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THE STRONG EMBEDDING THEOREM FOR 3-REPRESENTATIVE CHAIN GRAPHS

By

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Summary. Jaeger proposed the strong embedding conjecture: every 2-conncted simple graph has a strong embedding into some closed surface. In this paper, we show that the strong embedding conjecture for chain graphs is equivalent to the strong embedding conjecture. We solve the strong embedding conjecture for 3-representative chain graphs.

1. Introduction

By a *graph*, we mean a finite, undirected graph. A graph G is said to be *simple* if G has no loops and no multiple edges. A *cycle* is a regular connected graph of degree two. We denote the complete graph with n vertices by K_n.

A family $\mathcal{C}$ of cycles of a graph G is called a *cycle double cover* of G if each edge of G is contained in exactly two cycles in $\mathcal{C}$. In [5], Seymour proposed the cycle double cover conjecture: every 2-edge-connected simple graph has the cycle double cover conjecture: every 2-edge-connected simple graph has a cycle double cover. Many mathematicians have been studying the cycle double cover conjecture, but it is still open.

By a *surface*, we mean a compact connected 2-dimensional manifold (possibly with boundary). A surface S is said to be *closed* if the boundary of S is empty. Let $f: G \to S$ be a 2-cell embedding of a graph G into a closed surface S. For any face r of f, the boundary circuit of r is denoted by $W(r)$. Then $W(r)$ is not always a cycle of G. A 2-cell embedding f is said to be *strong* if $W(r)$ is a cycle of G for any face r of f. In [3], Jaeger presented the strong embedding conjecture as a strengthening of the cycle double cover conjecture (it is unsolved whether the strong embedding conjecture and the cycle double cover conjecture are equivalent or not).

Conjecture 1.1. (The Strong Embedding Conjecture) *Every 2-connected simple graph has a strong embedding into some closed surface.*

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For any planar 2-connected simple graph G, we can easily prove that every planar embedding of G is strong. Some results are known concerning the strong embedding conjecture (see [4, Lemma 2.1] and [8]). In this paper, we present the definition of chain graphs. We consider the strong embedding conjecture for chain graphs.

Let G be a 2-connected simple graph and v a vertex of G. A connected simple graph H is called a *vertex-splitting* of G at v if there exist two non-adjacent vertices v' and v'' of H such that $H-\{v', v''\}=G-v$ and G is obtained from H by identifying v' with v''. Two vertices v' and v'' of H are called the *splittings* of v. We denote a vertex-splitting of G at v by $\mathrm{Sp}(G; v)$.

Let G be a connected graph. A *block* of G is a maximal subgraph B of G such that there exists no cut-vetex of B. Therefore every block of G is either a 2-connected graph or the complete graph K_2. Let $\mathcal{B}$ be a sequence of some blocks $B_1, \cdots, B_m$ of G. For any two distinct vertices v' and v'' of G, $\mathcal{B}$ is called a *block-path* of G from v' to v'' if $\mathcal{B}$ satisfies the following conditions:

(B1) $B_i \cap B_{i'} \neq \emptyset$ for $|i-i'| \leqq 1$,

(B2) $B_i \cap B_{i'} = \emptyset$ for $|i-i'| > 1$ and

(B3) $v' \in V(B_1) - V(B_2)$ and $v'' \in V(B_m) - V(B_{m-1})$.

The block-path of G may have just a single block. We denote $\bigcup_{i=1}^{m} B_i$ by $|\mathcal{B}|$. We note that there exists the unique block-path of a connected graph G from v' to v'' for any two distinct vertices v' and v'' of G. Thus, for any 2-connected simple graph G and any vertex v of G, every vertex-splitting $\mathrm{Sp}(G; v)$ has the unique block-path $\mathcal{B}$ from v' to v'' such that $\mathrm{Sp}(G; v) = |\mathcal{B}|$, where v' and v'' are the splittings of v.

A vertex-splitting $\mathrm{Sp}(G; v)$ is said to be *strong* if, for any block B of $\mathrm{Sp}(G; v)$, either $B=K_2$ or B has a strong embedding into some closed surface. A 2-connected simple graph G is called a *chain graph* if there exists a strong vertex-splitting of G at some vertex of G. We propose the strong embedding conjecture for chain graphs.

Conjecture 1.2. *Every chain graph has a strong embedding into some closed surface.*

In Section 3, we shall show that Conjecture 1.1 and Conjecture 1.2 are equivalent. In [8], the author proved the following proposition.

Proposition 1.3. *Every chain graph of genus one has either a troidal strong embedding or a strong embedding into some closed non-orientable surface.*

Let f be a 2-cell embedding of a graph G into a closed surace S. In [7], Vitray defined the representativity of f. A 2-cell embedding f is said to be *k-representative* if every essential closed curve in S which does not intersect edges of $f(G)$ must contain at least k vertices of $f(G)$. A simple graph G is said to be *k-representable* into a closed surface S if G has a k-representative embedding into S.

