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Periodic solutions for evolution equations in Hilbert spaces

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Abstract. We consider the existence of periodic solutions of the problem $g(t,u) \in u' + Au$, where A is a maximal monotone operator defied in a Hilbert space and $g : R \times H \rightarrow H$ is a Caratheodory function periodic with respect to the first variable.

1. Introduction

In this paper, H is Hilbert space with an inner product $\langle \cdot, \cdot \rangle$, denote by $\| \cdot \|$ the norm induced by $\langle \cdot, \cdot \rangle$. Let $A : D(A) \subseteq H \rightarrow 2^H$ be a maximal monotone operator, we consider the existence of periodic solutions for nonlinear evolution equations of the form

$$g(t,u) \in \frac{du}{dt} + Au, t \in R \tag{1.1}$$

where $g : R \times H \rightarrow H$ is a Caratheodory function.

If g is Lipschitz or continuous with respect to the second variable, existence of periodic solutions for (1.1) has been studied by many authors. See [3], [5], [8], [9], [11], [13], [14]. When g is a Caratheodory function, periodic solutions for (1.1) has been studied by [12], [14] under other assumptions on g. In this paper, we shall give different assumptions on A and g than that of [12], [14]. Now, we state our result.

Theorem 1 *Let $A : D(A) \subseteq H \rightarrow 2^H$ be a maximal monotone operatar. $g : R \times H \rightarrow H$ is a Caratheodory operator and g is T-periodic with respect to the first variable. Suppose the following conditions are satisfied*

1. For some $\lambda > 0$, $J_\lambda = (I + \lambda^{-1} A)^{-1}$ is a compact operator on H;

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2. There exist $M_1, M_2 > 0$, such that

$$\|g(t,v)\| \leq M_1\|v\| + M_2, \forall (t,v) \in R \times H;$$

$\langle z - g(t, J_\lambda v), v\rangle \geq -\frac{a}{\lambda}\|J_\lambda v\|$, $\forall (t,v) \in R \times D(A)$, $z \in A_v; \lambda > 0$, where $a > 0$ is a constant;

Then (1.1) *has a T–periodic solution.*

Our approach is different from that of [12] and [14], the assumptions 1 and 2 are the same as in [12], but 3 is different from (*) of [12]. In [12], A is assumed to be a subdifferential of a lower semicontinuous convex function, but we do not require this condition.

2. Proof of Theorem

Let $L^2(0,T;H)$ be the space of functions $v : [0.T] \to H$ such that $\int_0^T \|v\|^2 dt < +\infty$. The norm and the inner product of $L^2(0,T;H)$ are denoted by $\|.\|_T$ and $\ll \cdot,\cdot \gg$ respectively. We identify the function in $L^2(0,T,H)$ *with T–periodic functions.* $W^{1,2}(0,T;H)$ represents the spacce of functions $v : [0,T] \to H$ such that $v, v^{(1)} \in L^2(0,T;H)$, where $v^{(1)}$ denotes the 1st derivative in the sense of distribution.

Let $A_\lambda = \lambda(I - J_\lambda)$ *for* $\lambda > 0$, *then* $A_\lambda x \in AJ_\lambda x$ *for* $x \in H$. It is well known that J_λ is nonexpansive and $J_\lambda x = J_u(J_\lambda x + u^{-1}A_\lambda x)$ *for* $\lambda, u > 0$ *and* $x \in H$(see [4]).

In this paper, a T–periodic solution $u(\cdot)$ of (1.1) means that $u \in W^{1,2}(0,T;H)$, $u(t) \in D(A)$ for almost all $t > 0$ and there exists $v(\cdot) \in L^2(0,T;H)$, such that $v(t) \in Au(t)$ for almost all $t > 0$, and
$\ll \frac{du}{dt} + v(t) - g(t,v), y \gg = 0$ *for all* $y \in W^{1,2}(0,T;H)$ *verifying* $y(0) = y(T)$.

Without loss of generality, we assume $0 \in D(A)$ *and* $0 \in A0$. Let $\mathcal{A} : L^2(0,T;H) \to L^2(0,T;H)$ be as following

$$\mathcal{A}u(\cdot) = \{v(\cdot) \in L^2(0,T;H) | u(t) \in D(A), v(t) \in Au(t); \text{a.e.}t \in [0,T]\}$$

It's known that $\mathcal{A} : L^2(0,T;H) \to L^2(0,T;H)$ is maximal monoton (see [1]).

