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# THREE-CYCLE REVERSIONS IN ORIENTED PLANAR TRIANGULATIONS

By

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**Abstract.** A planar *triangulation* $G$ is a simple graph embedded in the plane so that each face of $G$ is triangular and that any two faces share at most one edge. A $(*)$-*orientation* $D^*(G)$ of $G$ is an orientation of $G$ such that the outdegree of each vertex on $\partial G$ is 1 and that of each vertex not on $\partial G$ is 3, where $\partial G$ denotes the outer 3-cycle of $G$. In this paper, we shall show that for any planar triangulation $G$, there exists at least one $(*)$-orientation and that any two $(*)$-orientations of $G$ can be transformed into each other by a sequence of 3-cycle reversions, where the *3-cycle reversion* is a transformation in an oriented graph which replaces an oriented 3-cycle with the one with the inverse orientation. Finally, we shall show that in order to transform two $(*)$-orientations of $G$, we need at most $\left\lfloor \frac{1}{2}n^2 - 5n + \frac{27}{2} \right\rfloor$ 3-cycle reversions, where $n = |V(G)|$. The order of our estimation cannot be improved.

## 1. Introduction

Let $G$ be a graph. We denote the sets of the vertices and the edges of $G$ by $V(G)$ and $E(G)$, respectively. We also denote the degree of $x \in V(G)$ by $\deg_G(x)$, the set of the neighbors of $x$ in $G$ by $N_G(x)$, and the distance in $G$ of $x, y \in V(G)$ by $d_G(x, y)$. In this paper, we deal with only *simple graphs*, that is, graphs with no loops and no multiple edges. When a graph $G$ is embedded in the plane, $G$ is called the *plane graph*. We denote the set of the faces of $G$ by $F(G)$. For a plane graph $G$, there is a closed walk $W$ of $G$ such that each edge of $W$ is incident with the outer infinite face of $G$. We call such a closed walk the *outer closed walk* of $G$. In particular, when all the vertices of $W$ are distinct, $W$ is called the *outer cycle* of $G$.

Let $G$ be a plane graph. The union of faces $f_1, \ldots, f_m$ of $G$ is called the *region* of $G$ if the subgraph of the dual $G^*$ of $G$ induced by $\{f_1^*, \ldots, f_m^*\}$ is connected, where $f_i^*$ is the vertex of $G^*$ corresponding to $f_i$ for $i = 1, \ldots, m$. If the closure of a region $R$ is homeomorphic to a disk, then it is called the *closed 2-cell region*

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of $G$. Then its boundary consists of precisely one cycle. Otherwise, $R$ has at least two boundary cycles, say $C_1, \ldots, C_l$, where $C_i$ and $C_j$ might share at most one vertex for $1 \leq i < j \leq l$. In particular, when $C_1 \cup \cdots \cup C_l$ is connected, $R$ is called an *open 2-cell region*. In this case, since $G$ is embedded in the plane, there is precisely one cycle, say $C_1$, which includes the others, $C_2, \ldots, C_l$, inside. We call $C_1$ the *outer boundary cycle* of $R$, and call each of $C_2, \ldots, C_l$ the *inner boundary cycle* of it. By the definition of regions, the inside of an inner boundary cycle is empty. Note that among $\{C_2, \ldots, C_l\}$, $(\mathrm{Int}\, C_i) \cap (\mathrm{Int}\, C_j) = \emptyset$ for any $i$ and $j$. Otherwise, $C_i$ and $C_j$ were not contained in the boundary of $R$.

A *triangulation* $G$ of a closed surface $F^2$ is a simple graph embedded in $F^2$ so that each face of $G$ is triangular and that any two faces share at most one edge. In this paper, we shall mainly deal with triangulations of the plane. We simply call them *planar triangulations*.

Let $D$ be an oriented graph and $x, y \in V(D)$. We denote an oriented edge from $x$ to $y$ by $xy$. Here, we have to notice that $xy \neq yx$ in $D$. The *outdegree* of $x \in V(D)$ is the number of outgoing edges from $x$ and denoted by $\mathrm{od}_D(x)$. The *indegree* of $x$ is the number of incoming edges into $x$ and denoted by $\mathrm{id}_D(x)$. Clearly, $\sum_{v \in V(D)} \mathrm{od}_D(v) = \sum_{v \in V(D)} \mathrm{id}_D(v) = |E(D)|$.

Let $G$ be a planar triangulation with three vertices $v_1, v_2$ and $v_3$ on the outer cycle of length 3. The *$(*)$-oriented triangulation* $D^*(G)$ of the plane is defined as an oriented triangulation with the underlying graph $G$ such that $\mathrm{od}_{D^*(G)}(v_i) = 1$ $(i = 1, 2, 3)$ and $\mathrm{od}_{D^*(G)}(v) = 3$ for any vertex $v \neq v_i$. We also call such an oriented planar triangulation simply a *$(*)$-orientation*.

Let $D$ be an oriented graph and $C := v_1 v_2 \cdots v_n v_1$ an oriented cycle in $D$. The *cycle reversion* of $C$ in $D$ is defined to replace the oriented edges $v_1v_2, v_2v_3, \ldots, v_nv_1$ with $v_2v_1, v_3v_2, \ldots, v_1v_n$, respectively. In particular, the cycle reversion of a cycle of length $n$ is said to be the *$n$-cycle reversion*. Note that a cycle reversion in $D$ changes neither outdegree nor indegree of each vertex of $D$.

In this paper, we first show the following two theorems in Section 2.

**Theorem 1.** *Every planar triangulation has at least one $(*)$-orientation.*

**Theorem 2.** *Any two $(*)$-orientations of a fixed planar triangulation can be transformed into each other by a sequence of 3-cycle reversions.*

Let $G$ be a planar triangulation and let $D_1^*(G), D_2^*(G), \ldots, D_k^*(G)$ be all the $(*)$-orientations of $G$. Define $d(G, D_i^*, D_j^*)$ to be the minimum number of 3-cycle reversions needed to transform $D_i^*(G)$ and $D_j^*(G)$ into each other. Let

$$d(G) := \max\{d(G, D_i^*, D_j^*) \ : \ 1 \leq i < j \leq k\}.$$

In Section 3, we shall estimate the value of $d(G)$ by $|V(G)|$, as follows.

**Theorem 3.** *Let $G$ be a planar triangulation with $n$ vertices where $n \geq 7$. Then,*

$$d(G) \leq \left\lfloor \frac{1}{2}n^2 - 5n + \frac{27}{2} \right\rfloor.$$

To get a good estimation, we shall introduce an invariant for the difference between given two $(*)$-orientations of a fixed planar triangulation, which is equal to the minimum number of 3-cycle reversions needed to transform the two $(*)$-orientations.