Let G be a chain graph and v a vertex of G. A strong vertex-splitting $\mathrm{Sp}(G;v)$ of G at v is said to be *k-representative* if each block of $\mathrm{Sp}(G;v)$ is either planar or k-representable into some closed surface. A chain graph G is said to be *k-representative* if there exists a k-representative strong vertex-splitting of G at some vertex of G. By the definition, every chain graph is 2-representative. In Section 5, we shall show the strong embedding theorem for 3-representative chain graphs, as follows.

Theorem 1.4. *Every 3-representative chain graph has a strong embedding into some closed surface.*

2. Definitions and notation

In this section, we define the terminology used in this paper. We refer to [1] and [2] for the basic terminology and notation in graph theory and topological graph theory respectively. The vertex set and the edge set of a graph G are denoted by $V(G)$ and $E(G)$ respectively. We shall define the terminology for 2-cell embeddings. Let f be a 2-cell embedding of a graph G into a closed surface S. We denote the set of all faces of f by $R(f)$. For any $r\in R(f)$, the closure of r in S is denoted by $\bar{r}$. A closed walk W of G is called the *boundary circuit* of a face r of f if a closed walk $f(W)$ is obtained from $\bar{r}\cap f(G)$ by traversing a simple closed curve just inside r (see [2], p. 101). We denote the boundary circuit of a face r of f by $W(r)$.

We shall define the basic terminology in PL-topology. Let K be a simplicial n-complex for $n=1, 2$. The polyhedron of K is denoted by $|K|$. The *boundary* of K is the simplicial $(n-1)$-subcomplex of K consisting of all $(n-1)$-simplexes each of which is contained in exactly one n-simplex of K, and it is denoted by ∂K. The polyhedron $|\partial K|$ is called the *boundary* of $|K|$. The boundary of $|K|$ is denoted by $\partial|K|$. We suppose that $n=2$. Then $|K|$ is called a *pseudo-surface* if each 1-simplex of K is contained in one or two 2-simplexes of K. By the definition, a surface is a pseudo-surface.

We suppose that $\partial K\neq\emptyset$. Let σ^2 be a 2-simplex of K and σ^1 a 1-simplex of σ^2 which is contained in ∂K. The simplicial complex $K-\{\sigma^1, \sigma^2\}$ is said to be obtained from K by an *elementary collapsing*. A simplicial 2-complex K is said to *collapse* to a subcomplex K' of K if K' is obtained from K by a finite sequence of elementary collapsings.

Let S be a surface and X a topological subspace of S. The *interior* of X is denoted by $\mathrm{int}\,X$. A compact 2-manifold N with $\partial N \neq \emptyset$ is called a *regular neighborhood* of X in S if S has a triangulation which has subcomplexes L and L' satisfying that $N = |L|$, $X = |L'|$ and L collapses to L'. A regular neighborhood of X in S is denoted by $\mathrm{N}(X; S)$.

We shall define curves in a pseudo-surface. Let Σ be a pseudo-surface. A continuous map ω from the unit interval $[0, 1]$ to Σ is called a *path*. The image $\omega([0, 1])$ is called a *curve* in Σ. For any curve γ in Σ, we denote the path whose image is γ by ω_γ. A path ω is said to be *closed* if $\omega(0)=\omega(1)$, and a curve γ in Σ is said to be *closed* if ω_γ is closed. We denote the set of all closed curves in Σ by $\Gamma(\Sigma)$. A path ω is said to be *simple* if ω is one-to-one, and a curve γ in Σ is said to be *simple* if ω_γ is simple.

We shall define the homotopy in a pseudo-surface (see [6]). Let γ_1 and γ_2 be two curves in Σ. A curve γ_1 is said to be *homotopic* to γ_2 in Σ, denoted by '$\gamma_1 \simeq \gamma_2$ in Σ', if ω_{γ_1} is freely homotopic to ω_{γ_2} in Σ. For any $\gamma \in \Gamma(\Sigma)$, the homotopy class of γ in Σ is denoted by $[\gamma]$.

A point x in Σ is a closed curve in Σ such that ω_x is a constant map. A closed curve γ in Σ is said to be *inessntial* in Σ if there exists a point x in Σ such that $\gamma \simeq x$ in Σ. Otherwise, a curve γ is said to be *essential* in Σ. It is well-known that a simple closed curve γ is inessential in Σ if and only if γ bounds a disk in Σ. If γ is inessential in Σ, then the homotopy class of γ in Σ is denoted by $[\gamma]=1$. Let S be a closed surface and γ a simple closed curve in S. If $S-\gamma$ is not connected, then γ is said to be *separating* in S.

