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ASYMPTOTIC PROPERTIES OF THE SOLUTIONS OF A CLASS OF OPERATOR-DIFFERENTIAL EQUATIONS

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Summary. Some asymptotic properties of the nonoscillating solutions of operator-differential equations of arbitrary order are investigated.

1. Introduction

The goal of the present paper is by means of a single approach to investigate some asymptotic properties of the nonoscillating solutions of differential equations with "maximd", with distributed delay, with autoregulable deviation, integro-differential equations, etc. To realize this single approach an operator with certain properties is introduced, as well as appropriately chosen operator-differential equations and inequalities. In the paper results obtained in [2]-[8] are generalized.

2. Preliminary notes

Consider the operator-differential equation

$$[c_{n-1}(t)[c_{n-2}(t)[\cdots[c_0(t)x(t)]'\cdots]']']'+\delta F(t, (\mathcal{A}x)(t))=c(t) \tag{1}$$

for $t\geqq t_0$, where $t_0\in \boldsymbol{R}$ is a fixed number, $n\geqq 1$ is an integer, $\mathcal{A}$ is an operator with certain properties, $\delta=\pm 1$ and

$$c_i\in C^{n-i}([t_0, \infty); (0, \infty)) \ (i=0, 1, \cdots, n-1)$$

Introduce the following notation:

$$(L_0x)(t)=c_0(t)x(t)$$

$$(L_ix)(t)=c_i(t)[(L_{i-1}x)(t)]', \qquad 1\leqq i\leqq n, \qquad c_n(t):\equiv 1,$$

where $c_i\in C^{n-i}([t_0, \infty); (0, \infty))$, $0\leqq i\leqq n$; $x: [T_X, \infty)\rightarrow \boldsymbol{R}$, $T_X\geqq t_0$.

Denote by $\mathscr{D}_n$ the set of all functions $x \in C([T_X, \infty); \boldsymbol{R})$ such that the functions $L_i x$ $(0 \leqq i \leqq n)$ exist and are continuous in $[T_X, \infty)$.

Definition 1. The function x is said to be a *solution* of equation (1) if $x \in \mathscr{D}_n$ and x satisfies equation (1) for $t \geqq \max\{T_x, T_{Ax}\}$

Definition 2. A given function $u: [t_0, \infty) \to \boldsymbol{R}$ is said to *eventually enjoy the property* P if there exists a point $t_{P,u} \geqq t_0$ such that for $t \geqq t_{P,u}$ it enjoys the property P.

Definition 3. The solution x of equation (1) is said to be *regular* if $\sup\{|x(t)|\} > 0$ eventually.

Definition 4. The regular solution x of equation (1) is said to *oscillate* if $\sup\{t: x(t)=0\} = \infty$. Otherwise, the regular solution is said to be *nonoscillating*.

Introduce the following conditions:

H1. $c_i \in C^{n-i}([t_0, \infty); (0, \infty))$, $0 \leqq i \leqq n$.

H2. $\delta = \pm 1$.

H3. $\displaystyle\int^{\infty} \frac{dt}{c_i(t)} = \infty$, $1 \leqq i \leqq n-1$.

H4. $c \in C([t_0, \infty); \boldsymbol{R})$.

H5. $\mathcal{A}: \mathscr{D}_n \to C([T_{AX}, \infty); \boldsymbol{R})$, $T_{AX} \geqq t_0$.

H6. If $u, v \in \mathscr{D}_n$ and $u(t) \leqq v(t)$ for $t \geqq t_0$, then $(\mathcal{A}u)(t) \leqq (\mathcal{A}v)(t)$ for $t \geqq T_{AX}$

H7. If u_p, $u \in \mathscr{D}_n$ $(p = 1, 2, \cdots)$ and $\{u_p\}_{p=1}^{\infty}$ is a monotone sequence and $\lim_{p\to\infty} u_p(t) = u(t)$ for $t \geqq t_0$, then $\lim_{p\to\infty} (\mathcal{A}u_p)(t) = (\mathcal{A}u)(t)$ for each $t \geqq t_0$.

H8. If $u \in \mathscr{D}_n$ and u is an eventually of constant sign and nonzero function, then the function $\mathcal{A}u$ is also eventually of constant sign and nonzero, and they are of the same sign.

H9. $F \in C([t_0, \infty) \times (\boldsymbol{R}_+ \cup \boldsymbol{R}_-))$, $\boldsymbol{R}_+ = (0, \infty)$, $\boldsymbol{R}_- = (-\infty, 0)$.

LEMMA 1 [8]. *Let the following conditions hold:*

1. *Conditions* H1-H3 *are met.*
2. $x \in \mathscr{D}_n$, $x(t) > 0$ *for* $t \geqq T$ $(T \geqq t_0)$.
3. $(L_n x)(t)$ *is of constant sign in* $[T, \infty)$.

Then there exists an integer l *such that:*

1. *For* $(L_n x)(t) \leqq 0$, $n+l$ *is an odd number.*
2. *For* $(L_n x)(t) \geqq 0$, $n+l$ *is an even number.*
3. $(-1)^{l+j}(L_j x)(t) \geqq 0$ *for* $l \leqq n-1$, $j = l, \cdots, n-1$, $t \geqq T$
4. $(L_i x)(t) > 0$ *for* $l > 1$, $1 \leqq i \leqq l-1$, $t \geqq T$

LEMMA 2 [8]. *Let the following conditions hold:*

1. *Condition* H1 *is satisfied.*

2. $0<\liminf_{t\to\infty} c_i(t)\leqq\limsup_{t\to\infty} c_i(t)<\infty,\ 1\leqq i\leqq n-1.$

3. $x\in\mathscr{D}_n$.