In Section 4, with respect to Theorem 3, we shall actually construct a planar triangulation $T$ with $n \geq 7$ vertices and two $(*)$-orientations $D_1^*(T)$ and $D_2^*(T)$ such that

$$d(T, D_1^*, D_2^*) = \frac{1}{3}n^2 - 3n + \frac{23}{3},$$

which gives a lower bound for the estimation in Theorem 3. Thus, the order of our estimation of $d(G)$ for a planar triangulation $G$ cannot be improved though the coefficient does not seem to be the best.

The $(*)$-orientation of a *planar quadrangulation* (i.e., a simple plane graph whose faces are all quadrilateral) has already been defined, similarly to that of a planar triangulation, so that the outdegree of each vertex on the outer boundary cycle is 1 and those of other vertices are all 2 [1]. In the same paper, they pointed out that there is some relation between the $(*)$-orientations of planar quadrangulations and the *orthogonal plane partitions*. An orthogonal plane partition is a partition of a square into rectangles by horizontal and vertical segments, as shown in Figure 1. Notice that any two segments, except the four segments of the outer square, intersect in the orthogonal plane partition if and only if one of the endpoints of one segment coincides with an inner point of the other segment. Corresponding a vertex to each segment of an orthogonal partition $S$, we can make a digraph $D_S$ from $S$ such that if an endpoint of a segment $x_i$ and an inner point of a segment $x_j$ coincides, then we put a directed edge from the vertex $x_i$ to the vertex $x_j$. For the outer square of $S$, we suppose that the oriented cycles of length 4 corresponds to it in $D_S$. Then, the resulting digraph $D_S$ is a $(*)$-orientation of a planar quadrangulation. (Each inner vertex in $D_S$ has outdegree 2 since each inner segment has both endpoints on two other distinct segments.) Here, we do not describe the details for them.

In the present paper, we shall show that the phenomena of the $(*)$-orientations of planar quadrangulations, shown in [1], also hold for the $(*)$-orientations of planar triangulations, and estimate the minimum number of 3-cycle reversions

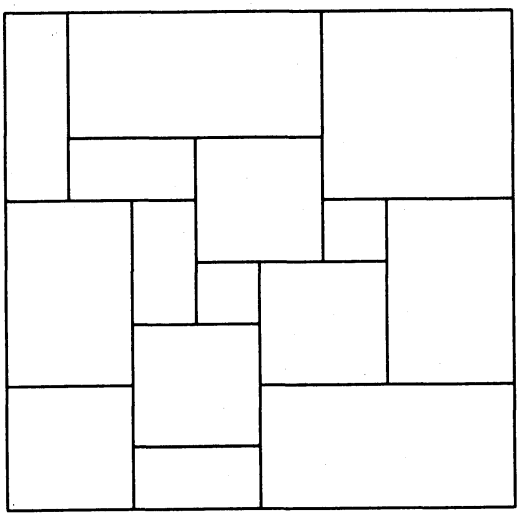

**Figure 1** An orthogonal plane partition

needed to transform given two (∗)-orientations of planar triangulations. By this, one can easily estimate the minimum number of 4-cycle reversions needed to transform given two (∗)-orientations of planar quadrangulations. Moreover, one will be able to find a relation between (∗)-orientations of planar triangulations and some partition of a triangle by several triangular disks.

## 2. Proof of Theorems 1 and 2

Let $G$ be a triangulation of a closed surface $F^2$ and $e$ an edge of $G$. The *contraction* of $e$ is defined to delete $e$, identify its two ends and replace the two pairs of multiple edges with two single edges, respectively. If the contraction of $e$ breaks the simpleness of a graph, then we do not apply it, and otherwise, $e$ is said to be *contractible*. When $e$ is contractible, the resulting graph obtained by the contraction of $e$ is also a triangulation. A triangulation $G$ is said to be *contractible* to a triangulation $T$ if $T$ is obtained from $G$ by a sequence of contractions of edges.

The following result proved by Steinitz and Rademacher is useful to establish the (∗)-orientability of planar triangulations.

**Lemma 4. (Steinitz and Rademacher [2])** *Every planar triangulation is contractible to* $K_4$.

We shall show Theorem 1.

**Proof of Theorem 1.** Let $G$ be a planar triangulation with $|V(G)| = n$. We use induction on $|V(G)|$. If $G$ has no contractible edge, that is, $G$ is isomorphic to $K_4$ by Lemma 4, then $G$ clearly has a $(*)$-orientation. Suppose that $G$ has a contractible edge $e$. By the contraction of $e$, $G$ is deformed into the triangulation, denoted by $G/e$, with $|V(G/e)| = n-1$. In this case, the union of two faces of $G$ sharing $e$ is deformed into a path of length 2. By the hypothesis of induction, $G/e$ has a $(*)$-orientation and the path has some orientation. By applying one of the operations in Figure 2 depending on the orientation of the edges incident with $[e] \in V(G/e)$ which is the contraction image of $e$ in $G$, we can construct a $(*)$-orientation of $G$. (In Figure 2, (1) represents the case when none of the three edges starting from $[e]$ are toward the right.) Even if the outdegree of $[e]$ is 1 (i.e., $[e]$ is on the boundary cycle of $G/e$), we can conclude the similar argument as above. Thus, the theorem follows. ■

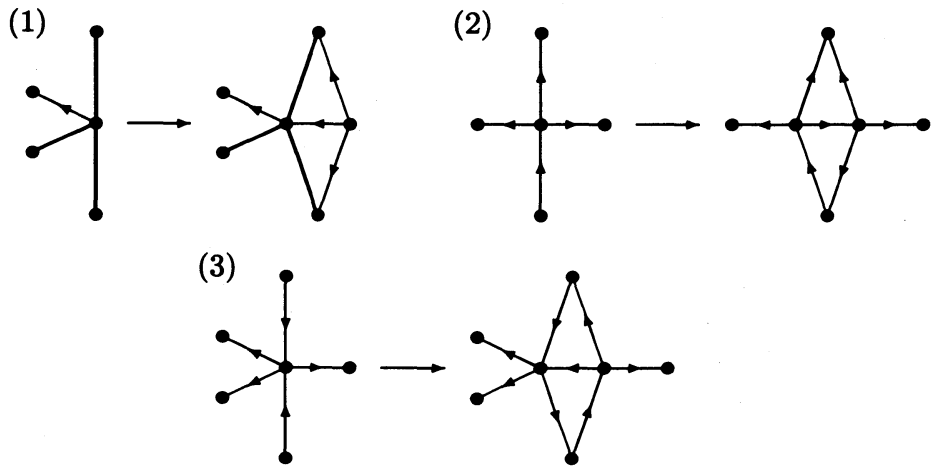


**Figure 2**

Let $G$ be a planar triangulation and $D_1^*(G)$ and $D_2^*(G)$ two $(*)$-orientations of $G$. Define the *subtraction* $D := D_1^*(G) - D_2^*(G)$ so that $V(D) := V(G)$ and $E(D) := \left\{uv \in E\left(D_1^*(G)\right) : uv \notin E\left(D_2^*(G)\right) \text{ for } uv \in E(G)\right\}$. It is clear that $D$ is a spanning subgraph of $D_1^*(G)$.