We suppose that Σ is a pseudo-surface in a closed surface S with $\partial\Sigma \neq \emptyset$. A closed curve C in $\partial\Sigma$ is called a *boundary component* of Σ in S if there exists a 1-sphere component C' of $\partial\mathrm{N}(\Sigma; S)$ such that $C \simeq C'$ in $\mathrm{N}(\Sigma; S) - \mathrm{int}\,\Sigma$.

3. Conjectures

We have the strong embedding theorem for planar graphs: every planar 2-connected simple graph has a planar strong embedding. In this secion, our goal is to prove the following proposition.

Proposition 3.1. *Conjecture* 1.1 *and Conjectrue* 1.2 *are equivalent.*

Proof. It is obvious that if Conjecture 1.1 holds, then Conjecture 1.2 holds. We assume that Conjecture 1.2 holds. By induction on the genus of a graph, we shall show that Conjecture 1.1 holds. Let G be a 2-connected simple graph and $g(G)$ the genus of G. If $g(G)=0$, then every planar embedding of G is strong. Thus Conjecture 1.1 holds. We suppose that if $g(G)<g$ $(g \geqq 1)$, then Conjecture 1.1 holds. If $g(G)=g$, then there exists a 2-cell embedding f of G into a closed orientable surface S with genus g. We may assume that f is not strong.

Then there exists a face r of f such that $W(r)$ is not a cycle of G. There exists a vertex v of $W(r)$ such that v occurs twice in $W(r)$. We can choose a simple closed curve γ in $\bar{r}$ such that $\gamma \cap f(G)=f(v)$ and $[\gamma]\neq 1$ in $\bar{r}$. Since G is 2-connected, $[\gamma]\neq 1$ in S and γ is not separating in S. We obtain the surface $\bar{S}$ by cutting S along γ. Let $\hat{S}$ be the closed surface obtained from $\bar{S}$ by capping off each 1-sphere component of $\partial\bar{S}$ with a 2-cell and let $\hat{f}$ be a 2-cell embedding of a connected graph $\hat{G}$ into $\hat{S}$ naturally induced by this construction of $\hat{S}$. Then $\hat{G}$ is a vertex-splitting of G at v. Since $\hat{S}$ has genus $g-1$, $g(B)\leqq g-1$ for any block B of $\hat{G}$. If a block B is 2-conneted, then B has a strong embedding into a closed surface by induction hypothesis. Therefore $\hat{G}$ is a strong vertex-splitting. Thus G is a chain graph. By the assumption that Conjecture 1.2 holds, G has a strong embedding into some closed surface. □

By Proposition 1.3 and Proposition 3.1, we have the following corollary.

Corollary 3.2. *Every 2-connected simple graph of genus one has a strong embedding into some closed surface.*

By Proposition 3.1, it is sufficient to consider only chain graphs for the strong embedding conjecture.

4. Ladder face-paths

Let f be a strong embedding of a graph G into a surface S. Let P be a sequence of faces $r_1, \cdots, r_n$ of f. For any two distinct vertices v' and v'' of G, P is called a *face-path* of f from v' to v'' if P satisfies the following conditions:

(F1) $\partial\bar{r}_i \cap \partial\bar{r}_{i'} \neq \emptyset$ for $|i-i'|\leqq 1$,

(F2) $\partial\bar{r}_i \cap \partial\bar{r}_{i'} = \emptyset$ for $|i-i'|>1$ and

(F3) $v'\in V(W(r_1))-V(W(r_2))$ and $v''\in V(W(r_n))-V(W(r_{n-1}))$.

We denote $\bigcup_{i=1}^{n}\bar{r}_i$ by $|P|$. We note that $|P|$ is a connected pseudo-surface in S. We can easily prove the following proposition. We omit the proof (see [8]).

Proposition 4.1. *Let f be a strong embedding of a 2-connected graph G into a surface S. For any two distinct vertices v' and v'' of G, there exists a face-path of f from v' to v''.*

A face-path P of f from v' to v'' is said to be *ladder* if P satisfies the following conditions:

(F4) a regular neighborhood $\mathrm{N}(|P|; S)$ is planar and

(F5) $f(v')$ and $f(v'')$ are contained in the same boundary component of $|P|$ in S

In this section, our goal is to prove Theorem 4.2. Let G be a chain graph and $\mathrm{Sp}(G; v)$ a strong vertex-splitting of G at a vertex v of G, and let v' and v'' be the splittings of v. Then $\mathrm{Sp}(G; v)$ has the unique block-path $B_1 \cdots B_m$ from v' to v''. Let v_i be the cut-vertex of $\mathrm{Sp}(G: v)$ in $V(B_i) \cap V(B_{i+1})$ for $1 \leqq i < m$. We set $v_0 = v'$ and $v_m = v''$. Let f_i be a 2-cell embedding of B_i into a closed surface S_i such that f_i is strong if $B_i \neq K_2$. Then the following theorem holds.