Then, if one of the following two conditions hold:

1. L_0x *is a bounded function in* $[T, \infty)$ *and* $\lim_{t\to\infty}(L_nx)(t)=0$.

2. L_nx *is a bounded function in* $[T, \infty)$ *and* $\lim_{t\to\infty}(L_0x)(t)\in\boldsymbol{R}$,

then $\lim_{t\to\infty}(L_ix)(t)=0,\ 1\leqq i\leqq n-1.$

For any function $y\in C([T, \infty); \boldsymbol{R})$ and for any integer l, $0\leqq l\leqq n$ define the function,

$$(\varphi_c^l y)(t)=\begin{cases} \dfrac{1}{c_0(t)}\displaystyle\int_t^\infty\frac{1}{c_1(s_1)}\int_{s_1}^\infty\frac{1}{c_2(s_2)}\cdots\int_{s_{n-2}}^\infty\frac{1}{c_{n-1}(s_{n-1})}\int_{s_{n-1}}^\infty y(s)ds\,ds_{n-1}\cdots ds_2ds_1 & \text{for } l=0\\[2ex] \dfrac{1}{c_0(t)}\displaystyle\int_T^t\frac{1}{c_1(s_1)}\int_T^{s_1}\frac{1}{c_2(s_2)}\cdots\int_T^{s_{l-1}}\frac{1}{c_l(s_l)}\int_{s_l}^\infty\frac{1}{c_{l+1}(s_{l+1})}\cdots\int_{s_{n-1}}^\infty y(s)ds\cdots ds_l ds_{l-1}\cdots ds_2ds_1 & \text{for } 0<l\leqq n-1\\[2ex] \dfrac{1}{c_0(t)}\displaystyle\int_T^t\frac{1}{c_1(s_1)}\int_T^{s_1}\frac{1}{c_2(s_2)}\cdots\int_T^{s_{n-2}}\frac{1}{c_{n-1}(s_{n-1})}\int_T^{s_{n-1}} y(s)ds\,ds_{n-1}\cdots ds_2ds_1 & \text{for } l=n\end{cases}$$

3. Main results

Theorem 1. *Let the following conditions hold:*

1. *Conditions* H1-H9 *are met.*

2. *There exists a function* w *defined in* $[t_0, \infty)$ *such that* $w\in\mathscr{D}_n$ *and* $(L_nw)(t)=c(t)$.

3. *The function* L_0w *is bounded below in the interval* $[t_0, \infty)$.

4. *There exists a positive solution* y *of the inequality*

$$\delta(L_nx)(t)+F(t, (\mathscr{A}x)(t))\leqq\delta c(t) \tag{2}$$

such that $\liminf_{t\to\infty}(L_0y)(t)>0$.

5. $F(t, u)>0$ *for* $(t, u)\in[t_0, \infty)\times\boldsymbol{R}_+$ *and* $F(t, u)$ *is an increasing function with respect to* $u\in\boldsymbol{R}_+$.

Then there exists a positive solution x *of equation* (1) *with the following properties:*

1. $\liminf_{t\to\infty}(L_0x)(t)>0$

2. $x(t)\leqq y(t)$ *eventually.*

Proof. Let $y(t)>0$ be a solution of inequality (2) in the interval $[T_0, \infty)$ $(T_0\geqq t_0)$ and $\liminf_{t\to\infty}(L_0y)(t)>0$. Then $(\mathscr{A}y)(t)>0$ eventually.

Introduce the following notation:

$$w_0(t)=w(t)-\frac{1}{c_0(t)}\liminf_{t\to\infty}(L_0w)(t)$$

$$u(t)=y(t)-w_0(t)$$

Then $0<F(t,(\mathcal{A}y)(t))\leqq-\delta(L_nu)(t)$ teventually, i.e., the function $(L_nu)(t)$ is of constant sign for $t\geqq T_0$. Hence the function L_0u is monotone in $[T_0, \infty)$. This fact implies the existence of

$$\lim_{t\to\infty}(L_0u)(t)\in \boldsymbol{R}\cup\{-\infty, +\infty\}$$

But $\lim_{t\to\infty}(L_0u)(t)=\liminf_{t\to\infty}(L_0y)(t)>0$. Thus we obtained that u is an eventually positive function. Let $[\tau, \infty)$, $\tau\geqq T_0$ be the largest interval in which the function u is positive.

From Lemma 1 it follows that there exists an integer l $(0\leqq l\leqq n)$ such that

1. $n+l$ is an odd number for $\delta=1$.
2. $n+l$ is an even number for $\delta=-1$.
3. $(-1)^{l+j}(L_ju)(t)\geqq 0$ for $l\leqq n-1$; $j=l, \cdots, n-1$; $t\geqq\tau$.
4. $(L_iu)(t)>0$ for $l>1$, $1\leqq i\leqq l-1$, $t\geqq\tau^*$, $\tau^*\geqq\tau$.

Introduce the following notation:

$$T=\begin{cases}\tau & \text{for } l=0 \text{ or } l=1\\ \tau^* & \text{for } l>1\end{cases}$$

$$K=\begin{cases}\lim_{t\to\infty}(L_0u)(t), & l=0\\ (L_0u)(T), & l>0\end{cases}$$

From condition H1 and the fact that the function u is eventually positive it follows that $K>0$. From (2) we obtain that $-K<(L_0w_0)(t)$ eventually, i.e., $(K/c_0(t))+w_0(t)>0$ for $t\geqq T$.

After a repeated integration of inequality (2) we obtain that

$$y(t)\geqq\frac{K}{c_0(t)}+w_0(t)+(\varphi_c^l(F(\cdot, \mathcal{A}y))(t)$$

Let X be the set of all continuous functions x for $t\geqq T$ such that

$$\frac{K}{c_0(t)}+w_0(t)\leqq x(t)\leqq y(t)$$

For any function $x\in X$ define the function $\bar{x}(t)$:

$$\bar{x}(t)=\begin{cases}x(t), & t\geqq T\\ \dfrac{x(T)}{y(T)}y(t), & T_0\leqq t\leqq T\end{cases}$$

From the definition of $\bar{x}(t)$ it follows that

$$\frac{K}{c_0(t)}+w_0(t)\leqq \bar{x}(t)\leqq y(t), \qquad t\geqq T_0$$

Define the operator $S: X\to E$ by the formula

$$(Sx)(t)=\frac{K}{c_0(t)}+w_0(t)+(\varphi_c{}^l(F(\cdot, \mathcal{A}\bar{x})))(t)$$

where E is the set of all continuous functions in $[T_0, \infty)$.