**Proposition 5.** *The subtraction $D$ is a union of pairwise edge-disjoint oriented cycles.*

**Proof.** For any $v \in V(G)$, we have that $\mathrm{od}_{D_1^*(G)}(v) = \mathrm{od}_{D_2^*(G)}(v)$ and $\mathrm{id}_{D_1^*(G)}(v) = \mathrm{id}_{D_2^*(G)}(v)$. Thus, for each vertex $v$ of $G$, $\mathrm{od}_D(v) = \mathrm{id}_D(v)$, and

hence each component of $D$ is eulerian. Therefore, the subtraction $D$ is a union of pairwise edge-disjoint oriented cycles in $D_1^*(G)$. ■

Let $D^*(G)$ be a $(*)$-orientation of a planar triangulation $G$ and $F$ a union of faces of $D^*(G)$ whose interior is an open 2-cell. We denote the boundary walk of $F$ by $\partial F$, and the length of $\partial F$ by $|\partial F|$. (In this case, $\partial F$ might not be an oriented closed walk.) Also, we denote the interior of $F$ by $\operatorname{Int} F$. We write $V(\operatorname{Int} F)$ for $V(G) \cap \operatorname{Int} F$ and $E(\operatorname{Int} F)$ for $E(G) \cap \operatorname{Int} F$.

**Lemma 6.** *Suppose that $|\partial F| = k \geq 3$. Then, there are exactly $k-3$ oriented edges $e_1, \ldots, e_{k-3}$ in $F$ such that each $e_i$ starts from a vertex on $\partial F$, where $e_i \notin E(\partial F)$.*

**Proof.** We denote the number of vertices and edges in $\operatorname{Int} F$ by $p$ and $q$, respectively. Applying Euler's formula to $F$, we have that $(k+p)-(k+q)+f = 1$, where $f$ denotes the number of faces in $F$. Since $F$ is internally triangulated and since $|\partial F| = k$, we have that $3f = 2q + k$ and hence we can obtain that $q = 3p + k - 3$. Here we have that $\sum_{v \in V(\operatorname{Int} F)} \operatorname{od}(v) = 3p$ since $\operatorname{od}(v) = 3$ for any $v \in V(\operatorname{Int} F)$. Thus, the number of edges in $F$ which start from $\partial F$ but are not contained in $\partial F$ is equal to $q - \sum_{v \in V(\operatorname{Int} F)} \operatorname{od}(v) = k - 3$. ■

The following lemma is the key to prove Theorem 2.

**Lemma 7.** *Suppose that $|\partial F| > 3$. Then, there exists a pair of vertices $u, v \in V(\partial F)$ such that there is an oriented path $P$ in $F$ from $u$ to $v$ with $E(P) \cap E(\partial F) = \emptyset$ and $V(P) \cap V(\partial F) = \{u, v\}$.*

**Proof.** By Lemma 6, there are $k-3$ oriented edges, say $e_1, \ldots, e_{k-3}$, toward inside from $\partial F$, where $k = |\partial F|$. Here we regard $\partial F$ as a cycle, even if $\partial F$ has some repeated vertices. Let $v_i$ $(i = 1, \ldots, k-3)$ be the starting point of $e_i$. Suppose that $v_1, \ldots, v_{k-3}$ lie on $\partial F$ in this clockwise order. Note that $v_i$ and $v_{i+1}$ may coincide for some $i$ and that $v_i$ and $v_{i+1}$ are not always adjacent in $\partial F$. Cutting $\partial F$ at each $v_i$, we decompose $\partial F$ into $k-3$ (possibly non-oriented) paths $L_{v_1,v_2}, L_{v_2,v_3}, \ldots, L_{v_{k-3},v_1}$, where the path $L_{v_i,v_{i+1}}$ joins $v_i$ and $v_{i+1}$. If $v_i = v_{i+1}$, then we define $L_{v_i,v_{i+1}}$ as an isolated vertex, that is, a path of length 0.

Let $B_{v_i}$ $(i = 1, \ldots, k-3)$ be the maximal vertex set of $V(F)$ such that $v_i \in B_{v_i}$ and that for any $x \in B_{v_i}$, there exists an oriented path $P$ from $v_i$ to $x$ with $E(P) \subset E(\operatorname{Int} F)$. We denote by $[B_{v_i}]$ the subgraph of $F$ induced by $B_{v_i}$. Since $[B_{v_i}] \subset F$, we can define the boundary of $[B_{v_i}]$, denoted by $\partial[B_{v_i}]$. If $|V(\partial F) \cap B_{v_i}| \geq 2$ for some $i$, then we can take the required path in $[B_{v_i}]$, by the definition of $[B_{v_i}]$. So, we assume that for any $i$, $V(\partial F) \cap B_{v_i} = \{v_i\}$.

Consider the average length of $L_{v_i,v_{i+1}}$'s. Since $|\partial F| = k$ with $k > 3$, we have $\frac{k}{k-3} = 1 + \frac{3}{k-3} > 1$. Thus, for some $i$, the length of $L_{v_i,v_{i+1}}$ is at least 2. Let $L_{v_m,v_{m+1}}$ be the path on $\partial F$ whose length is at least 2. Let $K$ be the union of faces in $\operatorname{Int} F$ but not in $\bigcup_{i=1}^{k-3} \operatorname{Int}[B_{v_i}]$, where $\operatorname{Int}[B_{v_i}] := [B_{v_i}] - \partial[B_{v_i}]$. Since $K$ may be disconnected, we take the component $K'$ of $K$ which contains $L_{v_m,v_{m+1}}$ on $\partial K'$. By the maximality of $B_{v_i}$, $K'$ has no oriented edge toward inside from a vertex on $\partial K'$.