Theorem 4.2. *Let G be a chain graph as above. We suppose that if $B_i \neq K_2$, then there exists a ladder face-path of f_i from v_{i-1} to v_i for $1 \leqq i \leqq m$.*

(1) *If $m=1$ and if there exists at most one face r_1 of f_1 such that $v', v'' \in V(W(r_1))$, then G has a strong embedding into some closed surface.*

(2) *If $m>1$, then G has a strong embedding into some closed surface.*

The following proposition holds for planar embeddings.

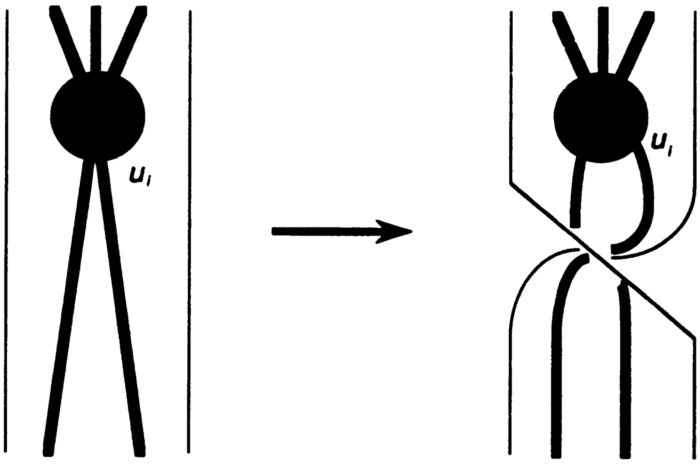
Proposition 4.3. *Let G be a planar 2-connected simple graph and f a strong embedding of G into either a disk or a sphere S. Then there exists a ladder face-path of f from v' to v'' for any two distinct vertices v' and v'' of G.*

Proof. It is sufficient to show the proposition when S is a disk. Since G is 2-connected, there exists a cycle C of G containing v' and v'' and bounding a disk D in S. By Proposition 4.1, there exists a face-path P of f from v' to v'' such that $|P| \subset D$. Then P is a face-path of f from v' to v'' which satisfies Condition (F4). Because D is a disk, there exists a boundary component C of $|P|$ in S such that $C \simeq \partial D$ in $\mathrm{N}(D; S) - \mathrm{int}\,|P|$. Since $f(v'), f(v'') \in \partial D$, P satisfies Condition (F5). Therefore P is ladder. □

The following lemma is necessary to prove Theorem 4.2.

Lemma 4.4. *Let B be a 2-connected simple graph and let v' and v'' be two non-adjacent vertices of B. We assume that there exists a strong embedding f_0 of B into some closed surface S_0 such that there exists a ladder face-path of f_0 from v' to v''. We set the edge $e = v'v''$ and the graph $G_0 = B + e$. Then G_0 has a strong embedding into some closed surface.*

Proof. Let $P = r_1 \cdots r_n$ be a ladder face-path of f_0 from v' to v'' with $n \geqq 2$. Let $\bar{S}_0 = (S_0 - \bigcup_{i=1}^{n} r_i) \cup \mathrm{N}(f_0(B); S_0)$. We choose a vertex $u_i \in V(W(r_i)) \cap V(W(r_{i+1}))$ for $1 \leqq i < n$. We set $u_0 = v'$ and $u_n = v''$. We deform $\mathrm{N}(f_0(u_i); \bar{S}_0)$ as shown in Figure 1 for $1 \leqq i < n$.



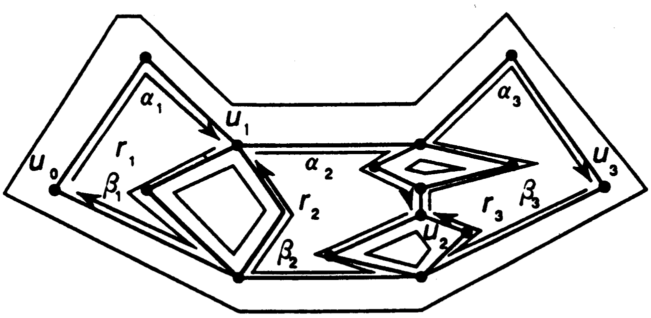
$N(f_0(u_l);\bar{S}_0)$

Figure 1

By the above deformation, we obtain the surface $\bar{S}$ from $\bar{S}_0$. Then $\partial\bar{S}$ is a 1-sphere. We obtain the closed surface S from $\bar{S}$ by capping off $\partial\bar{S}$ with a 2-cell D and a 2-cell embedding $\bar{f}$ of B into S. By embedding e into $\mathrm{N}(D\,;S)$, we get a 2-cell embedding f of G_0 into S such that $f|_B=\bar{f}$.