Let $W = \{v : R \to H, v(t+T) = v(t) \quad \textit{for} \quad t \in R \quad \textit{and} \quad v|_{[0,T]} \in W^{1,2}(0,T;H)\}$ endowed with the norm $\|\cdot\|_{1,T}$ *of* $W^{1,2}(0,T;H)$, *i.e.* $\|v\|^2_{1,T} = \|\frac{dv}{dt}\|^2_T + \|v\|^2_T$ for $v \in W$;

Now, we define an operator $T_n : W \to W^*$ by

$$\begin{aligned} \ll T_n u, v \gg &= \ll \frac{1}{n}\frac{du}{dt}, \frac{dv}{dt} \gg + \ll \frac{1}{n}u, v \gg \\ &+ \ll \frac{du}{dt} - g(t, J_n u), v \gg \quad \text{for} \quad u, v \in W. \end{aligned} \tag{2.1}$$

For a reflexive Banach space E, an operator $T : E \to E^*$ is said to be an operator of (S_+), if u_n converges to u weakly in E and $\overline{\lim_{i\to\infty}}(Tu_n, u_n - u) \leq 0$ imply that u_n converges to strongly in E and Tu_n has a subsequence converging weakly to Tu.(see [6])

Lemma 2. 1 *For $n \geq 1$, the operator T_n defined by* (2.1) *is an operator of* (S_+).

Proof. For fixed $n \geq 1$, let $(u_i) \subset W$ be a sequnence such that $u_i \rightharpoonup u \in W$ weakly in W and

$$\overline{\lim_{i\to\infty}}[\ll \frac{1}{n}\frac{du_i}{dt}, \frac{d(u_i - u)}{dt} \gg + \ll \frac{1}{n}u_i, u_i - u \gg + \ll \frac{du_i}{dt} - g(t, J_n u_i), u_i - u \gg] \leq 0$$

Since J_n is compact and nonexpansive, so $J_n u_i$ is relatively compact in $L^2(0, T; H)$, and further that $\{g(t, J_n u_i)\}$ is also relatively compact in $L^2(0, T; H)$.

So $\lim_{i\to\infty} \ll g(t, J_n u_i), u_i - u \gg = 0.$

Since $\lim_{i\to\infty} \ll \frac{du_i}{dt}, u_i \gg = \lim_{i\to\infty}\int_0^T \frac{1}{2}\frac{d}{dt}\|u_i\|^2 dt = 0,$
$\lim_{i\to\infty} \ll \frac{du_i}{dt}, u_i \gg == \lim_{i\to\infty} \ll u_i, \frac{du}{dt} \gg = 0.$

Therefore, we have

$$\overline{\lim_{i\to\infty}}[\ll \frac{1}{n}\frac{du_i}{dt}, \frac{d(u_i - u)}{dt} \gg + \ll \frac{1}{n}u_i, u_i - u \gg] \leq 0.$$

Hence

$$\overline{\lim_{i\to\infty}}[\int_0^T \frac{1}{n}\|\frac{du_i}{dt} - \frac{du}{dt}\|^2 dt + \int_0^T \frac{1}{n}\|u_i - u\|^2 dt] \leq 0.$$

So we get $u_i \to u$ strongly in W.

Lemma 2. 2 *There exists $R_0 > 0$, suh that*
$\ll T_n u, u \gg + \ll v, u \gg > 0$, $\forall u \in W, \|u\|_{1,T} = R_0$ *and* $u(\cdot) \in D(\mathcal{A}), v(\cdot) \in \mathcal{A}u(\cdot)$.

Proof. Suppose $u \in W$, and $u \in D(\mathcal{A}), v \in \mathcal{A}_u$.

$$\text{Since} \ll T_n u, u \gg + \ll v, u \gg \; = \; \frac{1}{n}\int_0^T \|\frac{du}{dt}\|^2 dt + \frac{1}{n}\int_0^T \|u\|^2 dt$$

$$
\begin{aligned}
&+ \int_0^T \langle \frac{du}{dt}, u\rangle dt + \int_0^T \langle v - g(t, J_n u), u\rangle dt \\
&= \frac{1}{n}\|u\|^2_{1,T} + \int_0^T \langle v - g(t, J_n u), u\rangle dt
\end{aligned}
$$