Here, if $|\partial K'| = 3$, then the length of $L_{v_m,v_{m+1}}$ has to be 2 and there exists an oriented edge $v_m v_{m+1}$ or $v_{m+1} v_m$ through $\operatorname{Int} F$. In this case, this edge is the required oriented path. So, we may suppose that $|\partial K'| > 3$. However, this situation contradicts to Lemma 6. Thus, for some $i$, $|V(\partial F) \cap B_{v_i}| \geq 2$. Therefore, we can take the required path. ■

Now, we shall prove Theorem 2.

**Proof of Theorem 2.** By Theorem 1, for any planar triangulation $G$, there is a $(*)$-orientation of $G$. Let $D_1^*(G)$ and $D_2^*(G)$ be two $(*)$-orientations of $G$, and $D := D_1^*(G) - D_2^*(G)$. By Proposition 5, we let $D := C_1 \cup C_2 \cup \cdots \cup C_k$, where $C_1, \ldots, C_k$ are pairwise edge-disjoint oriented cycles in $D_1^*(G)$. Observe that the cycle reversions of $C_i$ for $i = 1, \ldots, k$ transform $D_1^*(G)$ into $D_2^*(G)$. Thus, it suffices to show that the cycle reversion of each $C_i$ can be obtained by a sequence of 3-cycle reversions. We fix $i$ and denote $C_i$ by $C$ in the following argument.

We use induction on the number of faces in the 2-cell region bounded by $C$. If $|C| = 3$, then we can apply the 3-cycle reversion of $C$. Note that $C$ might not bound a face in this case. Now suppose that $|C| > 3$. By Lemma 7, since $|C| > 3$, there is an oriented path $P$ (through the 2-cell region bounded by $C$) from $u \in V(C)$ to $v \in V(C)$. Since $C$ is an oriented cycle, $C$ is decomposed into the two oriented paths, say $P_1$ and $P_2$, connecting $u$ and $v$, where we suppose that $P_1$ is oriented from $v$ to $u$ and $P_2$ is from $u$ to $v$. Now, we can find an oriented cycle $P \cup P_1$. Since $P \cup P_1$ bounds a less number of faces than $C$, the cycle reversion of $P \cup P_1$ can be obtained by a sequence of 3-cycle reversions, by the hypothesis of induction. After this reversion, the union of the reversed $P$ and $P_2$ also forms an oriented cycle bounding a less number of faces than $C$. This can be reversed similarly. By the two operations, we can reverse the direction of only $C$ since $P$ is reversed twice. Thus, the cycle reversion of $C$ can be obtained by a sequence of 3-cycle reversions. Therefore, the theorem follows. ■

In Theorem 2, the cycles applied 3-cycle reversions do not always bound faces. In fact, we need such 3-cycle reversions in general. We can see this fact by the following examples. Let $G$ be a planar triangulation which has several $(*)$-

oreintations, say $D_1^*(G)$, $D_2^*(G)$, .... Let $\tilde{G}$ be the planar triangulation obtained from $G$ by putting a vertex of degree 3 into each face (except the infinite face) of $G$. Since each vertex not on the boundary of $G$ has outdegree 3 in $(*)$-orientations, the three edges incident with the added vertex of degree 3 must be outgoing in $(*)$-orientations of $\tilde{G}$. Thus, we can construct the two $(*)$-orientations $D_1^*(\tilde{G})$ and $D_2^*(\tilde{G})$ of $\tilde{G}$ from $D_1^*(G)$ and $D_2^*(G)$ by adding a vertex with three outgoing edges to each finite face of $D_1^*(G)$ and $D_2^*(G)$, respectively. Since both $D_1^*(\tilde{G})$ and $D_2^*(\tilde{G})$ have no facial oriented cycle of length 3, $D_1^*(\tilde{G})$ and $D_2^*(\tilde{G})$ cannot be transformed by a sequence of facial 3-cycle reversions. (However, if we assume that $G$ is 4-connected, any two $(*)$-orientations can be transformed by a sequence of facial 3-cycle reversions since all the 3-cycles of $G$ are facial.)

By the inductive algorithm used in the proof of Theorem 2, we can roughly estimate the minimum number of 3-cycle reversions needed to transform two $(*)$-orientations into each other.

Let $G$ be a planar triangulation with $n$ vertices. Let $D_1^*(G)$ and $D_2^*(G)$ be two $(*)$-orientations of $G$ and $D := D_1^*(G) - D_2^*(G)$. By Euler's formula, we obtain that $|E(G)| = 3n - 6$ and $|F(G)| = 2n - 4$. By Proposition 5, $D$ is a union of pairwise edge-disjoint oriented cycles. Let $k_1$ and $k_2$ be the numbers of oriented cycles in $D$ of length more than 3 and that of length exactly 3 respectively, and put $D := C_1 \cup \cdots \cup C_{k_1} \cup C_1' \cup \cdots \cup C_{k_2}'$, where $|C_i| \geq 4$ and $|C_i'| = 3$ for each $i$. Since $D \subset D_1^*(G)$ and since any two oriented cycles in $D$ share no edge, we have that $4k_1 + 3k_2 \leq |E(G)|$.

By the inductive argument in the proof of Theorem 2, observe that in order to reverse an oriented cycle $C_i$, we need at most $|F(C_i)|$ 3-cycle reversions, where $|F(C_i)|$ denotes the number of faces in the 2-cell bounded by $C_i$. Thus, we have

$$\begin{aligned} d(G, D_1^*, D_2^*) &\leq \sum_{i=1}^{k_1} |F(C_i)| + k_2 \\ &\leq k_1(|F(G)| - 1) + k_2 \\ &\leq \frac{1}{4}(|F(G)| - 1)(4k_1 + 3k_2) \\ &\leq \frac{1}{4}(|F(G)| - 1)|E(G)| \\ &= \frac{3}{4}(2n - 5)(n - 2) \\ &= \frac{3}{2}n^2 - \frac{27}{4} + \frac{15}{2}. \end{aligned}$$

In order to get a better estimation, we need more elaborate arguments as in the following section.

## 3. Estimation of the number of 3-cycle reversions

In this section, we first define a nice invariant for the subtraction of the two $(*)$-orientations $D_1^*(G)$ and $D_2^*(G)$ of a planar triangulation $G$, which is equal to $d(G, D_1^*, D_2^*)$.

