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BAHADUR-TYPE REPRESENTATION OF SAMPLE CONDITIONAL QUANTILES BASED ON WEAKLY DEPENDENT DATA

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Summary. Bahadur-type representations of nearest-neighbor and kernel estimators of the conditional p-quantile $\theta_p(x)$ are obtained when the sample forms a strictly stationary absolutely regular sequence of two dimensional random vectors.

1. Introduction

Let $\{(X_i, Z_i)\}$ be a strictly stationary sequence of two dimensional random vectors defined on a probability space $(\Omega, \mathcal{F}, P)$. For $0<p<1$, let $\theta_p(x)$ denote the p-quantile of Z given $X=x$. Bhattacharya and Gangopadhyay [2] obtained Bahadur-type representations of nearest-neighbor and kernel estimators of $\theta_p(x)$, when $\{(X_i, Z_i)\}$ are independent and identically distributed random vectors.

In this paper, we will consider the analogous problem when $\{(X_i, Z_i)\}$ satisfies some mixing condition.

2. The main results

Let (X, Z) be a two-dimensional random vector. Let f be the probability density function of X and $g(\cdot|x)$ the conditional density function of z and $G(\cdot|x)$ the conditional density function of Z given $X=x$. The object is to estimate $\theta_p(x_0)$, the conditional p-quantile of Z given $X=x_0$. Since $p\in(0, 1)$ and x_0 will remain fixed, we shall write $\theta_p(x_0)=\theta$.

We will consider the following conditions:

Condition I.
(i) $f(x_0)>0$

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(ii) $f''(x_0)$ exists in a neighborhood of x_0, and there exist positive constants ε and C such that $|x-x_0|\leqq\varepsilon$ implies $|f''(x)-f''(x_0)|\leqq C|x-x_0|$.

Condition II.

(i) $g(\theta|x_0)>0$ and $G(\theta|x_0)=p$.

(ii) The partial derivatives $g_z(z|x)$ and $g_{xx}(z|x)$ of $g(z|x)$ and $G_{xx}(z|x)$ of $G(z|x)$ exist in a neighborhood of (x_0, θ) and there exist positive constants ε and C such that $|x-x_0|\leqq\varepsilon$ and $|z-\theta|\leqq\varepsilon$ together imply the following:

$$
\begin{aligned}
&|g_z(z|x)|\leqq C, \qquad |g_x(z|x_0)|\leqq C, \qquad |g_{xx}(z|x_0)|\leqq C,\\
&|g_{xx}(z|x)-g_{xx}(z|x_0)|\leqq C|x-x_0|,\\
&|G_{xx}(z|x)-G_{xx}(z|x_0)|\leqq C|x-x_0|.
\end{aligned}
\tag{2.1}
$$

By Condition II, θ is uniquely defined by $G(\theta|x_0)=p$.

Now, let $\{(X_i, Z_i): -\infty<i<\infty\}$ be an $\boldsymbol{R}^2$-valued strictly stationary stochastic sequence which satisfies the following condition.

Condition III. There exists a nonrandom sequence $\{\beta(n)\}$ for which the following requirements are satisfied:

(i) $\beta(n)\downarrow 0$ as $n\to\infty$, and

$$\sum_{n=1}^{\infty} n^2\beta^{1/5}(n)<\infty; \tag{2.2}$$

(ii) $\{X_i\}$ is absolutely regular, i.e.,

$$E \sup_{B\in\mathcal{M}_n^{\infty}(X)} |P(B|\mathcal{M}_{-\infty}^{0}(X))-P(B)|\leqq\beta(n); \tag{2.3}$$

(iii) $\{Z_i\}$ satisfies the conditional absolutely regular condition given $\mathcal{M}_{-\infty}^{\infty}(X)$, i.e.,

$$E\{\sup_{D\in\mathcal{M}_n^{\infty}(Z)} |P(D|\mathcal{M}_{-\infty}^{0}(Z)\times\mathcal{M}_{-\infty}^{\infty}(X))-P(D|\mathcal{M}_{-\infty}^{\infty}(X))\}\leqq\beta(n). \tag{2.4}$$

Here, $\mathcal{M}_a^b(W)$ denotes the σ-algebra generated by $W_a, \cdots, W_b$ $(a\leqq b)$.