The inclusion $SX\subset X$ is valid since:

1. From the definition of the operator S it follows that

$$\frac{K}{c_0(t)}+w_0(t)\leqq (Sx)(t), \qquad t\geqq T\,.$$

2. From condition 3 of Theorem 1 and condition H6 we obtain that

$$y(t)\geqq (Sx)(t)\,.$$

Let x_1, $x_2\in X$ and $0<x_1(t)\leqq x_2(t)$. From the definition of the operator S it follows that $0<(Sx_1)(t)\leqq (Sx_2)(t)$ for $t\geqq T_0$, i.e., S is a monotone increasing mapping of the set X into itself. Let $\{x_n(t)\}_{n=0}^{\infty}$ be a montone decreasing sequence of elements of the set X for $t\geqq T$ obtained by the following recurrent formula:

$$x_0(t)=y(t), \qquad t\geqq T$$

$$x_n(t)=(Sx_{n-1})(t), \qquad t\geqq T \tag{4}$$

Let $\lim\limits_{n\searrow\infty} x_n(t)=x(t)$ for $t\geqq T$. Then $\lim\limits_{n\to\infty}(\mathcal{A}x_n)(t)=(\mathcal{A}x)(t)$. From the Lebesgue dominated convergence theorem we obtain that $\lim\limits_{n\to\infty}(Sx_n)(t)=(Sx)(t)$ for $t\geqq T$. But from (4) it follows that $\lim\limits_{n\to\infty}(Sx_n)(t)=x(t)$. Then we obtain that $(Sx)(t)=x(t)$, i.e., $x(t)$ is the positive solution sought of equation (1) such that $\liminf\limits_{t\to\infty}(L_0x)(t)>0$, $x(t)\leqq y(t)$ eventually. □

Theorem 2. *Let the following conditions hold*:

1. *Conditions* 1 *and* 2 *of Theorem* 1 *are met.*
2. *The function* L_0w *is bounded above for* $t\geqq t_0$.
3. *There exists a negative solution* y *of the inequality*

$$\delta(L_nx)(t)+F(t, (\mathcal{A}x)(t))\geqq \delta c(t)$$

such that $\limsup\limits_{t\to\infty}(L_0y)(t)<0$.

4. $F(t, u)<0$, $(t, u)\in([t_0, \infty)\times \boldsymbol{R}_-)$.

$F(t, u)$ *is an increasing function with respect to* $u\in\boldsymbol{R}_-$.

Then there exists a negative solution x *of equation* (1) *with the properties*:

1. $\limsup_{t\to\infty}(L_0x)(t)<0$.
2. $x(t)\geqq y(t)$ eventually. (5)

The proof of Theorem 2 is carried out along the scheme of the proof of Theorem 1.

Theorem 3. *Let the following conditions hold:*

1. *Condition* 1 *of Theorem* 2 *is met.*
2. *There exists* $\lim_{t\to\infty}(L_0w)(t)\in \boldsymbol{R}$.
3. $F(t, u)>0$ *for* $(t, u)\in[t_0, \infty)\times\boldsymbol{R}_+$ $(F(t, u)<0$ *for* $(t, u)\in[t_0, \infty)\times\boldsymbol{R}_-)$.

Then each positive (negative) solution of equation (1) *enjoys the property*

$$\lim_{t\to\infty}(L_0x)(t)\in\boldsymbol{R}\cup\{-\infty, +\infty\} \tag{6}$$

Proof. Let x be a positive solution of equation (1) in the interval $[T, \infty)$, $T\geqq t_0$.

Introduce the notation:

$$w_0(t)=w(t)-\frac{1}{c_0(t)}\lim_{t\to\infty}(L_0w)(t)$$

$$u(t)=x(t)-w_0(t), \qquad t\geqq T$$

Then $(L_nu)(t)=-\delta F(t, (\mathcal{A}x)(t))$, i.e., L_nu is of constant sign in the interval $[T, \infty)$. This implies that L_0u is a monotone function for $t\geqq T$, i.e., there exists

$$\lim_{t\to\infty}(L_0u)(t)\in\boldsymbol{R}\cup\{-\infty, +\infty\}$$

But $\lim_{t\to\infty}(L_0u)(t)=\lim_{t\to\infty}(L_0x)(t)$, i.e., there also exists

$$\lim_{t\to\infty}(L_0x)(t)\in\boldsymbol{R}\cup\{-\infty, +\infty\}$$

Theorem 4. *Let the following conditions hold:*

1. *Conditions* 1, 2 *and* 5 *of Theorem* 1 *are satisfied.*
2. *There exists a positive solution* y *of the equation*

$$(L_nx)(t)+\delta F(t, (\mathcal{A}x)(t))=0 \tag{7}$$

such that $\lim_{t\to\infty}(L_0y)(t)>0$.

Then there exists a positive solution x *of equation* (1) *with the properties:*

1. $\lim_{t\to\infty}(L_0x)(t)>0$.
2. $x(t)\leqq y(t)$ *eventually.*

Proof. Let y be a positive solution of equation (7) in the interval $[T_0, \infty)$, $T_0\geqq t_0$ and $\lim_{t\to\infty}(L_0y)(t)>0$.

Introduce the notation:

$$w_0(t)=w(t)-\frac{1}{c_0(t)}\lim_{t\to\infty}(L_0w)(t)$$

$$u(t)=y(t)+w_0(t).$$

Then $\lim_{t\to\infty}(L_0u)(t)=\lim_{t\to\infty}(L_0y)(t)>0$.