Let $G$ be a planar triangulation. We shall decompose $G$ into some plane graphs $G_1, \ldots, G_k$, as follows. Let $G'_0 := G$. Let $\Delta_i$ be the *innermost* triangular region of $G'_{i-1}$ (say bounded by $a_ib_ic_i$ where $a_i, b_i, c_i \in V(G_i)$) but is not a face of $G'_{i-1}$, that is, there is no such a region in $\Delta_i$. Let $G_i := \Delta_i - \{a_i, b_i, c_i\}$ and let $G'_i := G'_{i-1} - V(G_i)$. Note that each $G_i$ is either an isolated vertex or a 2-connected plane graph. Here, each $G'_i$ is either a planar triangulation or a cycle of length 3. Repeating this procedure, if we obtain the outer cycle of $G$ of length 3 as $G'_{k-1}$, then put $G_k := G'_{k-1}$. It is easy to see that each $G_i$ contains no 3-cycle which does not bound a face. Also, any two triangular regions of $G$ are disjoint or otherwise one is contained in the other. Hence this decomposition is unique, up to labeling of subscripts. We call such a decomposition of $G$ the *$\Delta$-decomposition.*

As is mentioned above, we can $\Delta$-decompose a planar triangulation $G$ into $G_1, \ldots, G_k$ so that $G_k$ is the outer 3-cycle of $G$. By Proposition 5, for two $(*)$-orientations $D_1^*(G)$ and $D_2^*(G)$ of $G$, their subtraction $D$ is a union of pairwise edge-disjoint oriented cycles, say $D := C_1 \cup \cdots \cup C_m$, where each $C_i$ is an oriented cycle in $D_1^*(G)$.

Let $D_j^*(G_i)$ be the subgraph of $D_j^*(G)$ $(j = 1, 2)$ induced by $V(G_i)$ for $i = 1, \ldots, k$.

**Lemma 8.** *For any $i$ $(i = 1, \ldots, m)$, $C_i$ is completely contained in some $D_1^*(G_j)$.*

**Proof.** Observe that for each $i$ $(i = 1, \ldots, k-1)$, $G_i$ was surrounded by some 3-cycle in $G$. For a triangular region $\Delta$ of $G$ but is not a face of $G$, there is no oriented edge in $\Delta$ toward inside from a vertex on its boundary, by Lemma 6. Thus, for any $(*)$-orientation of $G$, there is no oriented cycle which contains $u \in V(\mathrm{Int}\,\Delta)$ and $v \in V(G) - V(\Delta)$ simultaneously. Thus, the lemma follows. ■

Let $G$ be a plane graph and $G^*$ its dual. Let $D$ be an orientation of a subgraph of $G$. For a path $p$ of $G^*$ with its starting point and endpoint prescribed, $e \in E(D)$ is called a *right* edge if for the direction of $p$, $e$ crosses $p$ from the left to the right. Similarly, $e$ is called a *left* edge for the path $p$ if $e$ crosses $p$ from the right to the left for the direction of $p$. See Figure 3.

Let $G$ be a planar triangulation and $D_1^*(G)$ and $D_2^*(G)$ two $(*)$-orientations

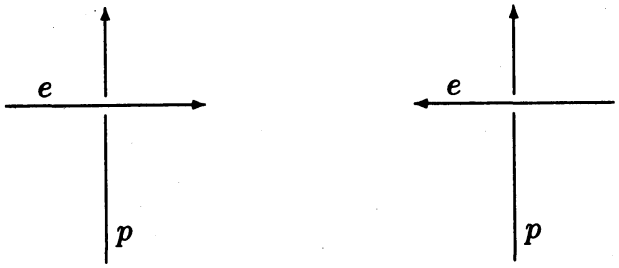


**Figure 3** A right and left edges

of $G$. Let $\{G_1, \ldots, G_k\}$ be the $\Delta$-decomposition of $G$. By Lemma 8, if we put $D_i = D_1^*(G_i) - D_2^*(G_i)$ for $i = 1, \ldots, k$, then $D_i$ consists of pairwise edge-disjoint oriented cycles.

Now, suppose that $G_i$ is 2-connected. Let $f$ and $S$ be a finite face and the infinite face of $G_i$, and $f^*$ and $S^*$ the vertices of $G_i^*$ corresponding to $f$ and $S$, respectively. For $f$ and a fixed path $p$ from $f^*$ to $S^*$, let

$$\phi_{D_i}(f,p) := \Big|\{e \in E(D_i) \ : \ e \text{ is right for } p\}\Big| - \Big|\{e \in E(D_i) \ : \ e \text{ is left for } p\}\Big|.$$

For $S$, we define $\phi_{D_i}(S,p) = 0$.

**Proposition 9.** *For each face $f$, $\phi_{D_i}(f,p)$ does not depend on the choice of the path $p$.*

**Proof.** By Lemma 8, $D_i$ consists of several pairwise edge-disjoint oriented cycles, say $C_1, \ldots, C_m$, in $D_1^*(G_i)$, respectively. Let $f$ and $S$ be a finite face and the infinite face of $G_i$. Let $p$ be a path in the dual $G_i^*$ of $G_i$ from $f^*$ to $S^*$, where $f^*$ and $S^*$ are the vertices of $G_i^*$ corresponding to $f$ and $S$ in $G_i$, respectively. If the path $p$ from $f^*$ to $S^*$ crosses $C_j$ once to enter inside, then the path must cross it again to exit outside. The directions of these two edges of $C_j$ crossing $p$ are different and hence $C_j$ is not counted in $\phi_{D_i}(f,p)$. Thus, by the definition of the depth, $C_j$ is counted in $\phi_{D_i}(f,p)$ as 1 or $-1$ (depending on the orientation of $C_j$) only if $f$ is inside $C_j$. Therefore, $\phi_{D_i}(f,p) = |B(f)| - |B'(f)|$, where $B(f)$ (resp., $B'(f)$) $\subset \{C_1, \ldots, C_m\}$ is the set of clockwise (resp., counterclockwise) oriented cycles containing $f$ inside. ■

For each face $f$, we simply write $\phi_{D_i}(f)$ instead of $\phi_{D_i}(f,p)$ for any path $p$. We call $|\phi_{D_i}(f)|$ the *depth* of $f$ in $D_i$.

Let

$$\Phi(D_i) := \sum_{f \in F(G_i)} |\phi_{D_i}(f)|$$

for $i = 1, \ldots, k$. If $G_i$ is an isolated vertex, then we define that $\Phi(D_i) = 0$. Let

$$\Phi(D) := \sum_{i=1}^{k} \Phi(D_i).$$

We call this invariant $\Phi(D)$ the *total depth* of $D = D_1^*(G) - D_2^*(G)$. Note that $\Phi(D)$ is a non-negative integer.