Assume that for each i (X_i, Z_i) has the same distribution as that of (X, Z).

Next, let $Y=|X-x_0|$. Then, the probability density function f_Y of Y, the conditional probability density function $g^*(\cdot|y)$ and the conditional distribution function $G^*(\cdot|y)$ are given, respectively, by the following equalities:

$$
\begin{aligned}
&f_Y(y)=f(x_0+y)+f(x_0-y),\\
&g^*(z|y)=\frac{1}{f_Y(y)}\{f(x_0+y)g(z|x_0+y)+f(x_0-y)g(z|x_0-y)\},\\
&G^*(z|y)=\frac{1}{f_Y(y)}\{f(x_0+y)G(z|x_0+y)+f(x_0-y)G(z|x_0-y)\}.
\end{aligned}
\tag{2.5}
$$

Note that

$$g^*(z|0)=g(z|x_0), \qquad G^*(z|0)=G(z|x_0).$$

In what follows, we write

$$g(z|x_0)=g(z) \quad \text{and} \quad G(z|x_0)=G(z).$$

Let $Y_i=|X_i-x_0|$ $(i\geqq 1)$. Then, $\{(Y_i, Z_i): i\geqq 1\}$ is a strictly stationary absolutely regular sequence such that each (Y_i, Z_i) has the same distribution function as that of (Y, Z) and the sequence has the same coefficient $\beta(n)$ as that of $\{(X_i, Z_i)\}$.

Let $Y_{n,1}\leqq\cdots\leqq Y_{n,n}$ denote the order statistics of $Y_1, \cdots, Y_n$ and $Z_{n,1}, \cdots, Z_{n,n}$ the induced order statistics of $(Y_1, Z_1), \cdots, (Y_n, Z_n)$, i.e.,

$$Z_{n,i}=Z_j \quad \text{if} \quad Y_{n,i}=Y_j.$$

For any positive integer $k\leqq n$, the k-nearest neighbor (or k-NN) empirical distribution function of Z (with respect to x_0) is defined as

$$\hat{G}_{n,k}(z)=k^{-1}\sum_{i=1}^{k} I(Z_{n,i}\leqq z) \tag{2.6}$$

where $I(A)$ denotes the indicator of the event A.

Now, the k-NN estimator of θ can be expressed as the p-quantile of $\hat{G}_{n,k}$, i.e.,

$$\begin{aligned} \hat{\theta}_{n,k}&=\text{the } [kp]\text{th order statistic of } Z_{n,1}, \cdots, Z_{n,k} \\ &=\inf\left\{z:\ \hat{G}_{n,k}(z)\geqq\frac{[kp]}{k}\right\}. \end{aligned} \tag{2.7}$$

The kernel estimator of θ with uniform kernel and bandwidth h can also be expressed in the same manner, viz.,

$$\tilde{\theta}_{n,k}=\inf\left\{z:\ \hat{G}_{n,K_n(h)}(z)\geqq\frac{[K_n(h)p]}{K_n(h)}\right\} \tag{2.8}$$

where

$$K_n(h)=\sum_{i=1}^{n} I\left(Y_i\leqq\frac{h}{2}\right)=n\hat{F}_{Y,n}\left(\frac{h}{2}\right), \tag{2.9}$$

and $\hat{F}_{Y,n}$ denotes the empirical distribution function of $Y_1, \cdots, Y_n$. It is obvious that the kernel estimators are related to the k-NN estimators by

$$\tilde{\theta}_{n,k}=\hat{\theta}_{n,K_n(h)} \tag{2.10}$$

where $K_n(h)$ is the random integer given by (2.9).

Let

$$I_n(a, b)=\{k\,|\,[n^{4/5}a]\leqq k\leqq[n^{4/5}b]\} \qquad (0<a<b) \tag{2.11}$$

and

$$J_n(c, d)=[n^{-1/5}c,\ n^{-1/5}d] \qquad (0<c<d). \tag{2.12}$$

In the sequel, we write "a. s." instead of writing "ultimately holds almost surely".

