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A simple proof of Sarason's result for interpolation in H^∞

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Abstract. In [3], D. Sarason proved the following result : Let q be a non-constant inner function, and let Q be the orthogonal projection from L^2 onto $K = H^2 \ominus T_q H^2$. If $A \in \mathcal{B}(K)$ commutes with QL_zQ, then there is a function ψ in H^∞ such that $\|\psi\|_\infty = \|A\|$ and $A = QL_\psi Q$.

The proof is not so easy and simple. And, in this paper, I will give its simple proof by using some properties of Toeplitz and Hankel operators.

For φ in L^∞, the Laurent operator L_φ is the multiplication operator on L^2 given by $L_\varphi f = \varphi f$ for $f \in L^2$. And the Toeplitz operator T_φ is the operator on H^2 given by $T_\varphi f = PL_\varphi f$ for $f \in H^2$, where P is the orthogonal projection from L^2 onto H^2.

The following results are well known, but, for convenience's sake we state here them without proof.

Proposition 1. T_φ has the following properties.

(1) $T_z{}^* T_\varphi T_z = T_\varphi$

(2) $T_\varphi{}^* = T_{\overline{\varphi}}$, where the bar denotes the complex conjugate

(3) $\|T_\varphi\| = \|\varphi\|_\infty$

Proposition 2. ([2]) $A \in \mathcal{B}(H^2)$ is a Toeplitz operator if and only if $T_z{}^* A T_z = A$. And, in particular, $A \in \mathcal{B}(H^2)$ is analytic Toeplitz operator (i.e., $A = T_\varphi$ for some $\varphi \in H^\infty$) if and only if $T_z A = AT_z$.

Proposition 3. ([1]) If $\mathcal{M}$ is a non-zero invariant subspace of T_z, then there exists an isometric Toeplitz operator T_g such that $\mathcal{M} = T_g H^2$.

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For φ in L^∞, the Hankel operator H_φ is the operator on H^2 given by $H_\varphi f = J(I-P)L_\varphi f$ for $f \in H^2$, where J is the unitary operator on L^2 given by $J(z^{-n}) = z^{n-1}$, $n = 0, \pm 1, \pm 2, \cdots$.

And the following results are known.

Proposition 4. H_φ has the following properties.

(1) $T_z{}^* H_\varphi = H_\varphi T_z$
(Hence $\mathcal{N}_{H_\varphi} = \{x \in H^2\ ;\ H_\varphi x = o\}$ is invariant under T_z and $\mathcal{N}_{H_\varphi} = \{o\}$ or $\mathcal{N}_{H_\varphi} = T_q H^2$, where q is inner)

(2) $H_\varphi{}^* = H_{\varphi^*}$, where $\varphi^*(z) = \overline{\varphi(\overline{z})}$

(3) $H_\varphi = O$ if and only if $(I-P)\varphi = o$ (i.e., $\varphi \in H^\infty$)

Proposition 5. (Nehari) $A \in \mathcal{B}(H^2)$ is a Hankel operator if and only if $T_z{}^* A = AT_z$. Moreover we can choose the symbol $\varphi \in L^\infty$ of $A = H_\varphi$ such as $\|A\| = \|\varphi\|_\infty$.

Nextly, we have the following relations between Toeplitz and Hankel operators.

Theorem 1. For any $\psi \in H^\infty$, $H_\varphi T_\psi = H_{\varphi\psi}$.

Corollary 1. For any $\psi \in H^\infty$, $T_\psi{}^* H_\varphi = H_\varphi T_{\psi^*}$.

Proof.

$$\begin{aligned} T_\psi{}^* H_\varphi &= (H_\varphi{}^* T_\psi)^* = (H_{\varphi^*} T_\psi)^* \\ &= H_{\varphi^*\psi}{}^* = H_{\varphi\psi^*} = H_\varphi T_{\psi^*} \quad \text{by Theorem 1.} \end{aligned}$$

Concerning the invariant subspaces of $T_z{}^*$, we have the following.