Choose a constant c such that $0<c<\lim_{t\to\infty}(L_0u)(t)$. Let us choose $T\geqq T_0$ so that for $t\geqq T$, $(L_0u)(t)>c$, $(L_0w_0)(t)\leqq c$. Then for the function $\bar{u}(t)=u(t)-(c/c_0(t))$ we obtain that $0<\bar{u}(t)\leqq y(t)$, $t\geqq T$.

Consequently, $\bar{u}(t)>0$ is a solution of the inequality

$$\delta(L_n\bar{u})(t)+F(t, (\mathcal{A}\bar{u})(t))\leqq\delta c(t)$$

Moreover, $\lim_{t\to\infty}(L_0\bar{u})(t)=\lim_{t\to\infty}(L_0u)(t)>0$. From Theorem 1 it follows that there exists a positive solution x of equation (1) such that $\lim_{t\to\infty}(L_0x)(t)>0$, $x(t)\leqq\bar{u}(t)\leqq y(t)$ eventually. □

Theorem 5. *Let the following conditions hold:*

1. *Conditions* 1 *and* 2 *of Theorem* 3 *and condition* 4 *of Theorem* 2 *are met.*
2. *There exists a negative solution* y *of equation* (7) *such that* $\lim_{t\to\infty}(L_0y)(t)<0$.

Then there exists a negative solution x *of equation* (1) *with the properties:*

1. $\lim_{t\to\infty}(L_0x)(t)<0$.
2. $x(t)\geqq y(t)$ *eventually.*

The proof of Theorem 5 is carried out along the scheme of the proof of Theorem 4.

Consider the operator-differential equation

$$[c^*_{n-1}(t)[c^*_{n-2}(t)[\cdots[c^*_0(t)x(t)]'\cdots]']']'+\partial F(t, (\mathcal{A}x)(t))=0 \tag{8}$$

where $c^*_i\in C^{n-i}([t_0, \infty); \boldsymbol{R}_+)$, $(0\leqq i\leqq n-1)$.

Introduce the following notation:

$$(L^*_0x)(t)=c^*_0(t)x(t)$$

$$(L^*_ix)(t)=c^*_i(t)[(L^*_{i-1}x)(t)]', \qquad i=1, 2, \cdots, n;\ c^*_n(t)\equiv1.$$

Theorem 6. *Let the following conditions hold:*

1. $\int^{\infty}\frac{dt}{c^*_i(t)}=\infty$, $1\leqq i\leqq n-1$.
2. $c^*_i(t)\leqq c_i(t)$ *for* $t\geqq t_0$, $0\leqq i\leqq n-1$.
3. *Conditions* H1-H9 *and condition* 5 *of Theorem* 1 *are met.*

4. *There exists a positive solution* y *of equation* (8) *such that* $\lim_{t\to\infty}(L_0^*y)(t)>0$.

Then there exists a positive solution x *of equation* (7) *with the following properties*:

$$\begin{aligned}&1.\ \lim_{t\to\infty}(L_0x)(t)>0.\\ &2.\ x(t)\leqq y(t)\ \textit{eventually}.\end{aligned}\tag{9}$$

Proof. Let y be a solution of equation (8) in $[T_0, \infty)$ for $T_0\geqq t_0$ and $\lim_{t\to\infty}(L_0^*y)(t)>0$. Consequently,

$$(L_n^*y)(t)=-\delta F(t, (\mathcal{A}y)(t)), \text{ i.e., } (L_n^*y)(t)\geqq 0 \quad \text{for } \delta=-1$$

and

$$(L_n^*y)(t)\leqq 0 \quad \text{for } \delta=1, \ t\geqq T_0.$$

From Lemma 1 it follows that there exists an integer l, $0\leqq l\leqq n$ such that $n+l$ is an odd number for $\delta=1$, $n+l$ is an even number for $\delta=-1$ and

$$(-1)^{l+j}(L_j^*y)(t)\geqq 0, \quad t\geqq T_0, \quad l\leqq n-1, \quad l\leqq j\leqq n-1.$$

$$(L_i^*y)(t)>0, \quad t\geqq T_1, \ T_1\geqq T_0, \quad l>1, \quad 1\leqq i\leqq l-1.$$

Introduce the following notation:

$$T=\begin{cases} T_0, & l=0 \ \text{ or } \ l=1\\ T_1, & l>1\end{cases}$$

$$K=\begin{cases} \lim_{t\to\infty}(L_0^*y)(t), & l=0\\ (L_0^*y)(T), & l>0\end{cases}$$

Then for $t\geqq T$ we obtain that

$$y(t)\geqq \frac{K}{c_0^*(t)}+(\varphi_{c^*}^l(F(\cdot, \mathcal{A}y)))(t)$$

But from condition 2 of Theorem 6 it follows that

$$y(t)\geqq \frac{K}{c_0(t)}+(\varphi_c^l F(\cdot, \mathcal{A}y)))(t)$$

Consider the set X of all continuous functions x in $[T, \infty)$ such that $K/c_0(t)\leqq x(t)\leqq y(t)$ and define

$$\bar{x}(t)=\begin{cases} x(t), & t\geqq T\\ \dfrac{x(T)}{y(T)}y(t), & T_0\leqq t\leqq T\end{cases}$$

for each function $x\in X$.