Theorem 1. *Put*

$$\hat{\theta}_{n,k}-\theta=\rho(\theta)\left(\frac{k}{n}\right)^2+\{kg(\theta)\}^{-1}\sum_{i=1}^{k}\{I(Z^*_{n,i}>\theta)-(1-p)\}+R_{n,k} \tag{2.13}$$

where

$$\rho(\theta)=-\frac{f(x_0)G_{xx}(\theta\,|\,x_0)+2f'(x_0)G_x(\theta\,|\,x_0)}{24f^3(x_0)g(\theta)} \tag{2.14}$$

and

$$Z^*_{n,i}=G^{-1}\circ G^*(Z_{n,i}\,|\,Y_{n,i}), \qquad 1\leqq i\leqq n, \qquad n\geqq 1. \tag{2.15}$$

Then

$$\max_{k\in I_n(a,b)}|R_{n,k}|=O(n^{-3/5}\log n) \quad a.s. \tag{2.16}$$

Theorem 2. *Put*

$$\begin{aligned}\tilde{\theta}_{n,h}-\theta&=\rho(\theta)f^2(x_0)h^2\\&\quad+\{[nhf(x_0)]g(\theta)\}^{-1}\sum_{i=1}^{[nhf(x_0)]}\{I(Z^*_{n,i}>\theta)-(1-p)\}+R^*_{n,h}\end{aligned} \tag{2.17}$$

where $\rho(\theta)$ and $Z^*_{n,i}$ are the ones defined by (2.14) *and* (2.15), *respectively.*

Suppose that instead of Condition III *the following more stringent condition than Condition* III *is satisfied*:

Condition IV. *There exists a function $\phi(n)$ of n ($\geqq 1$) for which the next requirements are satisfied*:

(i) *$\phi(n)\downarrow 0$ as $n\to\infty$ and*

$$\sum_{n=1}^{\infty}n^2\phi^{1/5}(n)<\infty\,; \tag{2.18}$$

(ii) *$\{X_i\}$ is ϕ-mixing, i.e.,*

$$\sup_{B\in\mathcal{M}_n^{\infty}(X)}|P(B\,|\,\mathcal{M}_{-\infty}^{0}(X))-P(B)|\leqq\phi(n) \tag{2.19}$$

holds almost surely;

(iii) *$\{Z_i\}$ satisfies the conditional ϕ-mixing condition given $\mathcal{M}_{-\infty}^{\infty}(X)$, i.e.,*

$$\sup_{D\in\mathcal{M}_n^{\infty}(Z)}|P(D\,|\,\mathcal{M}_{-\infty}^{0}(Z)\times\mathcal{M}_{-\infty}^{\infty}(X))-P(D\,|\,\mathcal{M}_{-\infty}^{\infty}(X))|\leqq\phi(n) \tag{2.20}$$

holds almost surely.

Then

(2.21) $$\sup_{h\in J_n(c,d)} |R^*_{n,h}| = O(n^{-3/5}\log n) \qquad a.e.$$

Remark. It is known that if $\{X_i\}$ is ϕ-mixing, then $\{X_i\}$ is absolutely regular.

3. Preliminaries

In this section, we prove lemmas which correspond to Lemmas 1-9 in [2]. We introduce the following notations:

(3.1) $$\begin{aligned} &g^*(\cdot\,|Y_{n,i}) = g_{n,i}(\cdot), && G^*(\cdot\,|Y_{n,i}) = G_{n,i}(\cdot), \\ &\bar{g}_{n,k}(\cdot) = k^{-1}\sum_{i=1}^{k} g_{n,i}(\cdot), && \bar{G}_{n,k}(\cdot) = k^{-1}\sum_{i=1}^{k} G_{n,i}(\cdot). \end{aligned}$$

Define $\bar{\theta}_{n,k}$ by

(3.2) $$\bar{G}_{n,k}(\bar{\theta}_{n,k}) = p = G(\theta).$$

Lemma 1. *For* $B > b/f(x_0)$

(3.3) $$Y_{n,[n^{4/5}b]} \leqq Bn^{-1/5} \qquad a.s.$$

Proof. Put $r = n^{4/5}$ and

$$\eta_j = I(Y_j \leqq n^{-1}rB) \qquad (1 \leqq j \leqq r).$$

Since $Y_{n,[rb]} > n^{-1}rB$ if and only if $\sum_{j=1}^{n} \eta_j < [rb]$ we have