By the assumption 3, we have

$$
\int_0^T \langle v - g(t, J_n u), u\rangle dt \geq -\frac{a}{n}\int_0^T \|J_n u\| dt.
$$

Since $0 \in A0$, so $\|J_n u\| \leq \|u\|$. Therefore, we have

$$
\begin{aligned}
\int_0^T \langle v - g(t, J_n u), u\rangle dt \geq -\frac{a}{n}\int_0^T \|u\| dt &\geq -\frac{a}{n}(\int_0^T \|u\|^2 dt)^{\frac{1}{2}}\sqrt{T} \\
&\geq -\frac{a}{n}\sqrt{T}\|u\|_{1,T}.
\end{aligned}
$$

Hence $\ll T_n u, u \gg + \ll v, u \gg \geq \frac{1}{n}\|u\|_{1,T}(\|u\|_{1,T} - a\sqrt{T})$. Let $R_0 > a\sqrt{T}$, we get the desired result.

In the following, we denote by $\mathcal{A}_\lambda = (\lambda I + \mathcal{A}^{-1})^{-1}$, $R_\lambda = I - \lambda\mathcal{A}_\lambda$, $\lambda > 0$.

Lemma 2. 3 *Let R_0 be the same as in Lemmma 2. 2. Then for each $n \geq 1$, there exists $\lambda_0^n > 0$, such that*

$$
0 \notin \underset{i\in[0,1]}{U} [t(T_n + A_\lambda) + (1-t)J](\partial B_{R_0}), \forall\lambda \in (0, \lambda_0^n) \tag{2.2}
$$

where $J : W \to W^*$ is the dual mapping, $B_{R_0} = \{u \in W, \|u\|_{1,T} < R_0\}$.

Proof. Suppose (2.2) is not true. There exist $\lambda_i \to 0^+$, $t_j \to t_0$, $u_j \in \partial B_{R_0}$, with $u_j \rightharpoonup u_0$ and $\frac{du_j}{dt} \rightharpoonup \frac{du_0}{dt}$ weakly in $L^2(0,T;H)$, such that

$$
t_j(T_n u_j + A_{\lambda_j} u_j) + (1-t_j)Ju_j = 0 \tag{2.3}
$$

Case(i). $t_0 = 0$. Since $\mathcal{A}_{\lambda_j} 0 = 0$, $\ll \mathcal{A}_{\lambda_j} u_j, u_j \gg \geq 0$, and $\ll t_j T_n u_j, u_j \gg \to 0$ as $j \to \infty$.

So $\overline{\lim_{i\to\infty}} \ll Ju_j, u_j \gg = \overline{\lim_{i\to\infty}}\|u_j\|^2_{1,T} \leq 0$. Therefore, we have $u_j \to 0 \in \partial B_{R_0}$, a contradiction.

Case(ii). $t_0 \neq 0$. Times (2.3) by $\mathcal{A}_{\lambda_j} u_j$, we get

$$
t_j \ll T_n u_j, \mathcal{A}_{\lambda_j} u_j \gg + t_j \int_0^T \|\mathcal{A}_{\lambda_j} u_j\|^2 dt + (1-t_j) \ll Ju_j, \mathcal{A}_{\lambda_j} u_j \gg = 0
$$

$$\text{and } \ll T_n u_j, \mathcal{A}_{\lambda_j} u_j \gg \; = \; \ll \frac{1}{n}\frac{du_j}{dt}, \frac{d\mathcal{A}_{\lambda_j} u_j}{dt} \gg + \ll \frac{1}{n} u_j \mathcal{A}_{\lambda_j} u_j \gg$$
$$+ \; \ll \frac{du_j}{dt} - g(t, J_n u_j), \mathcal{A}_{\lambda_j} u_j \gg .$$

The monotonicity of $\mathcal{A}_{\lambda_j}$ implies that

$$\ll \frac{du_j}{dt}, \frac{d\mathcal{A}_{\lambda_j} u_j}{dt} \gg \geq 0, \quad \ll u_j \mathcal{A}_{\lambda_j} u_j \gg \geq 0.$$

So we have

$$\int_0^T \|\mathcal{A}_{\lambda_j} u_j\|^2 dt \leq -\frac{1-t_j}{t_j} \ll J u_j, \mathcal{A}_{\lambda_j} u_j \gg - \ll \frac{du_j}{dt} - g(t, J_n u_j), \mathcal{A}_{\lambda_j} u_j \gg . \tag{2.4}$$

By the assumption 2, we know $(\mathcal{A}_{\lambda_j} u_j)$ is bounded in $L^2(0,T;H)$.