By the definition of depths, we can see that any two faces of $D_1^*(G_i)$ sharing an edge in $D_i$ have different depths whose difference is exactly 1. So we can say that $D_i$ divides the face set of $D_1^*(G_i)$ into several regions so that any two faces belonging to the same region have the same depth. We call the region with the largest depth the *deepest region.*

Let $\{G_1, \ldots, G_k\}$ be the $\Delta$-decomposition of $G$. By Proposition 5, $D$ is a union of pairwise edge-disjoint oriented cycles of $D_1^*(G)$. Observe that $E(D) = \emptyset$ means $D_1^*(G) = D_2^*(G)$. Thus, when $D_1^*(G) \neq D_2^*(G)$, we have that $E(D) \neq \emptyset$ and hence for some $s$, $D_s := D_1^*(G_s) - D_2^*(G_s)$ is not empty and consists of some pairwise edge-disjoint oriented cycles in $D_1^*(G_s)$, by Lemma 8. Let $R$ be the deepest region of $D_1^*(G_s)$.

**Lemma 10.** *The deepest region $R$ has an outer oriented boundary cycle $C$ and inner oriented boundary cycles $C_1', \ldots, C_m'$ $(m \geq 0)$ such that for any $i$ and $j$, $(\mathrm{Int}\, C_i') \cap (\mathrm{Int}\, C_j') = \emptyset$ and that each $C_i'$ has the orientation different from $C$.*

**Proof.** Let $R$ be the deepest region of $D_1^*(G_s)$. Let $f$ and $S$ be a finite face in $R$ and the infinite face of $G_s$, respectively. We may suppose that $\phi_{D_s}(f) := M > 0$. (Otherwise, put $D := D_2^*(G) - D_1^*(G)$.) Consider a path $p$ of $G_s^*$ from $f^*$ to $S^*$, where $f^*$ and $S^*$ are the vertices of $G_s^*$ corresponding to $f$ and $S$, respectively. Trace $p$ from $f^*$ to $S^*$.

We claim here that the edge in $D_s$ which $p$ first meets is a right edge. Notice that when $p$ meets each $e \in E(D_s)$, if $e$ is right (resp., left), we get the value $+1$ (resp., $-1$), and finally get the depth of $f$ as the summation of these values. So, if the edge in $D_s$ which $p$ first meets was left, the vertex of $G_s^*$ then visited by $p$ would have the depth equal to $\phi_{D_s}(f) + 1$. This contradicts $f$ being in the deepest region.

Since $R$ is a region, it has the outer boundary cycle, say $C$, in $G$ and might have several inner boundary cycles, say $C_1', \ldots, C_m'$, in $G$. Notice that for any $i$ and $j$ $(1 \leq i < j \leq m)$, $(\mathrm{Int}\, C_i') \cap (\mathrm{Int}\, C_j') = \emptyset$ since $C \cup C_1' \cup \cdots \cup C_m'$ forms the boundary of $R$. Observe that each edge on $C, C_1', \ldots, C_m'$ is right in $D_1^*(G)$. Thus, $C$ is clockwise oriented and $C_1', \ldots, C_m'$ are counterclockwise oriented. ■

**Lemma 11.** *Let $D_1^*(G)$ and $D_2^*(G)$ be two $(*)$-orientations of a planar tri-*

*angulation $G$ and $D := D_1^*(G) - D_2^*(G)$. If $\Phi(D) > 0$, then there exists a 3-cycle $C$ in $D_1^*(G)$ whose 3-cycle reversion decreases $\Phi(D)$ by 1.*

**Proof.** Suppose that $D_s = D_1^*(G_s) - D_2^*(G_s)$ is not empty. Let $R$ be the deepest region of $D_1^*(G_s)$. We may assume that $\phi_{D_s}(f) > 0$ for any $f$ in $R$. Suppose that $R$ has an outer boundary cycle $C$ with the clockwise orientation and inner boundary cycles $C_1', \ldots, C_m'$ with the counterclockwise orientations as in Lemma 10.

We shall show that there exists a 3-cycle $K$ in $R$ with the same direction as $C$. Since any 3-cycle of $G_s$ bounds a face, $K$ must bound a face, say $f$, in $G_s$. If there is $K$, the 3-cycle reversion of $K$ decreases $\phi_{D_s}(f)$ by 1, since for any path $p$ from $f^*$ to $S^*$ in the dual of $G_s$, the reversion either decreases one right edge or increases one left edge of $D_s$ On the other hand, for any face $f' \neq f$, since $f'$ is not inside $K$, $\phi_{D_s}(f')$ is not changed by the 3-cycle reversion of $K$. Thus, this decreases the depth of only $f$ by 1 and the total depth of $D$ by 1.

In order to find such $K$, we use induction on the number of faces in the region $R$. If the outer boundary cycle $C$ of $R$ bounds only one face, then $|C| = 3$ and hence $K = C$. Then we consider the case of $|C| > 3$. By Lemma 7, since $|C| > 3$, there exist $u, v \in V(C)$ such that $u$ and $v$ are joined by an oriented path $P$ in the closed 2-cell bounded by $C$ with $V(C) \cap V(P) = \{u, v\}$ and $E(C) \cap E(P) = \emptyset$. If $P$ runs through Int $C_i'$ for some $i$, then replace the subgraph of $P$ inside $C_i'$ with the oriented path on $C_i'$, fixing its ends on $C_i'$, so that the resulting path forms an oriented path between $u$ and $v$. Repeating this operation, we can deform $P$ to run in the closure of $R$.

Decompose $C$ into the two oriented paths joining $u$ and $v$. Either of the two paths and $P$ forms the oriented cycle, say $C'$, with the same orientation as $C$, which bounds a less number of faces than $C$. Since $C$ and each $C_i'$ have the different orientations, $C'$ bounds a part of $R$. By the hypothesis of induction, there is the required 3-cycle $K$ in the 2-cell region bounded by $C'$. ■

Note that Lemma 11 is an alternative proof of Theorem 2. If we want to show only Theorem 2, the method used in the proof of Theorem 2 is enough. However, to get a good estimation of the minimum number of 3-cycle reversions needed to transform given two $(*)$-orientations, we need a minute argument as in Lemma 11.

The following is the most important theorem.