(3.4) $$P_n = P(Y_{n,[rb]} > n^{-1}rB) = P\Big(\sum_{j=1}^{n} (\eta_j - E\eta_j) < -n(E\eta_j - n^{-1}[rb])\Big).$$

We note here that

(3.5) $$\begin{aligned} n(E\eta_j - n^{-1}[rb]) &= n\Big(\int_{|t-x_0|<n^{-1}rB} f(t)dt - n^{-1}[rb]\Big) \\ &> r(Bf(x_0) - b) = c_0 n^{4/5}. \end{aligned}$$

On the other hand, by (2.2) and the fact that $|\eta_j| \leqq 1$ we have

(3.6) $$E\Big|\sum_{j=1}^{n} (\eta_j - E\eta_j)\Big|^4 \leqq cn^2$$

(cf. [7]). Hence, by (3.4)-(3.6) we have

(3.7) $$P_n \leqq cn^{-6/5}$$

for all n sufficiently large.

Now, (3.3) follows from (3.7) and the Borel-Cantelli lemma. □

Next, let F_Y be the distribution function of Y. Then, by Condition I the following facts are obtained:

(i) $h(u)=F_Y^{-1}(u)$ is defined for $0\leqq u\leqq\varepsilon$ (ε being some positive number) as the uinque solution of $F_Y(h(u))=u$,

(ii) h''' is continuous at 0,

(iii) $h(0)=h''(0)=0, \qquad h'(0)=\{2f(x_0)\}^{-1}, \qquad h'''(0)=-f''(x_0)\{8f^4(x_0)\}^{-1}$.

(See Lemma 1 in [3]).

Put

$$U_i=F_Y(Y_i) \qquad (i\geqq 1).$$

Then, it is obvious that $\{U_i: i\geqq 1\}$ is a strictly stationary absolutely regular sequence of random variables such that $\{U_i: i\geqq 1\}$ has the same mixing coefficient as that of $\{(X_i, Z_i): i\geqq 1\}$ and each U_i has the uniform distribution on (0, 1). Further, let

$$U_{n,i}=F_Y(Y_{n,i}) \qquad (1\leqq i\leqq n).$$

Since $Y_{n,1}\leqq\cdots\leqq Y_{n,n}$ are the order statistics in Y_i, $i=1,\cdots,n$, $0\leqq U_{n,1}\leqq\cdots\leqq U_{n,n}\leqq 1$ are the order statistics in U_i, $i=1,\cdots,n$.

Lemma 2. *Put*

$$k^{-1}\sum_{i=1}^{k} Y_{n,i}^2=\{12f(x_0)\}^{-2}\left(\frac{k}{n}\right)^2+R_{n,k}. \tag{3.8}$$

Then

$$\max_{k\leq[rb]}|R_{n,k}|=O(n^{-3/5}) \qquad a.s. \tag{3.9}$$

where $r=[n^{4/5}]$.

Proof. We note first that for i $(1\leqq i\leqq[rb])$

$$\begin{aligned} Y_{n,i}&=h(U_{n,i})\\ &=h(0)+h'(0)U_{n,i}+\frac{h''(0)}{2}U_{n,i}^2+\frac{h'''(\zeta_i U_{n,i})}{6}U_{n,i}^3 \qquad (3.10)\\ &=\{2f(x_0)\}^{-1}U_{n,i}+\frac{h'''(\zeta_i U_{n,i})}{6}U_{n,i}^3 \end{aligned}$$

where ζ_i is a random variable such that $|\zeta_i|\leqq 1$. Furthermore, letting $F_{U,n}$ denote the empirical distribution function of $U_1,\cdots,U_n$,

$$\begin{aligned}\max_{1\leq i\leq[rb]}\left|U_{n,i}-\frac{i}{n}\right|&=\max_{1\leq i\leq[rb]}\left|U_{n,i}-F_{U,n}(U_{n,i})\right|\\ &\leqq \sup_{0\leq u\leq n^{-1/5}b}|F_{U,n}(u)-u|,\end{aligned}$$

so that by Theorem in [6] (cf. Theorem 2 in [1]) and the property of the Kiefer process

$$\max_{1\leq i\leq [rb]}\left|U_{n,i}-\frac{i}{n}\right|=O(n^{-7/10}(\log\log n)^{1/2}) \qquad \text{a. s.} \tag{3.11}$$