Without loss of generality, we may assume $\mathcal{A}_{\lambda_j} u_j \rightharpoonup f_0$ weakly in $L^2(0,T;H)$, $T_n u_j \rightharpoonup f_1, J u_j \rightharpoonup f_2$ weakly in W^* (otherwise, taking a subsequence).

By (2.3), we have

$$t_0(f_1 + f_0) + (1 - t_0) f_2 = 0. \tag{2.5}$$

Again, by (2.3), we get

$$\ll T_n u_j, u_j - u_0 \gg + \ll \mathcal{A}_{\lambda_j} u_j, u_j - u_0 \gg + \ll \frac{1-t_j}{t_j} J u_j, u_j - u_0 \gg = 0.$$

Since T_n, J are operators of (S_+),

$$\text{so } \overline{\lim_{j\to\infty}} \ll \mathcal{A}_{\lambda_j} u_j, u_i - u_0 \gg \leq 0 \quad \text{i.e.} \quad \overline{\lim_{j\to\infty}} \ll \mathcal{A}_{\lambda_j} u_j, u_i \gg \leq \ll f_0, u_0 \gg . \tag{2.6}$$

By the monotonicity of $\mathcal{A}$, we have

$$\ll z - \mathcal{A}_{\lambda_j} u_j, v - R_{\lambda_j} u_j \gg \geq 0 \quad , \quad v \in D(\mathcal{A}), z \in \mathcal{A}v;$$
$$\ll z - \mathcal{A}_{\lambda_j} u_j, v - u_j + \lambda_j \mathcal{A}_{\lambda_j} u_j \gg \geq 0 \quad , \quad v \in D(\mathcal{A}), z \in \mathcal{A}v;$$

Letting $j \to \infty$, we get

$$\ll f_0, u_0 \gg \geq \overline{\lim_{j\to\infty}} \ll \mathcal{A}_{\lambda_j} u_j, u_0 \gg \geq \ll -z, v - u_0 \gg + \ll f_0, v \gg , \; v \in D(\mathcal{A}), z \in \mathcal{A}_v;$$
$$\text{i.e.} \quad \ll f_0 - u_0 \gg \geq , \; v \in D(\mathcal{A}), z \in \mathcal{A}_v.$$

The maximality of $\mathcal{A}$ implies that

$$u_0 \in D(\mathcal{A}), \text{and} \quad f_0 \in \mathcal{A}_v. \tag{2.7}$$

Now, we have $\mathcal{A}_{\lambda_j} u_0 \to \mathcal{A}^0 u_0$ strongly in $L^2(0,T;H)$ (see[10], Th23.3). Since $\ll \mathcal{A}_{\lambda_j} u_j - \mathcal{A}_{\lambda_j} u_0, u_j - u_0 \gg \geq 0$, so it follows from (2.3), we get

$$\overline{\lim_{j\to\infty}} \ll t_j T_n u_j + (1-t_j) J u_j, u_j - u_0 \gg \leq 0.$$

Both T_n and J are operators of (S_+). So $u_j \to u_0$ strongly in W, and

$$T_n u_j \rightharpoonup F_1 = T_n u_0, J u_j \to J u_0 = f_2.$$

The maximal monotonicity of $\mathcal{A}$ also implies that $\mathcal{A}_{\lambda_j} u_j \rightharpoonup f_0 \in \mathcal{A} u_0$. In view of (2.5), we get

$$0 \in t_0 (T_n + \mathcal{A}) u_0 + (1 - t_o) J u_0, u_0 \in \partial B_{R_0} \cap D(\mathcal{A}).$$

It is a contradiction to Lemma 2. 2 . We complete the proof.

Lemma 2. 4 *For each $n \geq 1$, there exists an integer $P_n \geq n$, and $u_n \in W$, $\|u_n\|_{1,T} < R_0$, such that*

$$T_n u_n + \mathcal{A}_{\frac{1}{P_n}} u_n = 0, \textit{where} \quad R_0 \quad \textit{is the same as in } \text{lemma 2. 3}.$$

Proof. By lemma 2. 3 and [6], we have

$$\deg(T_n + \mathcal{A}_\lambda, B_{R_0}, 0) = \deg(J, B_{R_0}, 0) = 1, \quad \forall \lambda \in (0, \lambda_0^n).$$

So $(T_n + \mathcal{A}_\lambda) u = 0$ has a solution in B_{R_0} for each $\lambda \in (0, \lambda_0^n)$. Taking an integer $P_n \geq n$, such that $\frac{1}{P_n} < \lambda_0^n$, then there is a $u_n \in B_{R_0}$,such that

$$T_n u_n + \mathcal{A}_{\frac{1}{P_n}} u_n = 0.$$

This ends the proof.