We shall show that f is strong. Let r' and r'' be two faces of f satisfying that $e\in E(W(r'))\cap E(W(r''))$. Then $D\subset\bar{r}'\cup\bar{r}''$. By the construction of f, we note that $W(r)$ is a cycle of G_0 for any $r\in R(f)-\{r', r''\}$ because $r\in R(f_0)$. We shall check that $W(r')$ and $W(r'')$ are cycles of G_0.

Since P is a ladder face-path of f_0 from v' to v'', $\mathrm{N}(|P|\,;S_0)$ is planar. We regard $\mathrm{N}(|P|\,;S_0)$ as an oriented planar surface. Giving the orientation over $\mathrm{N}(|P|\,;S_0)$, we regard $W(r_i)$ as a directed cycle for $1\leqq i\leqq n$. We can take the directed path α_i in $W(r_i)$ from u_{i-1} to u_i and the directed path β_i in $W(r_i)$ from u_i to u_{i-1} for $1\leqq i\leqq n$ (see Figure 2).



$N(|P|;S_0)$

Figure 2

Since $E(W(r'))\cup E(W(r''))=(\bigcup_{i=1}^{n} E(W(r_i)))\cup e$, we may assume that $\alpha_1\subset W(r')$ and $\beta_1\subset W(r'')$. By the construction of S,

$$W(r')=\left(\bigcup_{i=1}^{\lceil n/2\rceil}\alpha_{2i-1}\right)\cup\left(\bigcup_{i=1}^{\lfloor n/2\rfloor}\beta_{2i}\right)\cup e$$

and

$$W(r'')=\left(\bigcup_{i=1}^{\lfloor n/2\rfloor}\alpha_{2j}\right)\cup\left(\bigcup_{i=1}^{\lceil n/2\rceil}\beta_{2i-1}\right)\cup e.$$

Since P is a face-path of f_0 from v' to $v_{\prime\prime}$, $\alpha_i\cap\beta_{i'}=\emptyset$ for $|i-i'|>1$. It suffices to show that $\alpha_i\cap\beta_{i+1}=\alpha_{i+1}\cap\beta_i=u_i$ for $1\leqq i\leqq n$.

We can choose a simple curve γ_i in $\bar{r}_i$ satisfying that $\gamma_i\cap\partial\bar{r}_i=\partial\gamma_i=\{f_0(u_{i-1}), f_0(u_i)\}$ for $1\leqq i\leqq n$. We extend $\bigcup_{i=1}^{n}\gamma_i$ to a simple curve γ in $\mathrm{N}(|P|\,;S_0)$ satisfying that $\gamma\cap\partial\mathrm{N}(|P|\,;S_0)=\partial\gamma\subset\mathrm{N}(f_0(u_0)\,;S_0)\cup\mathrm{N}(f_0(u_n)\,;S_0)$. Since P is a ladder face-path of f_0 from v' to v'', two points of $\partial\gamma$ are contained in the same connected component of $\partial\mathrm{N}(|P|\,;S_0)$ by Condition (F5). There exist two planar surfaces N_α and N_β in $\mathrm{N}(|P|\,;S_0)$ such that $N_\alpha\cup N_\beta=\mathrm{N}(|P|\,;S_0)$ and $N_\alpha\cap N_\beta=\gamma$. We assume that $f_0(\alpha_1)\subset N_\alpha$ and $f_0(\beta_1)\subset N_\beta$. By the consideration of the orientation of $\mathrm{N}(|P|\,;S_0)$, $f_0(\alpha_i)\subset N_\alpha$ and $f_0(\beta_i)\subset N_\beta$ for $1\leqq i\leqq n$. Therefore $f_0(\alpha_i)\cap f_0(\beta_{i+1})\subset\gamma$ and $f_0(\alpha_{i+1})\cap f_0(\beta_i)\subset\gamma$ for $1\leqq i<n$. It follows that $\alpha_i\cap\beta_{i+1}=\alpha_{i+1}\cap\beta_i=u_i$ for $1\leqq i<n$. Thus $W(r')$ and $W(r'')$ are cycles of G_0. □

To prove Theorem 4.2, we make use of an edge-contraction of a graph. Let G be a graph and e an edge of G which is not a loop. We denote the graph obtained from G by the edge-contraction of e by G/e. Let f be a 2-cell embedding of G into a closed surface S. We can obtain a 2-cell embedding of G/e into S by contracting $f(e)$ to a vertex in S. Such a 2-cell embedding is denoted by f_e. We can easily prove the following proposition. We omit the proof (see [8]).

Proposition 4.5. *Let G be a 2-connected graph and e an edge of G with two distinct endvertices v' and v'', and let f be a strong embedding of G into a closed surface S. We suppose that there exists no face r of f such that $e\notin E(W(r))$ and $v', v''\in V(W(r))$. Then f_e is strong.*

By Lemma 4.4 and Proposition 4.5, we can show Theorem 4.2 (1).