Define the operator S: $X\to C([t_0, \infty); \boldsymbol{R})$ by the formula

$$(Sx)(t)=\frac{K}{c_0(t)}+(\varphi_c^l F(\cdot, \mathcal{A}\bar{x}))(t)$$

It is immediately verified that

$$\frac{K}{c_0(t)} \leqq (Sx)(t) \leqq y(t), \quad t \geqq t_0, \text{ i.e., } S: X \to X$$

Let $x_1, x_2 \in X$ and $x_1(t) \leqq x_2(t)$, $t \geqq T$. Then

$$(Sx_1)(t) \leqq (Sx_2)(t), \qquad t \geqq T.$$

Consider the convergent sequence $\{x_k(t)\}_{k=0}^{\infty}$, $t \geqq T$ such that

$$x_0(t) = y(t)$$

$$x_k(t) = (Sx_{k-1})(t), \qquad K = 1, 2, \cdots$$

i.e. the sequence $\{x_k(t)\}_{k=0}^{\infty}$ is decreasing for $t \geqq T$. If $x(t) = \lim_{n\to\infty} x_n(t)$ then from the Lebesgue dominated convergence theorem it follows that $x(t) = (Sx)(t)$, i.e., $x(t)$ is a positive solution of equation (7) with the properties (9).

Theorem 7. *Let the following conditions hold:*

1. *Conditions* 1, 2 *and* 3 *of Theorem* 6 *and condition* 4 *of Theorem* 2 *are met.*
2. *There exists a negative solution* y *of equation* (8) *with the properties* $\lim_{t\to\infty} (L_0^* y)(t) < 0$.

Then there exists a negative solution x *of equation* (7) *with the following properties:*

1. $\lim_{t\to\infty} (L_0 x)(t) < 0$.
2. $x(t) \geqq y(t)$ *eventually.*

The proof of Theorem 7 is carried out along the scheme of the proof of Theorem 6.

Theorem 8. *Let the following conditions hold:*

1. *Conditions* H1, H2, H4-H9 *and condition* 5 *of Theorem* 1 *are met.*
2. *There exists a positive solution* y *of the inequality*

$$\delta(L_n x)(t) + F(t, (\mathcal{A}x)(t)) \leqq 0$$

such that $\lim_{t\to\infty} (L_0 y)(t) > 0$.

Then there exists a positive solution x *of equation* (7) *with the following properties:*

$$\lim_{t\to\infty} (L_0 x)(t) > 0$$

$$\begin{aligned} &0 < (L_i x)(t) \leqq (L_i y)(t) \quad \textit{eventually}, \ 0 \leqq i \leqq l-1, \quad l \geqq 1 \\ &0 \leqq (-1)^{l+i}(L_i x)(t) \leqq (-1)^{l+i}(L_i y)(t) \quad \textit{eventually}, \quad l \leqq i \leqq n-1, \quad l \leqq n-1, \end{aligned} \tag{10}$$

where l is an integer, $0 \leqq l \leqq n$, such that $n+l$ is odd for $\delta=1$ and $n+l$ is even for $\delta=-1$.

Theorem 8 is a corollary of Theorem 1 and Lemma 1.

Theorem 9. *Let the following conditions hold:*

1. *Conditions* H1, H2, H4-H9 *and condition* 4 *of Theorem* 2 *are met.*
2. *There exists a negative solution y of the inequality*

$$\delta(L_n x)(t)+F(t, (\mathcal{A}x)(t)) \geqq 0$$

such that $\lim_{t\to\infty}(L_0 y)(t)<0$.

Then there exists a negative solution x of equation (7) *with the properties:*

$$(L_i y)(t) \leqq (L_i x)(t) \quad \textit{eventually}, \quad l>0, \quad 0 \leqq i \leqq l-1.$$

$$\lim_{t\to\infty}(L_0 x)(t)<0 \tag{11}$$

$$(-1)^{l+i}(L_i y)(t) \leqq (-1)^{l+i}(L_i x)(t) \leqq 0 \quad \textit{eventually}, \quad l \leqq n-1, \quad l \leqq i \leqq n-1.$$

where l is an integer $(0 \leqq l \leqq n)$ such that $n+l$ is odd for $\delta=1$ and $n+l$ is even for $\delta=-1$.

Theorem 9 is a corollary of Theorem 2 and Lemma 1.

Theorem 10. *Let the following conditions hold:*

1. *Conditions* 1, 2 *and* 3 *of Theorem* 1 *are met.*
2. *There exists a positive solution y of equation* (1) *such that* $\liminf_{t\to\infty}(L_0 y)(t)$ >0.

Then there exists a positive solution x of equation (7) *with the following properties:*

1. $\lim_{t\to\infty}(L_0 x)(t)>0$.
2. $x(t) \leqq y(t)$ *eventually.* (12)

Proof. Let y be a positive solution of equation (1) in $[T_0, \infty)$, $T_0 \geqq t_0$ such that $\liminf_{t\to\infty}(L_0 y)(t)>0$.

Introduce the folloing notation:

$$w_0(t)=w(t)-\frac{1}{c_0(t)}\liminf_{t\to\infty}(L_0 y)(t)$$

$$v(t)=y(t)-w_0(t)$$

Then $\lim_{t\to\infty}(L_0 v)(t)=\liminf_{t\to\infty}(L_0 y)(t)>0$. From the fact that $\lim_{t\to\infty}(L_0 v)(t)>0$ it follows that we can choose a constant c such that $0<c<\lim_{t\to\infty}(L_0 v)(t)$. Choose

$T \geqq T_0$ so that for $t \geqq T$ the following inequalities be valid

$$(L_0 v)(t) > c > 0 \quad \text{and} \quad (L_0 w)(t) \geqq -c$$

If we denote $\bar{v}(t) = v(t) - (c/c_0(t))$ for $t \geqq T$, then we obtain that

$$0 < \bar{v}(t) \leqq y(t), \qquad t \geqq T$$

Then $\delta(L_n \bar{v})(t) + F(t, (\mathcal{A}\bar{v})(t)) \leqq 0$. Since $\lim_{t\to\infty}(L_0 \bar{v})(t) = \lim_{t\to\infty}(L_0 v)(t) - c > 0$, then from Theorem 8 it follows that there exists a positive solution x of equation (7) for which $\lim_{t\to\infty}(L_0 x)(t) > 0$ and $x(t) \leqq \bar{v}(t) \leqq y(t)$.