**Theorem 12.** *Let $D_1^*(G)$ and $D_2^*(G)$ be two $(*)$-orientations of a planar triangulation $G$ and $D := D_1^*(G) - D_2^*(G)$. Then, $d(G, D_1^*, D_2^*) = \Phi(D)$.*

**Proof.** It is easy to see that $d(G, D_1^*, D_2^*) \leq \Phi(D)$, by Lemma 11. We shall

show that $d(G, D_1^*, D_2^*) \geq \Phi(D)$. Let $\{G_1, \ldots, G_k\}$ be the $\Delta$-decomposition of $G$. Let $K$ be any 3-cycle in $G$. Then, by the definition of $\Delta$-decomposition, $K$ bounds a finite face $f$ in some $G_i$. Observe that the 3-cycle reversion of $K$ can reduce only $\phi_{D_i}(f)$ by 1 and preserves $\phi_{D_i}(f')$ for any face $f' \neq f$ of $G_i$. So, in order to transform $D_1^*(G_i)$ into $D_2^*(G_i)$, each face $f$ in $G_i$ $(i = 1, \ldots, n)$ must be applied at least $|\phi_{D_i}(f)|$ 3-cycle reversions. So, we have that

$$d(G_i, D_1^*, D_2^*) \geq \sum_{f \in F(G_i)} |\phi_{D_i}(f)| = \Phi(D_i),$$

where $d(G_i, D_1^*, D_2^*)$ denotes the minimum number of 3-cycle reversions needed to transform $D_1^*(G_i)$ into $D_2^*(G_i)$. Thus, we have that

$$d(G, D_1^*, D_2^*) = \sum_{i=1}^{k} d(G_i, D_1^*, D_2^*) \geq \sum_{i=1}^{k} \Phi(D_i) = \Phi(D).$$

Therefore, the theorem follows. ■

Now, using Theorem 12, we shall estimate the maximum value of $d(G)$ for a planar triangulation $G$. Let $D_1^*(G)$ and $D_2^*(G)$ be two $(*)$-orientations of $G$ and $D := D_1^*(G) - D_2^*(G)$. The value $\Phi(D)$ is obtained as $\sum_{i=1}^{k} \Phi(D_i)$, where $\{G_1, \ldots, G_k\}$ is the $\Delta$-decomposition of $G$ and $D_i = D_1^*(G_i) - D_2^*(G_i)$. The following lemma estimates the value of each $\Phi(D_i)$.

**Lemma 13.** *Let $n_i = |V(G_i)|$. If $n_i = 1$, $\Phi(D_i) = 0$, and if $n_i \geq 3$, then*

$$\Phi(D_i) \leq \left\lfloor \frac{n_i^2}{2} - 2n_i + 3 \right\rfloor.$$

**Proof.** By the definition of $\Delta$-decomposition, each $G_i$ is either an isolated vertex or a 2-connected plane graph, and hence either $n_i = 1$ or $n_i \geq 3$ for $i = 1, \ldots, k$. If $n_i = 1$, then $G_i$ has no finite face and hence $\Phi(D_i) = 0$. If $n_i = 3$, then $G_i$ has at most one finite face, and hence $\Phi(D_i) \leq 1 = \lfloor \frac{3^2}{2} - 2 \cdot 3 + 3 \rfloor$. ¿From now, we suppose that $n_i \geq 4$. Then $G_i$ has the outer cycle of length $t \geq 4$. (For if $t = 3$, $G_i$ was $\Delta$-decomposable.) Consider the dual $G_i^*$ of $G_i$, and let $f^*$ and $S^*$ be the vertices corresponding to a face $f$ in the deepest region and the infinite face $S$, respectively. It is obvious that $\deg_{G_i^*}(S^*) = t \geq 4$ and for any $g^* \in V(G_i^*) - \{S^*\}$, $\deg_{G_i^*}(g^*) = 3$. Since $G_i$ has at least two faces, there is a face, say $f' (\neq S)$, adjacent with $f$ in $G_i$. Let $\tilde{G}_i^*$ be the plane graph obtained from $G_i^*$ by contracting the edge $f^* f'^*$, where $f'^*$ is the vertex of $G_i^*$ corresponding to $f'$. (Here, contracting an edge is to delete it and identify its both endpoints. This definition is different from that for triangulatons.) We

denote by $[f^*f'^*]$ the vertex of $\tilde{G}_i^*$ which is the image of the edge $f^*f'^*$ by the contraction. Observe that $G_i$ has no cycle of length 3 which does not bound a face. Thus, by the relation between the cycle set of a plane graph and the cut set of its dual, if an edge-set $E \subset E(\tilde{G}_i^*)$ separates $[f^*f'^*]$ and $S^*$, then $|E| \geq 4$. Moreover, since every vertex of $\tilde{G}_i^* - \{[f^*f'^*], S^*\}$ has degree 3, any vertex-set $V \subset \tilde{G}_i^* - \{[f^*f'^*], S^*\}$ separating $[f^*f'^*]$ and $S^*$ satisfies $|V| \geq 4$. Therefore, by well-known Menger's Theorem, $\tilde{G}_i^*$ has at least four pairwise disjoint paths, say $P_1$, $P_2$, $P_3$ and $P_4$, connecting $[f^*f'^*]$ and $S^*$.

Now consider the system of the four paths in $G_i^*$. Then, in $G_i^*$, $P_1 \cup P_2 \cup P_3 \cup P_4$ can be regarded as four paths between $\{f^*, f'^*\}$ and $S^*$, say $P_1' \cup P_2' \cup P_3' \cup P_4'$. Define the subgraph $\mathcal{P}$ of $G_i^*$ by

$$\mathcal{P} := P_1' \cup P_2' \cup P_3' \cup P_4' \cup f^*f'^*.$$

Suppose that $d_{\mathcal{P}}(f^*, S^*) = r$, where $d_T(a, b)$ denotes the distance in a graph $T$ of $a, b \in V(T)$. Since two adjacent faces $x$ and $y$ of $G_i$ satisfies $|\phi_{D_i}(x) - \phi_{D_i}(y)| \leq 1$ and since $\phi_{D_i}(S) = 0$, we have that $|\phi_{D_i}(f)| \leq r$. Moreover, for each $v^* \in V(\mathcal{P})$, we have that $|\phi_{D_i}(v)| \leq d_{\mathcal{P}}(v^*, S^*)$. Observe that for any $v^* \in V(G_i^*) - V(\mathcal{P})$, $|\phi_{D_i}(v)| \leq |\phi_{D_i}(f)| \leq r$. Since $G_i$ is internally triangulated on the plane with the outer cycle of length $t$, we have that $|F(G_i)| = 2n_i - t - 1 \leq 2n_i - 5$, by Euler's formula. Therefore, we have that

$$\begin{aligned}
\sum_{v \in F(G_i)} |\phi_{D_i}(v)| &= \sum_{v^* \in V(\mathcal{P})} |\phi_{D_i}(v)| + \sum_{v^* \in V(G_i^*) - V(\mathcal{P})} |\phi_{D_i}(v)| \\
&\leq \sum_{v^* \in V(\mathcal{P})} d_{\mathcal{P}}(v^*, S^*) + r|V(G_i^*) - V(\mathcal{P})| \\
&\leq 4\sum_{j=1}^{r-2} j + 3(r-1) + r + r(|F(G_i)| - 1 - 4(r-2) - 3 - 1) \\
&= -2r^2 + (|F(G_i)| + 1)r + 1 \\
&\leq -2r^2 + (2n_i - 4)r + 1 \\
&= -2\left(r - \frac{n_i - 2}{2}\right)^2 + \frac{n_i^2}{2} - 2n_i + 3 \\
&\leq \frac{n_i^2}{2} - 2n_i + 3.
\end{aligned}$$

■

Now, we shall show Theorem 3.

