} g_1 & f \\ f^* & g_2 \end{bmatrix}, n\begin{bmatrix} u_1 & v \\ v^* & u_2 \end{bmatrix} + \begin{bmatrix} u_1' & v' \\ v'^* & u_2' \end{bmatrix} \right\rangle \geq 2\alpha + \varepsilon$ and

$$\left\langle \begin{bmatrix} g_1 & f \\ f^* & g_2 \end{bmatrix}, \begin{bmatrix} u_1 & v \\ v^* & u_2 \end{bmatrix} \right\rangle + \frac{1}{n}\left\{ \left\langle \begin{bmatrix} g_1 & f \\ f^* & g_2 \end{bmatrix}, \begin{bmatrix} u_1' & v' \\ v'^* & u_2' \end{bmatrix} \right\rangle - (2\alpha + \varepsilon) \right\} \geq 0$$

for all $n \in \boldsymbol{N}$. Thus, $\left\langle \begin{bmatrix} g_1 & f \\ f^* & g_2 \end{bmatrix}, \begin{bmatrix} u_1 & v \\ v^* & u_2 \end{bmatrix} \right\rangle \geq 0$. Hence, our claim is proved.

This shows that $\begin{bmatrix} g_1 & f \\ f^* & g_2 \end{bmatrix} \in M_2(V)^+$. Also, by the definition of q_*, we have $\begin{bmatrix} g_1 & f \\ f^* & g_2 \end{bmatrix} \pm \begin{bmatrix} 0 & f_0 \\ f_0^* & 0 \end{bmatrix} \in M_2(V_*)^+$. We can show, as before, that $\begin{bmatrix} g_1 & f \\ f^* & g_2 \end{bmatrix} \in M_2(V_*)^+$. Now, let $v \in V$ with $\|v\| < 1$. Then there are $u_1, u_2 \in V^+$ with $\|u_k\| < 1$, $k = 1,2$ such that $\begin{bmatrix} u_1 & v \\ v^* & u_2 \end{bmatrix} \in M_2(V)^+$. It follows that $\begin{bmatrix} u_1 & e^{i\theta_v} \\ e^{-i\theta_v^*} & u_2 \end{bmatrix} \in M_2(V)^+$. Thus, choosing a suitable $\theta \in [0, 2\pi]$, we have,

$$0 \leq \left\langle \begin{bmatrix} g_1 & f_0 \\ f_0^* & g_2 \end{bmatrix}, \begin{bmatrix} u_1 & e^{i\theta_v} \\ e^{-i\theta_v^*} & u_2 \end{bmatrix} \right\rangle = g_1(u_1) + g_2(u_2) - 2|f_0(v)|.$$

Hence

$$\begin{aligned} 2\Big|f_0(v)\Big| &\leq g_1(u_1) + g_2(u_2) \\ &\leq \sup_{\substack{u_1' \in V_{sa} \\ \|u_1'\| \leq 1}} g_1(u_1') + \sup_{\substack{u_2' \in V_{sa} \\ \|u_2'\| \leq 1}} g_2(u_2') + \sup_{\substack{v' \in V \\ \|v'\| \leq 1}} (f_0(v') + \overline{f_0(v')}) \\ &= \sup\left\{ \left\langle \begin{bmatrix} g_1 & f_0 \\ f_0^* & g_2 \end{bmatrix}, \begin{bmatrix} u_1' & v' \\ v'^* & u_2' \end{bmatrix} \right\rangle : \begin{bmatrix} u_1' & v' \\ v'^* & u_2' \end{bmatrix} \in A \right\} \leq 2\alpha + \varepsilon. \end{aligned}$$

Since $\|f_0\| = \alpha$, we have, $2\alpha \leq \alpha_1 + \alpha_2 \leq 2\alpha + \varepsilon$, where $\alpha_k = \sup_{\substack{u_k' \in V_{sa} \\ \|u_k'\| \leq 1}} g_k(u_k') = \|g_k\|_*$, $k = 1, 2$.

Note that $\alpha_k \neq 0$, $k = 1, 2$. In fact, if $\alpha_1 = 0$, then $g_1 = 0$ so that $\begin{bmatrix} 0 & f_0 \\ f_0^* & g_2 \end{bmatrix} \in M_2(V_*)^+$. Being weakly closed, $M_2(V_*)^+$ is Archimedean. Also $M_2(V_*)^+ \subset M_2(V')^+$ and $M_2(V')^+$ is proper so that $M_2(V_*)^+$ is also proper, and hence almost-Archimedean. Thus $f_0 = 0$ a contradiction. Hence $\alpha_1 \neq 0$. Similarly, $\alpha_2 \neq 0$. Put $g_1^0 = \sqrt{\frac{\alpha_2}{\alpha_1}} g_1$, $g_2^0 = \sqrt{\frac{\alpha_1}{\alpha_2}} g_2$. Then $\|g_k^0\| = \sqrt{\alpha_1 \alpha_2}$, $k = 1, 2,$. Repeating the argument, given in the proof of 2.3, we can show that