Theorem 2. If $\mathcal{M} \neq H^2$ is an invariant subspace of $T_z{}^*$, then there exists an inner function g such that $\mathcal{M} = [H_{\overline{g}}{}^* H^2]^{\sim L^2} = H^2 \ominus T_g H^2$, where $[H_{\overline{g}}{}^* H^2]^{\sim L^2}$ denotes the L^2-closure of $H_{\overline{g}}{}^* H^2$.

Proof. Since $H^2 \ominus \mathcal{M}$ is a non-zero invariant subspace of T_z, there exists an inner function g such that $H^2 \ominus \mathcal{M} = T_g H^2$ by Proposition 3. For $f \in L^\infty$,

$$PL_f z^{-n} = H_{\overline{f}}{}^* z^{n-1} \quad \text{for} \quad n \geq 1.$$

Hence

$$\begin{aligned}\mathcal{M} &= H^2 \ominus T_g H^2 = \vee\{PL_z{}^n g \ :\ n = -1, -2, \cdots\}\\ &= \vee\{PL_g z^n \ :\ n = -1, -2, \cdots\}\\ &= \vee\{H_{\overline{g}}{}^* z^n \ :\ n = 0, 1, 2, \cdots\}\\ &= [H_{\overline{g}}{}^* H^2]^{\sim L^2}.\end{aligned}$$

Corollary 2. If $[H_\varphi H^2]^{\sim L^2} \neq H^2$, then there exists an inner function g such that $[H_\varphi H^2]^{\sim L^2} = [H_{\overline{g}}{}^* H^2]^{\sim L^2} = H^2 \ominus T_g H^2$ and $\mathcal{N}_{H_{\varphi^*}} = T_g H^2$.

Proof. By Proposition 4 (1), $[H_\varphi H^2]^{\sim L^2}$ is invariant under $T_z{}^*$ and, by Theorem 2, we have the conclusion.

Now we give a simple proof of the following Sarason's result which is our main purpose.

Theorem 3. ([3]) Let q be a non-constant inner function, and let Q be the orthogonal projection from L^2 onto $K = H^2 \ominus T_q H^2$. If $A \in \mathcal{B}(K)$ commutes with QL_zQ, then there is a function ψ in H^∞ such that $\|\psi\|_\infty = \|A\|$ and $A = QL_\psi Q$.

Proof. By Theorem 2, $K = [H_{\overline{q}}{}^* H^2]^{\sim L^2}$. Since $T_z{}^* = (QL_zQ)^*$ on K, by Proposition 4 (1)

$$T_z{}^* A^* H_{\overline{q}}{}^* = (QL_zQ)^* A^* H_{\overline{q}}{}^* = A^*(QL_zQ)^* H_{\overline{q}}{}^* = A^* T_z{}^* H_{\overline{q}}{}^* = A^* H_{\overline{q}}{}^* T_z.$$

Therefore by Proposition 5, $A^* H_{\overline{q}}{}^*$ is a Hankel operator and

$$A^* H_{\overline{q}}{}^* = H_\varphi$$

for some $\varphi \in L^\infty$ such as $\|H_\varphi\| = \|\varphi\|_\infty$. Let

$$\psi^* = \varphi q^*.$$

Then by Proposition 4 (2) and Theorem 1,

$$H_{\psi^*} = H_\varphi T_{q^*} = A^* H_{\overline{q}}{}^* T_{q^*} = A^* H_{\overline{q^*} q^*}.$$

Since q is inner, $\overline{q^*} q^* = 1$. Since $H_1 = O$, $H_{\psi^*} = O$. Then by Proposition 4 (3), $\psi^* \in H^\infty$, hence $\psi \in H^\infty$. Then by Theorem 1 and Corollary 1, we have

$$A^* H_{\overline{q}}{}^* = H_\varphi = H_{\overline{q^*}\psi^*} = H_{\overline{q}}{}^* T_{\psi^*} = T_\psi{}^* H_{\overline{q}}{}^*$$

and $A^* = QT_\psi{}^*Q$ and hence $A = QL_\psi Q$. And then $\|A\| \leq \|T_\psi\| = \|\psi\|_\infty$ and, conversely,

$$\|\psi\|_\infty = \|\varphi\|_\infty = \|H_\varphi\| = \|A^* H_{\overline{q}}{}^*\| \leq \|A^*\| \; \|q\|_\infty = \|A\|.$$

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