Proof of Theorem. By the above Lemmas, for each $n \geq 1$, there exists an integer $P_n \geq n$, $u_n \in W$, $\|u_n\|_{1,T} < R_0$, such that

$$T_n u_n + \mathcal{A}_{\frac{1}{P_n}} u_n = 0. \tag{2.8}$$

We multiply (2.8) by $\mathcal{A}_{\frac{1}{P_n}} u_n$ amd integrate over $[0,T]$, then we get

$$\int_0^T \langle \frac{1}{n} \frac{du_n}{dt}, \frac{d\mathcal{A}_{\frac{1}{P_n}} u_n}{dt} \rangle dt + \int_0^T \langle \frac{1}{n} u_n, \mathcal{A}_{\frac{1}{P_n}} u_n \rangle dt$$
$$+ \int_0^T \langle \frac{du_n}{dt} - g(t, J_n u_n), \mathcal{A}_{\frac{1}{P_n}} u_n \rangle dt + \int_0^T \langle \|\mathcal{A}_{\frac{1}{P_n}} u_n\|^2 dt = 0.$$

By the monotonicity of $\mathcal{A}_{\frac{1}{P_n}}$, we have $\langle \frac{du_n}{dt}, \frac{d\mathcal{A}_{\frac{1}{P_n}} u_n}{dt} \rangle \geq 0$, and $\langle u_n, \mathcal{A}_{\frac{1}{P_n}} u_n \rangle \geq 0$.

So we get

$$\int_0^T \|\mathcal{A}_{\frac{1}{P_n}} u_n\|^2 dt \leq \int_0^T \langle g(t, J_n u_n) - \frac{du_n}{dt}, \mathcal{A}_{\frac{1}{P_n}} u_n \rangle dt. \tag{2.9}$$

Since $\|u_n\|_{1,T} < R_0$, and the assumption 2 implies that

$$\sup_{n\geq 1} \int_0^T \|\mathcal{A}_{\frac{1}{P_n}} u_n\|^2 dt < +\infty. \tag{2.10}$$

Now, we prove

$$\sup_{n\geq 1} \int_0^T \|\mathcal{A}_{\frac{1}{n}} u_n\|^2 dt < +\infty. \tag{2.11}$$

By the monotonicity of $\mathcal{A}$, we have

$$\ll \mathcal{A}_{\frac{1}{n}} u_n - \mathcal{A}_{\frac{1}{P_n}} u_n, R_{\frac{1}{n}} u_n - R_{\frac{1}{P_n}} u_n \gg \geq 0,$$

$$\ll \mathcal{A}_{\frac{1}{n}} u_n - \mathcal{A}_{\frac{1}{P_n}} u_n, u_n - \frac{1}{n}\mathcal{A}_{\frac{1}{n}} u_n - R_{\frac{1}{P_n}} u_n \gg \geq 0,$$

i.e. $\ll \mathcal{A}_{\frac{1}{n}} u_n - \mathcal{A}_{\frac{1}{P_n}} u_n, -\frac{1}{n}\mathcal{A}_{\frac{1}{n}} u_n + \frac{1}{P_n}\mathcal{A}_{\frac{1}{P_n}} u_n \gg \geq 0.$

$$\frac{1}{n} \ll \mathcal{A}_{\frac{1}{n}} u_n, \mathcal{A}_{\frac{1}{n}} u_n u_n \gg \leq (\frac{1}{P_n} + \frac{1}{n}) \ll \mathcal{A}_{\frac{1}{n}}, \mathcal{A}_{\frac{1}{P_n}} u_n \gg$$
$$-\frac{1}{P_n} \ll \mathcal{A}_{\frac{1}{P_n}} u_n, \mathcal{A}_{\frac{1}{P_n}} u_n \gg .$$

So $\int_0^T \|\mathcal{A}_{\frac{1}{n}} u_n\|^2 dt \leq (1 + \frac{n}{P_n}) \int_0^T \langle \mathcal{A}_{\frac{1}{n}} u_n, \mathcal{A}_{\frac{1}{P_n}} u_n \rangle dt - \frac{n}{P_n} \int_0^T \|\mathcal{A}_{\frac{1}{P_n}} u_n\|^2 dt.$

Since $n \leq P_n$ and by (2.10), we get

$$\sup_{n\geq 1} \int_0^T \|\mathcal{A}_{\frac{1}{n}} u_n\|^2 dt < +\infty.$$

So (2.11) holds.