Proof of Theorem 4.2 (1). Since $m=1$, $\mathrm{Sp}(G\,;v)$ is 2-connected and f_1 is strong. We set the edge $e=v'v''$ and the graph $G_0=\mathrm{Sp}(G\,;v)+e$. We first suppose that there exists a face r_1 of f_1 such that $v', v''\in V(W(r_1))$. Then there exists a strong embedding f of G_0 into S_1 such that $f|_{\mathrm{Sp}(G;v)}=f_1$ and $f(e)\subset\bar{r}_1$. By the assumption of Theorem 4.2(1), there exists no face r in $R(f_1)-\{r_1\}$ such that $v', v''\in V(W(r))$. Thus there exists no face r of f such that $e\notin E(W(r))$ and $v', v''\in V(W(r))$. By Proposition 4.5, f_e is a strong embedding of G.

Now we may assume that there exists no face r_1 of f_1 such that $v', v''\in V(W(r_1))$. By Lemma 4.4, there exists a strong embedding f of G_0 into some closed surface. By the construction of f in Lemma 4.4, if a face r of f satisfies that $e\notin E(W(r))$ and $v', v''\in V(W(r))$, then $r\in R(f_1)$, contrary to our assumption. Therefore there exists no face r of f such that $e\notin E(W(r))$ and $v', v''\in V(W(r))$. By Proposition 4.5, f_e is a strong embedding of G. □

To prove Theorem 4.2 (2), we shall define a connected sum of surfaces and a connected sum of graphs. Let S_1 and S_2 be two closed surfaces. A surface S is called a *connected sum* of S_1 and S_2 if there exists a disk D_i in $\operatorname{int} S_i$ and a homeomorphism $h_i: S_i - \operatorname{int} D_i \to S$, for $i=1, 2$, such that $S=h_1(S_1-\operatorname{int} D_1)\cup h_2(S_2-\operatorname{int} D_2)$ and $h_1(S_1-\operatorname{int} D_1)\cap h_2(S_2-\operatorname{int} D_2)=h_1(\partial D_1)=h_2(\partial D_2)$. A connected sum of S_1 and S_2 is unique up to homeomorphisms, and it is denoted by $S_1\# S_2$.

Let G_1 and G_2 be two graphs, e_i an edge of G_i with endvertices v_i^1 and v_i^2 for $i=1, 2$. A graph G is called a *connected sum* of G_1 and G_2 associated with e_1 and e_2 if G is obtained from $(G_1-e_1)\cup(G_2-e_2)$ by adding two edges $v_1^1v_2^1$ and $v_1^2v_2^2$. We can easily prove the following proposition. We omit the proof (see [8]).

Proposition 4.6. *Let G_1 and G_2 be two 2-connected graphs as above. If G_i has a strong embedding into a closed surface S_i for $i=1, 2$, then G has a strong embedding f into $S_1\# S_2$ satisfying that there exists no face r of f such that v_1^j, $v_2^j\in V(W(r))$ and $v_1^jv_2^j\notin E(W(r))$ for $j=1, 2$.*

By Proposition 4.6, we shall complete the proof of Theorem 4.2.

Proof of Theorem 4.2 (2). Let $B_1^*, \cdots, B_m^*$ be disjoint copies of $B_1, \cdots, B_m$, and let v_i^1 and v_i^2 be the vertices of B_{i+1}^* and B_i^* corresponding to v_i for $0\leqq i\leqq m$. Since B_i^* is isomorphic to B_i, f_i is regarded as a 2-cell embedding of B_i^* into S_i. We set $\bar{B}_i^*=B_i^*+v_{i-1}^1v_i^2$. If $B_i=K_2$, then there exists a planar strong embedding f_j^* of $\bar{B}_i^*$ into S_i. We set $S_i^*=S_i$ in this case.

If $B_i\neq K_2$, then there exists a strong embedding f_i^* of $\bar{B}_i^*$ into some closed surface S_i^* by Lemma 4.4. Let G_1 be the connected sum of $\bar{B}_1^*$ and $\bar{B}_2^*$ associated with $v_0^1v_1^2$ and $v_1^1v_2^2$ such that $v_0^1v_2^2$, $v_1^1v_1^2\in E(G_1)$. For $1<i<m$, a 2-connected simple graph G_i is the connected sum of G_{i-1} and $\bar{B}_{i+1}^*$ associated with $v_0^1v_i^2$ and $v_i^1v_{i+1}^2$ such that $v_0^1v_{i+1}^2$, $v_i^1v_i^2\in E(G_i)$. We set $e_i=v_i^1v_i^2$, for $1\leqq i<m$, and $e_m=v_0^1v_m^2$. By Proposition 4.6, G_{m-1} has a strong embedding f into $S_1^*\#\cdots\# S_m^*$ satisfying that there exists no face r of f such that v_i^1, $v_i^2\in V(W(r))$ and $e_i\notin E(W(r))$ for $1\leqq i\leqq m$, where $v_m^1=v_0^1$. Since $(\cdots((G_{m-1}/e_1)/e_2)/e_m=G$, $(\cdots((f_{e_1})_{e_2})\cdots)_{e_m}$ is a strong embedding of G by Proposition 4.5. □

5. 3-Representative chain graphs

In this section, our goal is to prove Theorem 1.4 by applying Theorem 4.2 for 3-representative chain graphs.