$\begin{bmatrix} g_1^0 & f_0 \\ f_0^* & g_2^0 \end{bmatrix} \in M_2(V_*)^+$. Thus $\|g_1^0\|_* + \|g_2^0\|_* = 2\sqrt{\alpha_1\alpha_2} \leq \alpha_1 + \alpha_2 \leq 2\alpha + \varepsilon$ or equivalently, $\|g_k^0\|_* \leq \alpha + \varepsilon/2 = 1 - \varepsilon/2 < 1$, $k = 1, 2$. Hence $g_1^0, g_2^0 \in U_*^+$ whence $f_0 \in S(U_*)$. Therefore $S(U_*) = U_*$ and $\|\cdot\|_*$ is a Riesz norm on V_*. ■

2.6. Remark. (1) 2.5. can be regarded as the converse of 2.4. In fact, in 2.4., V'^+ , being a dual cone, is weak*-closed and hence norm closed.
(2) In 2.5, the closed unit ball of $(V, \|\cdot\|)$ is solid, which follows from 2.4.

2.7. Theorem. *Let $(V, \{M_n(V)^+\})$ be a matrix ordered space and suppose that $\|\cdot\|$ is an approximate order unit norm on V. Consider the matrix ordered Banach dual $(V', \{M_n(V')^+\})$. Then $\|\cdot\|'$ is a base norm on V'.*

Proof. In light of 1.18 and 2.4, it sufficies to show that $\|\cdot\|'$ is additive on V'^+. To see this, let $f, g \in V'^+$. Let $v \in V$, $\|v\| < 1$. Since $\|\cdot\|$ is Riesz norm, there are $u_1, u_2 \in V^+$ with $\|u_k\| < 1$, $k = 1, 2$ such that $\begin{bmatrix} u_1 & v \\ v^* & u_2 \end{bmatrix} \in M_2(V)^+$. Since $\|\cdot\|$ is an approximate order unit norm, we can find $u \in V^+$, $\|u\| < 1$, such that $u_1 \leq u$, $u_2 \leq u$. Then $\begin{bmatrix} u & v \\ v^* & u \end{bmatrix} \in M_2(V)^+$. Choose $\theta \in [0, 2\pi]$ such that $|f(v)| = -e^{i\theta} f(v)$. Then $\begin{bmatrix} u & e^{i\theta}v \\ e^{-i\theta}v^* & u \end{bmatrix} \in M_2(V)^+$. Now $\begin{bmatrix} f & f \\ f & f \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \end{bmatrix} (f) [1 \quad 1] \in M_2(V')^+$ so that $0 \leq \left\langle \begin{bmatrix} f & f \\ f & f \end{bmatrix}, \begin{bmatrix} u & e^{i\theta}v \\ e^{-i\theta}v^* & u \end{bmatrix} \right\rangle = 2f(u) - 2|f(v)|$.
Thus $|f(v)| \leq f(u)$. Similarly, given $v' \in V$ with $\|v'\| < 1$, we can find $u' \in V'$ with $\|u'\| < 1$ such that $|g(v)' \leq g(u')$. Choose $u_0 \in V^+$, $\|u_0\| < 1$, such that $u \leq u_0$, $u' \leq u_0$. Then

$$|f(v)| + |g(v)'| \leq f(u) + g(u') \leq (f + g)(u_0) \leq \|f + g\|'.$$

It follows that $\|f\|' + \|g\|' \leq \|f + g\|' \leq \|f\|' + \|g\|'$. Hence $\|\cdot\|'$ is additive on V'^+. ■

2.8. Theorem. *Let $(V, \{M_n(V)^+\})$ be a positively generated matrix ordered space and suppose that $\|\cdot\|$ is a base norm on V and that V^+ is norm-closed. Assume that $(V_*, \|\cdot\|_*)$ is a predual of $(V, \|\cdot\|)$ such that V_* is a self-adjoint subspace of V'. Then $\|\cdot\|_*$ is an approximate order unit norm on V_*.*

Proof. By 2.5, $\|\cdot\|_*$ is a Riesz norm on V_* . Hence, by 1.12, it is enough to show that if $f_1 + f_2 \in V_*^+$ with $\|f_k\|_* < 1$, $k = 1, 2$, then there is $f \in V_*^+$ with $\|f\|_* < 1$ such that $f_1 < f, f_2 \leq f$.