Since $\|u_n\|_{1,T} < R_0$, $n \geq 1$, so $\int_0^T \|\frac{du_n}{dt}\|^2 dt < R_0^2$, then it follows that

$$\int_0^T \|\frac{d(J_n u_n)}{dt}\|^2 dt \leq R_0^2, n \geq 1 \tag{2.12}$$

By (2.12), we get

$$\sup\{\|J_n u_n(t)\| : n \geq 1, 0 \leq t \leq T\} < +\infty \tag{2.13}$$

Let us remark that $\mathcal{A}_{\frac{1}{n}} u_n(t) = A_n u_n(t)$, $t \in [0, T]$, (2.11) becomes that

$$\int_0^T \|A_n u_n(t)\|^2 dt < +\infty \tag{2.14}$$

We show that $\{J_n u_n\}$ is relatively compact in $L^2(0,T;H)$. Let $\epsilon > 0$. By (2.12) and (2.13), using the same argument of [12], there exists an integer $m_0 > 0$ such that

$$\|J_n u_n(t) - J_n u_n(s)\|^2 < \frac{\epsilon}{6T}, \forall n \geq 1, |t-s| < \frac{2T}{m_0}. \tag{2.15}$$

On the other hand, there exists $D > 0$ such that

$$\inf\{\|A_n u_n(\tau)\| : t \leq \tau \leq t + \frac{T}{m_0}\} < D, \forall n \geq 1, 0 \leq t \leq (1 - \frac{1}{m_0})T.$$

We now choose $\{t_{m,n} : 1 \leq n; 1 \leq m \leq m_0\} \subset [0,T]$ such that $\frac{(m-1)T}{m_0} \leq t_{m,n} \leq \frac{mT}{m_0}$ and $\|A_n u_n(t_{m,n})\| \leq D$, $\forall n \geq 1$, $1 \leq m \leq m_0$.

For fixed $n_1 \geq 1$, we have

$$J_n(u_n(t_{m,n})) = J_{n_1}(J_n u_n(t_{m,n}) + n_1^{-1} A_n u_n(t_{m,n})).\forall n \geq 1.$$

By the assumption 1, we know $\{J_n u_n(t_{m,n})\}$ is relatively compact.

Without loss of generality, we may assume $\{J_n u_n(t_{m,n})\}$ is a convergent sequence for all $1 \leq m \leq m_0$, (otherwise, taking a subsequence).

Again, by (2.15), one can easily see $\{J_n u_n\}$ is convergent in $L^2(0,T;H)$.

$$\text{Let} \quad J_n u_n \to u \quad \text{strongly in} \quad L^2(0,T;H). \tag{2.16}$$

Since $A_n u_n = n(u_n - J_n u_n)$, by (2.14), (2.16), we know

$$u_n \to u \quad \text{strongly in} \quad L^2(0,T;H). \tag{2.17}$$

Since $\mathcal{A}_{\frac{1}{P_n}} u_n = P_n(u_n - R_{\frac{1}{P_n}} u_n)$, by (2.10), (2.17), we know

$$R_{\frac{1}{P_n}} u_n \to u \quad \text{strongly in} \quad L^2(0,T;H). \tag{2.18}$$

In view of (2.10), we may assume

$$\mathcal{A}_{\frac{1}{P_n}} u_n \to z \quad \text{weakly in} \quad L^2(0,T;H). \tag{2.19}$$

By (2.18), (2.19) and the maximl monotonicity of $\mathcal{A}$, we know $z \in \mathcal{A}u$ i.e. $z(t) \in Au(t)$, a.e. $t \in [0,t]$.

Since

$$\ll T_n u_n, v \gg + \ll \mathcal{A}_{\frac{1}{P_n}} u_n, v \gg = 0, \forall v \in W; \tag{2.20}$$

and $\|u_n\|_{1,T} < R_0$, $\frac{1}{n}\frac{du_n}{dt} \to 0$ strongly in $L^2(0,T;H)$.