Theorem 11. *Let the following conditions hold:*

1. *Conditions* 1, 2, 3 *and* 4 *of Theorem* 2 *are valid.*
2. *There extsts a negative solution y of equation* (1) *such that* $\limsup_{t\to\infty}(L_0 y)(t) < 0$.

Then there exists a negative solution x of equation (7) *with the following properties:*

1. $\lim_{t\to\infty}(L_0 x)(t) < 0$.
2. $x(t) \geqq y(t)$ *eventually.*

The proof of Theorem 11 *is carried out along the scheme of the proof of Theorem* 10.

4. Some particular realizations of the operator $\mathcal{A}$

1. Let $(\mathcal{A}x)(t) = \max_{S \in M(t)} x(s)$, where $M(t) = [p(t), q(t)]$ is a compact subset of the interval $[t_0, \infty)$, $t \geqq t_0$ and $\lim_{t\to\infty} p(t) = \infty$, $p(t) \leqq q(t)$ for $t \geqq t_0$, $p, q \in C([t_0, \infty); \mathbf{R})$.

We shall prove that for the so defined operator conditions H5-H8 are satisfied.

In fact, if $0 < x(t) \leqq y(t)$ for $t \geqq t_0$, then it is immediately verified that $0 < (\mathcal{A}x)(t) = \max_{s \in M(t)} x(s) \leqq \max_{s \in M(t)} y(s) = (\mathcal{A}y)(t)$ and $x(t)(\mathcal{A}x)(t) > 0$ for $t \geqq t_0$.

Let $x, x_k \in C([t_0, \infty); \mathbf{R})$, $k = 0, 1, \cdots$, $x_k(t) \leqq x(t)$ or $x_k(t) \geqq x(t)$ and $\lim_{k\to\infty} x_k(t) = x(t)$.

We shall prove that $\lim_{k\to\infty}[\max_{s \in M(t)} x_k(s)] = \max_{s \in M(t)} x(s)$.

To this end we shall use the inequality

$$\max_{s \in M(t)} x(s) - \max_{s \in M(t)} y(s) \leqq \max_{s \in M(t)} [x(s) - y(s)] \quad \text{(cf. [9]).}$$

From the fact that $x_k(t) \xrightarrow[k\to\infty]{} x(t)$ for $t \geqq t_0$ it follows that for each $\varepsilon > 0$ there exists $k_0 > 0$ such that if $k \geqq k_0$, then $|x_k(t) - x(t)| < \varepsilon$ for $t \geqq t$.

Then

$$\left|\max_{s\in M(t)} x_k(s)-\max_{s\in M(t)} x(s)\right|=\begin{cases}\max\limits_{s\in M(t)} x_k(s)-\max\limits_{s\in M(t)} x(s), & x_k(t)\geqq x(t)\\ \max\limits_{s\in M(t)} x(s)-\max\limits_{s\to M(t)} x_k(s), & x_k(t)\leqq x(t)\end{cases}$$

$$\begin{cases}\max\limits_{s\in M(t)}[x_k(s)-x(s)]<\varepsilon\\ \max\limits_{s\in M(t)}[x(s)-x_k(s)]<\varepsilon\end{cases}$$

If $x\in\mathcal{D}_n$, then $\mathcal{A}x\in C([T_{Ax},\infty);\boldsymbol{R})$ (cf. [1]).

Example 1. Consider the differential equation

$$(t^{-1}x'(t))'+\frac{1}{2}t^{-3}\max_{S\in[t-1,t]} x(s)=-t^{-2},\quad t\geqq 1 \tag{13}$$

and the differential inequality

$$(t^{-1}x'(t))'+\frac{1}{2}t^{-3}\max_{s\in[t-1,t]} x(s)\leqq -t^{-2},\quad t\geqq 1 \tag{14}$$

Here $(\mathcal{A}x)(t)=\max\limits_{s\in[t-1,t]} x(s)$. The functions $c_0(t)=1$, $c_1(t)=t^{-1}$, $F(t,u)=(1/2)ut^{-3}$ satisfy the conditions of Theorem 1 and $y(t)=4t>0$ is a solution of (14). Then there exists a positive solution x of equation (13) with the properties (3).

For instance, $x(t)=2t$ is such a solution.

Example 2. Consider the differential equation

$$(t^{-1}x'(t))'+\frac{1}{2}t^{-3}\max_{s\in[t,t+1]} x(s)=t^{-2},\quad t\geqq 1 \tag{15}$$

and the differential inequality

$$(t^{-1}x'(t))'+\frac{1}{2}t^{-3}\max_{s\in[t,t+1]} x(s)\geqq t^{-2},\quad t\geqq 1 \tag{16}$$

Here $(\mathcal{A}x)(t)=\max\limits_{s\in[t,t+1]} x(s)$. The functions $c_0(t)=1$, $c_1(t)=t^{-1}$, $F(t,u)=(1/2)t^{-3}u$ and $c(t)=t^{-2}$, $w(t)=-t$ for $t\geqq 1$ satisfy the conditions of Theorem 2 and $y(t)=-4t<0$ is a solution of (16). Then there exists a negative solution x of equation (15) with the properties (5). For instance, $x(t)=-2t$ is such a solution.

Example 3. Consider the differential equation

$$[e^{-t}[e^{-2t}[e^{-t}[e^{t}x(t)]']']']'-4e^{-3t}\max_{s\in[t-1,t]} x(s)=0,\quad t\geqq 1 \tag{17}$$

and the differential inequality

$$[e^{-t}[e^{-2t}[e^{-t}[e^{t}x(t)]']']']'-4e^{-3t}\max_{s\in[t-1,t]} x(s)\leqq 0,\quad t\geqq 1 \tag{18}$$

Here $(\mathcal{A}x)(t)=\max\limits_{s\in[t-1,t]} x(s)$. The functions $c_0(t)=e^t$, $c_1(t)=c_3(t)=e^{-t}$, $c_2(t)=e^{-2t}$, $F(t,u)=e^{-3t}u$ and $M(t)=[t-1,t]$ satisfy the conditions of Theorem 8 and $y(t)$

$=e^{4t}$ is a solution of (18) such that $\lim_{t\to\infty} e^{t}e^{4t}=\infty$. Then there exists a positive solution x of equation (17) with the properties (10). For instance, $x(t)=e^{t}$ is such a solution.