Let $\alpha = \max\left\{\|f_1\|_*, \|f_2\|_*\right\} = 1 - \varepsilon$, for some $\varepsilon > 0$. Define $q : V^+ \longrightarrow \boldsymbol{R}$ given by

$$q(u) = \sup\left\{f_1(u_1) + f_2(u_2) : u_1, u_2 \in V^+, u_1 + u_2 = u\right\}.$$

Let $u \in V^+$. If $u = u_1 + u_2$ for some $u_1, u_2 \in V^+$, then

$$\begin{aligned} f_1(u_1) + f_2(u_2) &\leq \|f_1\|_* \|u_1\| + \|f_2\|_* \|u_2\| \\ &\leq \alpha\left(\|u_1\| + \|u_2\|\right) = \alpha\|u_1 + u_2\| = \alpha\|u\|. \end{aligned}$$

Thus $q(u) \leq \alpha\|u\|$, for all $u \in V^+$. As in 2.5, we can show that q is a super linear functional on V^+ and that it is weak*-upper semi-continuous on Σ^+, where Σ is the closed unit ball of V. Consider the real ordered vector space (V_{sa}, V^+) and put $\Sigma_{sa} = \Sigma \cap V_{sa}$. Then Σ_{sa} is weak* compact convex in V_{sa} . Put $Q = \{u \in V^+ : q(u) \geq \alpha + \varepsilon/2\}$. Then Q is convex and weak*-closed. Hence we can find a weak*-continuous linear functional on V_{sa} (i.e. an element f in (V_{sa})) such that $f\,|\Sigma_{sa} < \alpha + \varepsilon/2, f|\,Q > \alpha + \varepsilon/2$. Using arguments parallel to those given in the proof of 2.5, we can show that $f \in (V_*)^+$, $\|f\|_* \leq \alpha + \varepsilon/2 < 1$. Also, it follows from the definition of q that $f_1 \leq f$, $f_2 \leq f$. Hence $\|\cdot\|_*$ is an approximate order unit norm on V_*. ■

2.9. Theorem. *Let $(V, \left\{M_n(V)^+\right\})$ be a positively generated matrix order space and suppose that $\|\cdot\|$ is a base norm on V. Then the dual norm on V' is an order unit norm.*

Proof. It only remains to show, from 1.14 , that there is a positive element g in V'^+ with $\|g\|' \leq 1$ which dominates the positive part of the open unit ball of V'. Let g be the strictly positive linear functional on V corresponding to the base norm $\|\cdot\|$. Let $v \in V$ with $\|v\| < 1$, then we can find $u_1, u_2 \in V^+$ with $\|u_k\| < 1$, $k = 1, 2$ such that $\begin{bmatrix} u_1 & v \\ v^* & u_2 \end{bmatrix} \in M_2(V)^+$. Choose $\theta \in [0, 2\pi]$ such that $|g(v)| = -e^{i\theta} g(v)$. Then, $\begin{bmatrix} u_1 & e^{i\theta} v \\ e^{-i\theta} v^* & u_2 \end{bmatrix} \in M_2(V)^+$ and we have,

$$0 \leq \left\langle \begin{bmatrix} g & g \\ g & g \end{bmatrix}, \begin{bmatrix} u_1 & e^{i\theta} v \\ e^{-i\theta} v^* & u_2 \end{bmatrix} \right\rangle = g(u_1 + u_2) - 2|g(v)|.$$

Thus, $|g(v)| \leq \frac{1}{2} g(u_1 + u_2) = \frac{1}{2}\|u_1 + u_2\| < 1$ and we have, $\|g\|' \leq 1$. Next, let $f \in V'^+$ with $\|f\|' \leq 1$. Then for any $u \in V^+$, we have, $f(u) \leq \|f\|'\|u\| \leq \|u\| = g(u)$, by 1.13. Thus $f \leq g$. Hence the theorem is proved. ■

2.10. Remarks. In the above proof, we note that $\|g\|' = 1$ and that g dominates the positive part of the closed unit ball of $(V', \|\cdot\|')$.

2.11. Theorem. *Let $(V, \{M_n(V)^+\})$ be a matrix ordered space and suppose that $\|\cdot\|$ is an order unit norm on V such that V^+ is norm-closed. Assume that $(V_*, \|\cdot\|_*)$ is a predual of $(V, \|\cdot\|)$ such that V_* is a self-adjoint subspace of V'. Then $\|\cdot\|_*$ is a base norm on V_*.*

Proof. By 2.5, $\|\cdot\|_*$ is a Riesz norm on V_*. Since an order unit is a particular type of an approximate order unit, it follows from 2.7, that $\|\cdot\|'$ is a base norm on V'. Also, using 1.18, it sufficies to show that $\|\cdot\|_*$ is additive on $(V_*)^+$. This is now evident, if we identify V_* in V'. ■

2.12. Theorem. *Let $(V, \{M_n(V)^+\})$ be a matrix ordered space and suppose that $\|\cdot\|$ is an order unit norm on V. Then the dual norm $\|\cdot\|'$ is a base norm on $(V', \{M_n(V')^+\})$ and the corresponding base is weak*-compact.*