Letting $n \to \infty$ in (2.20), by (2.16),(2.17),(2.19), we get

$$\ll \frac{du}{dt} - g(t,u), v \gg + \ll z, v \gg = 0, \forall v \in W.$$

So u is a T–perodic solution of (1.1). We complete the proof.

3. Examples

In this section, we give some exsamples that satisfy our condition 3.

Example 1 *Let $A : R^1 \to 2^{R^1}$ be as following*

$$Ax = \begin{cases} x+1, & x > 0; \\ [0,1], & x = 0; \\ x & x < 0. \end{cases}$$

Through a direct simple calculation, we get

$$J_\lambda x = \begin{cases} \frac{\lambda x - 1}{\lambda+1}, & x > \frac{1}{\lambda}; \\ 0, & x \in [0, \frac{1}{\lambda}]; \\ \frac{\lambda x}{\lambda+1}, & x < 0 \end{cases}$$

Let $g(t,x) : R^1 \times R^1 \to R^1$ be as following

$$g(t,x) = |\sin t| x + |\cos x|, \quad (t,x) \in R^1 \times R^1.$$

Then

$$(z - g(t, J_\lambda x), x) = \begin{cases} x^2(1 - \frac{\lambda}{\lambda+1}|\sin t|) + (\frac{\lambda+2}{\lambda+1} - |\cos\frac{\lambda x+1}{\lambda+1}|)x, & x > \frac{1}{\lambda}; \\ x^2, & x \in [0, \frac{1}{\lambda}]; \\ 0, & x = 0; \\ (1 - \frac{\lambda}{\lambda+1}|\sin t|)x^2 - x|\cos\frac{\lambda x}{\lambda+1}|, & x < 0. \end{cases} \quad \forall z \in Ax;$$

So we have $(z - g(t, J_\lambda x), x) \geq 0 > -\frac{a}{\lambda}|J_\lambda x|$, $a > 0$ is any constant.

Example 2 *Let $\Omega \subseteq R^N$ be a bounded domain with a smooth boundaary $\partial\Omega$. $Au = -\Delta u$, $D(A) = \{u \in H_0^1(\Omega) : Au \in L^2(\Omega)\}$; $g(t,u) : R \times L^2(\Omega) \to L^2(\Omega)$ is defined as following $g(t,u) = \lambda_1 u - \frac{u}{1+\|u\|^2}|\sin t|$, $(t,u) \in R^1 \times L^2(\Omega)$; where λ_1 is the first eignvalue of $-\Delta$ under the Dirichlet boundary condition, $\|\cdot\|$ is the L^2-norm.*

It's obvious that

$$\langle Au - g(t,u), u\rangle \geq \frac{\|u\|^2}{1+\|u\|^2}|\sin t|, \quad u \in D(A).$$

For any $a > 0$, $C_{t_0} > 0$, we can not prove

$$\langle Au - g(t,u), u\rangle \geq a\|u\|^2 - C_0$$

So A and g do not satisfy the third condition in [12], but we have

$$\langle Au - g(t, J_\lambda u), u\rangle = \|\nabla u\|^2 - \lambda_1\langle J_\lambda u, u\rangle + \frac{\langle J_\lambda u, u\rangle}{1+\|J_\lambda u\|^2}|\sin t|, \quad u \in D(A);$$

and $0 \leq \langle J_\lambda u, u\rangle \leq |u\|^2$.

So $\langle Au - g(t, J_\lambda u), u\rangle \geq \frac{\langle J_\lambda u, u\rangle}{1+\|J_\lambda u\|^2}|\sin t| \geq -\frac{a}{\lambda}\|J_\lambda u\|$ *for arbitrary constant* $a > 0$.

Remark 1 *In* [12], *condition* (3) *was replaced by* $(*) : \langle z-g(t,v), v\rangle \geq a\|v\|^2 - b$, $\forall v \in D(A)$, $z \in Av$; *where* a, b *are positive constants;*

For appropriately large v *in* $D(A)$, *we have*

$$a\|v\|^2 - b \geq c > 0$$

For $v \in D(A)$, *we have* $J_\lambda v \to v$ *as* $\lambda \to +\infty$. *So if* $(*)$ *holds, then for appropriately large* v *in* $D(A)$ *and sufficcently large* λ, 3 *holds.*

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