We shall show the existence of a ladder face-path of a 3-representative embedding of a 2-connected simple graph. The following lemma is necessary to prove Theorem 5.2.

Lemma 5.1. *Let S be a closed surface and D_i a disk in S for $i=1, 2$. We suppose that D_1 and D_2 satisfy the following conditions:*

(**D1**) $D_1 \cap D_2 = \partial D_1 \cap \partial D_2 \neq \emptyset$ *and*

(**D2**) *for any $\gamma \in \Gamma(D_1 \cup D_2)$, if $\gamma \cap (D_1 \cap D_2)$ consists of two points, then $[\gamma]=1$ in S.*

Then there exists a disk D in S such that $\mathrm{N}(D_1 \cup D_2; S) \subset D$.

Proof. We choose closed curves $\gamma_1, \cdots, \gamma_m$ in $D_1 \cup D_2$ such that $\gamma_i \cap (D_1 \cap D_2)$ consists of two points for $1 \leqq i \leqq m$. Then the fundamental group $\pi_1(D_1 \cup D_2)$ is generated by $[\gamma_1], \cdots, [\gamma_m]$. By Condition (D2), there exists a disk D_i' in S bounding γ_i for $1 \leqq i \leqq m$. Then $D_0 = D_1 \cup D_2 \cup \bigcup_{i=1}^{m} D_i'$ is compact and $\pi_1(D_0)=1$. By the classification theorem of compact surfaces, D_0 is either a disk or a sphere. Therefore we can choose a disk D in D_0 such that $\mathrm{N}(D_1 \cup D_2; S) \subset D$. □

By Lemma 5.1, we can obtain Lemma 5.2.

Lemma 5.2. *Let f be a strong embedding of a 2-connected simple graph G into a closed surface S, and let $P = r_1 \cdots r_n$ be a face-path of f from v' to v'' for any two distinct vertices v' and v'' of G. If $\bar{r}_i$ and $\bar{r}_{i+1}$ satisfy Conditions* (D1) *and* (D2) *of Lemma* 5.1 *for $1 \leqq i < n$, then there exists a ladder face-path of f from v' to v''.*

Proof of Lemma 5.2. First, we show the following proposition by induction on n:

(**P**) there exists a disk D in S such that $\mathrm{N}(|P|; S) \subset D$.

If $n=1$, then $\mathrm{N}(|P|; S)$ is a disk. Let $n>1$. We choose a vertex $u \in V(W(r_{n-1})) \cap V(W(r_n))$ and we set $P(n-1) = r_1 \cdots r_{n-1}$. Then $P(n-1)$ is a face-path of f from v' to u. By induction hypothesis, there exists a disk $D(n-1)$ in S such that $\mathrm{N}(|P(n-1)|; S) \subset D(n-1)$. If $r_n \subset D(n-1)$, then $\mathrm{N}(|P|; S) \subset D(n-1)$. Proposition (P) holds in this case.

If $r_n \not\subset D(n-1)$, then let $\Sigma = \bar{r}_{n-1} \cup \bar{r}_n$. By Lemma 5.1, there exists a disk $D(\Sigma)$ in S such that $\mathrm{N}(\Sigma; S) \cup D(\Sigma)$. We may assume that $\partial D(n-1) \subset$

$\partial \mathrm{N}(|P(n-1)|\,;S)$ and $\partial D(\Sigma) \subset \partial \mathrm{N}(\Sigma\,;S)$ since Conditions (D1) and (D2) hold for γ_{n-1} and γ_n. Let $D=D(n-1)\cup D(\Sigma)$. Since $D(n-1)\cap D(\Sigma)$ coincides with the disk $\mathrm{N}(\bar{r}_{n-1}\,;S)$, D is a disk. Proposition (P) holds.