Example 4. Consider the differential equation

$$[e^{-t}[e^{-2t}[e^{-t}[e^{t},\ x(t)]']']']'-4e^{-3t}\max_{s\in[t,t+1]} x(s)=0,\quad t\geqq 1 \tag{19}$$

and the differential inequality

$$[e^{-t}[e^{-2t}[e^{-t}[e^{t},\ x(t)]']']']'-4e^{-3t}\max_{s\in[t,t+1]} x(s)\leqq 0,\quad t\geqq 1 \tag{20}$$

Here $(\mathcal{A}x)(t)=\max_{s\in[t,t+1]} x(s)$. The functions $c_0(t)=e^{t}$, $c_1(t)=e^{-t}$, $c_2(t)=e^{-2t}$, $c_3(t)=e^{-t}$, $F(t, u)=4e^{-3}u$ satisfy the conditions of Theorem 9. Moreover, $y(t)=-e^{4t}$ is a solution of inequality (20) such that $\lim_{t\to\infty}(L_0 y)(t)<0$. Then there exists a negative solution x of equation (19) with the properties (11). For instance, $x(t)=-e^{t}$ is a solution of equation (19) for which $\lim_{t\to\infty}(L_0 x)(t)=-\infty$.

For $n=4$ and $\delta=-1$ we obtain that $l=2$. Then it is immediately verified that

$$(L_i y)(t)\leqq(L_i x)(t)<0,\qquad i=0, 1.$$

$$(-1)^{2+i}(L_i y)(t)\leqq(-1)^{2+i}(L_i x)(t)<0,\quad i=2, 3;\ t\geqq 1$$

Examlpe 5. Consider the differential equations

$$[t^{-1}[t^{-1}x'(t)]']'-3t^{-5}\max_{s\in[t-1,t]} x(s)=-3t^{-3},\quad t\geqq 2 \tag{21}$$

$$[t^{-1}[t^{-1}x'(t)]']'-3t^{-5}\max_{s\in[t-1,t]} x(s)=0,\qquad t\geqq 2 \tag{22}$$

Here $(\mathcal{A}x)(t)=\max_{s\in[t-1,t]} x(s)$. The functions $c_0(t)=1$, $c_1(t)=c_2(t)=t^{-1}$, $F(t, u)=3t^{-5}u$, $c(t)=3t^{-3}$, $w(t)=(3/4)[t^{2}\ln t-(t^{2}/2)]$ satisfy the conditions of Theorem 10. Moreoveer, $y(t)=t^{2}>0$ is a solution of equation (21) such that $\inf_{t\geq 2} y(t)=4>0$. Then there exists a solution x of equation (22) with the properties (12). For instance, $x(t)=t$ is a solution of equation (22) for which $\lim_{t\to\infty}(L_0 x)(t)=\infty$ and $x(t)=t\leqq t^{2}=y(t)$ for $t\geqq 2$.

2. Let $(\mathcal{A}x)(t)=x(g(t))$, where $g\in C([t_0, \infty);\ \boldsymbol{R})$, $\lim_{t\to\infty} g(t)=\infty$.

It is immediately verified that for the operator considered conditions H5–H8 are met.

Example 6. Consider the differential equation

$$[e^{-t}[e^{-t}x'(t)]']'+2e^{-4t}x(2t)=2e^{-2t},\quad t\geqq 2 \tag{23}$$

and the differential inequality

$$[e^{-t}[e^{-t}x'(t)]']'+2e^{-4t}x(2t)\leqq 2e^{-2t}, \quad t\geqq 2 \tag{24}$$

Here $(\mathcal{A}x)(t)=x(2t)$. The functions $c_0(t)=1$, $c_1(t)=c_2(t)=e^{-t}$, $w(t)=t$, $F(t, u)=2e^{-4t}u$, $c(t)=2e^{-2t}$ satisfy the conditions of Theorem 1. Moreover, $y(t)=te^t$ is a solution of inequality (24) such that $\liminf_{t\to\infty} y(t)>0$. Then there exists a positive solution x of equation (23) with the properties (3).

For instance, $x(t)=e^t$ is a solution of equation (23), for which

$$\liminf_{t\to\infty}(L_0x)(t)>0 \quad \text{and} \quad x(t)=e^t\leqq te^t=y(t) \qquad \text{for} \quad t\geqq 2$$

Example 7. Consider the differential equation

$$[e^{-t}[e^{-t}x'(t)]']'+2e^{-4t}x(2t)=-2e^{-2t}, \quad t\geqq 2 \tag{25}$$

and the differential inequality

$$[e^{-t}[e^{-t}x'(t)]']'+2e^{-4t}x(2t)\geqq -2e^{-2t}, \quad t\geqq 2 \tag{26}$$

Here $(\mathcal{A}x)(t)=x(2t)$. The functions $c_0(t)=1$, $c_1(t)=c_2(t)=e^{-t}$, $w(t)=-t$, $F(t, u)=2e^{-4t}\cdot u$, $c(t)=-2e^{-2t}$ satisfy the conditions of Theorem 2. Moreover, $y(t)=-te^t$ is a solution of inequality (26). Then there exists a negative solution x of equation (25) with the properties (5).

For instance, $x(t)=-e^t$ is such a solution.