**Proof of Theorem 3.** Let $G$ be a planar triangulation with $n$ vertices, and $\{G_1, \ldots, G_k\}$ the $\Delta$-decomposition of $G$ such that $G_k$ is the outer 3-cycle of

$G$. Let $n_i := |V(G_i)|$. Then $\sum_{i=1}^{k-1} n_i = n-3$. Suppose that $n_1 = \cdots = n_a = 1$ and $n_i \geq 3$ for $a+1 \leq i \leq k-1$. By Lemma 13, we have that

$$\begin{aligned}\Phi(D) &= \sum_{i=1}^{k} \Phi(D_i) \\ &\leq \sum_{i=a+1}^{k-1} \left(\frac{1}{2}n_i^2 - 2n_i + 3\right) + 1.\end{aligned}$$

It is easy to see that if $n_p \geq 3$ and $n_q \geq 3$, then

$$\left(\frac{1}{2}n_p^2 - 2n_p + 3\right) + \left(\frac{1}{2}n_q^2 - 2n_q + 3\right) < \frac{1}{2}(n_p + n_q)^2 - 2(n_p + n_q) + 3.$$

Therefore, since $n \geq 7$, we have that

$$\begin{aligned}\Phi(D) &\leq \frac{1}{2}\left(\sum_{i=a+1}^{k-1} n_i\right)^2 - 2\sum_{i=a+1}^{k-1} n_i + 3 + 1 \\ &\leq \frac{1}{2}(n-3)^2 - 2(n-3) + 3 + 1 \\ &= \frac{1}{2}n^2 - 5n + \frac{27}{2}.\end{aligned}$$

■

## 4. A lower bound in Theorem 3

Now, using Theorem 12, we shall consider the example which gives a lower bound of $d(G)$ in Theorem 3. See Figure 4, in which the planar triangulation $G$ with $n$ vertices is partially oriented, and is supposed to have $k$ pairwise edge-disjoint non-oriented cycles of length 4. Then we have that $n = 3k+4$. For $l \in \{0,1,2\}$, adding $l$ vertices into the innermost non-oriented quadrilateral, we can construct a planar triangulation with $n = 3k+4+l$ vertices. Giving the clockwise orientation to each non-oriented 4-cycle and the outer 3-cycle in Figure 4, we suppose to obtain $D_1^*(G)$. Giving the counterclockwise orientations to them, we obtain $D_2^*(G)$. It is easily checked that both of them are $(*)$-orientations of $G$. Clearly, $D := D_1^*(G) - D_2^*(G)$ is the union of the non-oriented cycles in Figure 4 with the clockwise orientation.

Consider the $\Delta$-decomposition of $G$. Since $G$ is 4-connected, $G$ can be $\Delta$-decomposed into the two graphs $G_1$ and $G_2$, where $G_2$ is the outer cycle of length 3 of $G$ and $G_1 = G - G_2$. Now consider the depths in $G_1$, in which

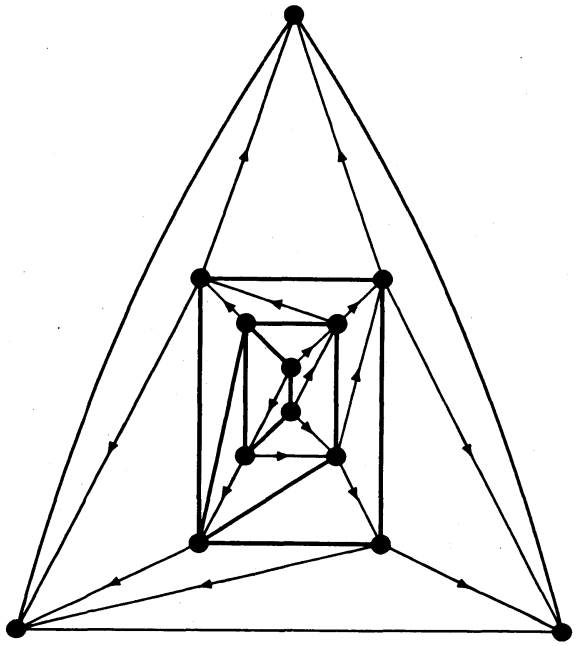

**Figure 4**

$D_1$ is $k$ pairwise edge-disjoint clockwise oriented 4-cycles. Observe that the innermost quadrilateral has $2+2l$ faces, each of which has depth $k$, and that the region bounded by $i$-th 4-cycle and $(i+1)$-th 4-cycle (counting from outside) for $i = 1, \ldots, k-1$ has six faces, in which each face has depth $i$. For $G_2$, $D_2$ is just a clockwise oriented 3-cycle, which has depth 1. Thus, we have that

$$\begin{aligned}
\Phi(D) &= \Phi(D_1) + \Phi(D_2) \\
&= \Big(6 + 6 \cdot 2 + \cdots + 6(k-1) + (2+2l)k\Big) + 1 \\
&= 3k^2 + (2l-1)k + 1 \\
&= 3\left(\frac{n-4-l}{3}\right)^2 - (2l-1)\left(\frac{n-4-l}{3}\right) + 1 \\
&= \frac{1}{3}n^2 - 3n + \frac{23}{3} + \frac{l-l^2}{3},
\end{aligned}$$

where $l \in \{0, 1, 2\}$.

**Proposition 14.** *There exists a planar triangulation $G$ with $n \geq 7$ vertices and its two $(*)$-orientations $D_1^*(G)$ and $D_2^*(G)$ such that*

$$d(G, D_1^*, D_2^*) = \left\lfloor \frac{1}{3}n^2 - 3n + \frac{23}{3} \right\rfloor .$$

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