Proof. It follows, from 2.7, that $\|\cdot\|'$ is a base norm on $(V', \{M_n(V')\})$. Let B be the corresponding base of V'^+. Then $B = \{f \in V'^+ : \|f\|' = 1\}$. Note that $\|f\|' = f(e)$, if $f \in V'^+$, where e is the order unit of V corresponding to $\|\cdot\|$. To see this, let $v \in V$, $\|v\| < 1$. Then $\begin{bmatrix} e & v \\ v^* & e \end{bmatrix} \in M_2(V)^+$. As before, we can show that $|f(v)| \leq f(e) \leq \|f\|'$. Hence, taking supremum over $v \in V$, $\|v\| < 1$, we have, $\|f\|' = f(e)$. Thus $B = \{f \in V'^+ : \|f\|' \leq 1, f(e) = 1\}$. Hence B is weak*-compact. ■

2.13. Theorem. *Let $(V, \{M_2(V)^+\})$ be a positively generated matrix ordered space and suppose that $\|\cdot\|$ is a base norm on V corresponding to the base B and that V^+ is norm closed. Let $(V_*, \|\cdot\|_*)$ be a predual of $(V, \|\cdot\|)$ such that V_* is a self-adjoint subspace of V'. Then $\|\cdot\|_*$ is an order unit norm on $(V_*, \{M_n(V_*)^+\})$, if B is weak*-compact.*

Proof. It follows, from 2.5, that $\|\cdot\|_*$ is a Riesz norm on V_*. It sufficies to show that there is g in V_*^+ with $\|g\|_* \leq 1$ such that $f \leq g$ for all $f \in V_*^+$ with $\|f\|_* < 1$. By 2.9, the dual norm $\|\cdot\|'$ on V' is an order unit norm. Let $g \in V'^+$ be the corresponding order unit for V'. Then, $\|g\|' \leq 1$ and $f \leq g$ for all $f \in V'^+$ with $\|f\|' < 1$. We show that g is weak*-continuous on V. Since B is weak*-compact it follows from 1.23, that $co(B \cup -B) = \Sigma_{sa}$, where Σ is the closed unit ball of $(V, \|\cdot\|)$ and $\Sigma_{sa} = \Sigma \cap V_{sa}$. Also, we have, $\frac{1}{2}B - \frac{1}{2}B = \Sigma_{sa} \cup g^{-1}(0)$. In fact, if $v \in \Sigma_{sa} \cap g^{-1}(0)$, then $v = \alpha b_1 - \beta b_2$ with $b_1, b_2 \in B$, $\alpha + \beta = 1$. Also $0 = g(v) = \alpha g(b_1) - \beta g(b_2) = \alpha - \beta$, so that $\alpha = \beta = \frac{1}{2}$. Thus $v \in \frac{1}{2}B - \frac{1}{2}B$. Now, that $\frac{1}{2}B - \frac{1}{2}B \subset \Sigma_{sa} \cap g^{-1}(0)$, is trivial. Since B is weak*-compact, $\Sigma_{sa} \cap g^{-1}(0)$ is also weak*-compact. Hence, by Krien-Smullyan Theorem, $g^{-1}(0)$ is a weak*-closed subspace of V_{sa}. Since V_{sa} is weak*-closed in V, g is weak*-continuous on

V and hence $g \in V_*^+$. Now the theorem is proved. ■

2.8. Remarks. (A) Let H be a complex Hilbert space. Suppose R is an ultra weakly closed operator system in $B(H)$. Then R has a predual $R_\star$ consisting of ultra weakly continuous linear functional on $B(H)$ restricted to R; or more precisely, $R_\star$ is identified with $B(H)_\star/R^\perp$ where $B(H)_\star$ is the space of ultra weakly continuous linear functionals on $B(H)$ [12]. Since R satisfies all the conditions of the converse part of the theorem 2.5, $R_\star$ is an mbn space. In particular, $B(H)_\star$ is an mbn space.

(B) Now let H be an infinite dimensional complex Hilbert space, so that $K(H)$, the space of compact operators on H, does not contain the unit element. In other words $K(H)$ is a strict amou space. Now corollary 2.7 shows that $K(H)$ is not a dual space. More precisely, $K(H)$ is a dual space if and only if it contains the identity of $B(H)$, which in turn is equivalent to the fact that H is finite-dimensional, in which case $K(H) = B(H)$.

(C) We generalize the situation by noting that every $C^\star$-algebra is an amou space (c.f. [8, 2.12]). Thus if a $C^\star$-algebra is a matrix dual space it is a dual mou space or equivalently, an ultraweakly closed self adjoint subalgebra of $B(H)$ or equivalently a von Neumann subalgebra of $B(H)$ for some complex Hilbert space H.

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