Example 8. Consider the differential equation

$$[t^{-1}[t^{-2}x(t)]']'+t^{-6}x(3t^2)=6t^{-4}, \qquad t\geqq 1 \tag{27}$$

Here $(\mathcal{A}x)(t)=x(3t^2)$. The functions $w(t)=2t$, $c(t)=6t^{-4}$, $F(t, u)=t^{-6}u>0$ for $u\in \boldsymbol{R}_+$, $c_0(t)=t^{-2}$, $c_1(t)=t^{-1}$ satisfy the conditions of Theorem 3. Then each positive solution of equation (27) enjoys the property (6). For instance, $x(t)=t$ is such a solution for which $\lim_{t\to\infty}(x(t)/t^2)=0$.

Example 9. Consider the differential equation

$$[e^{-t}[e^t x(t)]']'-2e^{-t}x(2t)=0, \quad t\geqq 2 \tag{28}$$

and the differential inequality

$$-[e^{-t}[e^t x(t)]^1]^1+2e^{-t}x(2t)\geqq 0, \quad t\geqq 2. \tag{29}$$

Here $(\mathcal{A}x)(t)=x(2t)$. The functions $c_0(t)=e^t$, $c_1(t)=e^{-t}$, $F(t, u)=2e^{-t}u$ satisfy the conditions of Theorem 9. Moreover, $y(t)=-e^{2t}$ is a solution of inequality (29) such that $\lim(L_0y)(t)=-\infty$. Then there exists a negative solution x of equation (28) with the properties (11).

For instance, $x(t)=-e^t$ is such a solution.

Example 10. Consider the differential equations

$$[t^{-2}[t^{-1}x'(t)]']'-4t^{-7}x(t^2)=-4t^{-3}, \quad t\geqq 1, \tag{30}$$

$$[t^{-2}[t^{-1}x'(t)]']'-4t^{-7}x(t^2)=0, \quad t\geqq 1. \tag{31}$$

Here $(\mathcal{A}x)(t)=x(t^2)$. The functions $c_1(t)=t^{-1}$, $c_2(t)=t^{-2}$, $F(t, u)=4t^{-t}u$, $C(t)=-4t^{-3}$, $w(t)=(2/3)t^3$ satisfy the conditions of Theorem 10. Moreover, $y(t)=t^2$ is a solution of equation (30) such that $\liminf_{t\to\infty} y(t)>0$. Then there exists a positive solution x of equation (31) with the properties (12). For instance, $x(t)=t>0$ is a solution of equation (31) for which $\lim_{t\to\infty}(L_0x)(t)=\infty$, $x(t)=t\leqq t^2=y(t)$, $t\geqq 1$.

3. Let $(\mathcal{A}x)(t)=\int_{t-a}^{t} k(t, s), x(s)ds$ where a is a positive constant, $k\in C([t_0+a, \infty)^2; (0, \infty))$ and there exists a constant $c>0$ such that $k(t, s)\leqq c$ eventually.

We shall prove that for the operator considered conditions H5-H8 are met. If $0<x(t)\leqq y(t)$, then

$$(\mathcal{A}y)(t)-(\mathcal{A}x)(t)=\int_{t-a}^{t} k(t, s)[y(s)-x(s)]ds\geqq 0$$

It is immediately verified that conditions H5 and H9 hold. $x_k, x\in\mathcal{D}_n$, $k=0, 1, \cdots$, $\lim_{k\to\infty} x_k(t)=x(t)$, i.e., for any $\varepsilon>0$ and each fixed number $t\geqq t_0$ there exists $k_0>0$ such that for $k\geqq k_0$ we have $|x_k(t)-x(t)|<\varepsilon/(ca)$.

Then

$$\lim_{k\to\infty}(\mathcal{A}x_k)(t)=(\mathcal{A}x)(t).$$

Example 11. Consider the differential equation

$$[e^{-t}[e^{-t}x'(t)]']'+\int_{t-1}^{t} e^{s-t}x(s)ds=\frac{e^2-1}{2e^2}e^t, \; t\geqq 1. \tag{32}$$

and the differential inequality

$$[e^{-t}[e^{-t}x'(t)]']'+\int_{t-1}^{t} e^{s-t}x(s)ds\leqq\frac{e^2-1}{2e^2}e^t, \quad t\geqq 1. \tag{33}$$

Here $(\mathcal{A}x)(t)=\int_{t-1}^{t} e^{s-t}x(s)ds$. The functions $c_0(t)=1$, $c_1(t)=c_2(t)=e^{-t}$, $F(t, u)=u$ $k(t, s)=e^{s-t}$, $w(t)=(e^2-1)/(12e^2)e^{3t}$ satisfy the conditions of Theorem 1. Moreover, $y(t)=t$, $e^t>0$ is a solution of inequality (33) such that $\liminf_{t\to\infty}(L_0y)(t)>0$. Then there exists a positive solution x of equation (32) with the properties (3).

For instance, $x(t)=e^t$ is such a solution.

Example 12. Consider the differential equation

$$[e^{-t}[e^{-t}x'(t)]']'+\int_{t-1}^{t} e^{s-t}x(s)ds=\frac{1-e^2}{2e^2}e^t, \quad t\geqq 1. \tag{34}$$

and the differential inequality

$$[e^{-t}[e^{-t}x'(t)]']'+\int_{t-1}^{t} e^{s-t}x(s)ds\geqq\frac{1-e^2}{2e^2}e^t, \quad t\geqq 1 \tag{35}$$

Here $(\mathcal{A}x)(t)=\int_{t-1}^{t} e^{s-t}x(s)ds$. The functions $c_0(t)=1$, $c_1(t)=c_2(t)=e^{-t}$, $F(t, u)=u$

$$k(t, s)=e^{s-t}\leqq 1, \quad c(t)=\frac{1-e^2}{2e^2}e^t, \quad w(t)=\frac{1-e^2}{12e^2}e^{3t},$$

satisfy the conditions of Theorem 2. Moreover, $y(t)=-te^t$ is a solution of inequality (35).

Then there exists a negative solution x of equation (34) with the properties (5).

For instance, $x(t)=-e^t$ is such a solution.

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