Let G_D be the subgraph of G induced by $f^{-1}(f(V(G))\cap D)$. Then G_D has the unique block-path $\mathcal{B}=B_1 \cdots B_m$ from v' to v''. Since f is strong, G_D has no cut edge and hence any B_j is not K_2. Let v_j be the cut-vertex of $V(B_j)\cap V(B_{j+1})$ for $1 \leqq j < m$. We set $v_0=v'$ and $v_m=v''$. For $1 \leqq j \leqq m$, there exists a disk D'_j in D such that $f(B_j) \subset D'_j$ and $\partial D'_j \subset f(B_j)$. Then $f|_{B_j}$ is a strong embedding of B_j into D'_j. By Proposition 4.3, there exists a ladder face-path P_j of f from v_{j-1} to v_j such that $|P_j| \subset D'_j$ for $1 \leqq j \leqq m$. Let $P_0=P_1 \cdots P_m$. Since f is strong, $D'_j \cap D'_{j+1}=f(v_j)$ for $1 \leqq j < m$. Thus P_0 is a ladder face-path of f from v' to v'' such that $|P_0| \subset D$. $\square$

Theorem 5.3. *Let f be a 3-representative embedding of a 2-connected simple graph into a closed surface. For any two distinct vertices v' and v'' of G, there exists a ladder face-path of f from v' to v''.*

Proof. By Proposition 4.1, there exists a face-path $P=r_1 \cdots r_n$ of f from v' to v''. Let r' and r'' be two distinct faces of f such that $\bar{r}' \cap \bar{r}''=\partial \bar{r}' \cap \partial \bar{r}'' \neq \emptyset$. For any $\gamma \in \Gamma(\bar{r}' \cup \bar{r}'')$, if $\gamma \cap (\bar{r}' \cap \bar{r}'')$ consists of two vertices of $f(G)$, then $[\gamma]=1$ in S because f is 3-representative. Therefore $\bar{r}'$ and $\bar{r}''$ satisfy Conditions (D1) and (D2) of Lemma 5.1. It follows that $\bar{r}_i$ and $\bar{r}_{i+1}$ satisfy Conditions (D1) and (D2) of Lemma 5.1 for $1 \leqq i < n$. By Lemma 5.2, there exists a ladder face-path of f from v' to v''. $\square$

We shall show Theorem 1.4 by Theorem 5.3.

Proof of Theorem 1.4. Let G be a 3-representative chain graph and $\mathrm{Sp}(G\,;v)$ a 3-representative strong vertex-splitting of G at a vertex v of G, and let v' and v'' be the splittings of v. Then $\mathrm{Sp}(G\,;v)$ has the unique block-path $B_1 \cdots B_m$ from v' to v''. Let v_i be the cut-vertex of $\mathrm{Sp}(G\,;v)$ in $V(B_i)\cap V(B_{i+1})$ for $1 \leqq i < m$. We set $v_0=v'$ and $v_m=v''$. Let f_i be a 2-cell embedding of B_i into a closed surface S_i such that f_i is either planar or 3-representative. By Proposition 4.3, if $B_i \neq K_2$ and f_i is planar, then there exists a ladder face-path of f_i from v_{i-1} to v_i. By Theorem 5.3, if f_i is non-planar, then there exists a ladder face-path of f_i from v_{i-1} to v_i. We may assume that $\mathrm{Sp}(G\,;v)$ is non-planar.

Assume that $m=1$, that is, $B_1=\mathrm{Sp}(G\,;v)$. We set the edge $e=v'v''$ and the 2-connected simple graph $\bar{B}_1=B_1+e$. We shall show that there exists at most one face r_1 of f_1 such that $v', v'' \in V(W(r_1))$. We suppose that there exist two faces r_1 and r_2 of f_1 such that $v', v'' \in V(W(r_j))$ for $j=1, 2$. Then there exists a 3-representative embedding f of $\bar{B}_1$ into S_1 such that $f|_{B_1}=f_1$ and $f(e) \subset \bar{r}_1$.

We note that $r_2 \in R(f)$. We take a simple curve γ_2 in $\bar{r}_2$ such that $\gamma_2 \cap \partial \bar{r}_2 = \partial \gamma_2 = \{f(v'), f(v'')\}$. Since f_1 is 3-representative, $[\gamma_2 \cup f(e)] = 1$ in S_1. Then there exists a disk D in S_1 such that $\partial D = \gamma_2 \cup f(e)$. We note that $f(\bar{B}_1) \cap \operatorname{int} D \neq \emptyset$. Contract $f(e)$ to get the 2-cell embedding f_e of G into S_1. Since $r_2 \in R(f_e)$, the simple closed curve γ_2^* in $\bar{r}_2$ is obtained from $\gamma_2 \cup f(e)$ by contracting $f(e)$ to $f_e(v)$ in S_1. Since $[\gamma_2^*] = [\gamma_2 \cup f(e)] = 1$ in S_1, we may assume that $\partial D = \gamma_2^*$ by the construction of f_e. Since $f_e(G) \cap \operatorname{int} D \neq \emptyset$, v is a cut-vertex of G. This contradicts that G is 2-connected. Therefore there exists at most one face r_1 of f_1 such that $v', v'' \in V(W(r_1))$. By Theorem 4.2(1), G has a strong embedding into some closed surface.

If $m>1$, then G has a strong embedding into some closed surface by Theorem 4.2 (2). □

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