Define random variables $R_{n,k}(1)$ and $R_{n,k}(2)$ by

$$k^{-1}\sum_{i=1}^{k}Y_{n,i}^2=\{2f(x_0)\}^{-2}k^{-1}\sum_{i=1}^{k}U_{n,i}^2+R_{n,k}(1), \tag{3.12}$$

$$k^{-1}\sum_{i=1}^{k}U_{n,i}^2=k^{-1}\sum_{i=1}^{k}\left(\frac{i}{n}\right)^2+R_{n,k}(2). \tag{3.13}$$

Then, by (3.10) and (3.11) it is easily shown that

$$\max_{1\leq k\leq [rb]}|R_{n,k}(1)|=O(n^{-4/5}) \qquad \text{a. s.,} \tag{3.14}$$

$$\max_{1\leq k\leq [rb]}|R_{n,k}(2)|=O(n^{-7/10}(\log\log n)^{1/2}) \qquad \text{a. s.,} \tag{3.15}$$

since

$$\max_{1\leq k\leq [rb]}\left|k^{-1}\sum_{i=1}^{k}\left(\frac{i}{n}\right)^2-\frac{1}{3}\left(\frac{k}{n}\right)^2\right|=O(n^{-6/5}). \tag{3.16}$$

Hence, (3.9) is obtained from (3.12)-(3.16). □

The following lemmas and a corollary may be proved by the methods in [2] and the proofs of them are omitted.

Lemma 3. *The following expansions hold for the conditional probability density function $g^*(z|y)$ and the conditional distribution function $G^*(z|y)$:*

$$\begin{aligned} g^*(z|y)&=g(z)+\frac{1}{2}y^2q(z)+y^3r(y,z), \\ G^*(z|y)&=G(z)+\frac{1}{2}y^2Q(z)+y^3R(y,z), \end{aligned} \tag{3.17}$$

where

$$\begin{aligned} g(z)&=g(z|x_0), \qquad G(z)=G(z|x_0), \\ q(z)&=g_{xx}(z|x_0)+\frac{2f'(x_0)g_x(z|x_0)}{f(x_0)}, \\ Q(z)&=G_{xx}(z|x_0)+\frac{2f'(x_0)G_x(z|x_0)}{f(x_0)}, \end{aligned} \tag{3.18}$$

and there exist positive constants ε and M such that $|q(z)|$, $|Q(z)|$, $|r(y,z)|$ and $|R(y,z)|$ are all bounded by M for $0\leqq y\leqq\varepsilon$ and $|z-\xi|\leqq\varepsilon$.

Lemma 4. *For every B, there exist N and C such that in the sample space of infinite sequences*

$$\{((y_1, z_1), (y_2, z_2), \cdots): y_i \geqq 0, z_i \textit{ real}\},$$

$Y_{n,[rb]} \leqq Bn^{-1/5}$ *implies*

$$\max_{k\in I_n(a,b)} |\bar{\theta}_{n,k}-\theta| \leqq Cn^{-2/5} \quad \textit{for all} \quad n \geqq N. \tag{3.19}$$

Corollary. *For* $0<a<b$

$$\max_{k\in I_n(a,b)} |\bar{\theta}_{n,k}-\theta| = O(n^{-2/5}) \quad a.s. \tag{3.20}$$

Lemma 5. *For* $0<a<b$

$$\max_{k\in I_n(a,b)} \left|\bar{\theta}_{n,k}-\theta-\rho(\theta)\left(\frac{k}{n}\right)^2\right| = O(n^{-3/5}) \quad a.s., \tag{3.21}$$

where $\rho(\theta) = -Q(\theta)\{24 f^2(x_0)g(\theta)\}^{-1}$.

The following lemma is easily proved by the method in [5].

Lemma 6. *Let* $\{\xi_i\}$ *be a nonstationary absolutely regular sequence such that* $|\xi_i| \leqq M_0$ *and* $E\xi_i=0$. *Suppose* $\sigma_0^2 = \max_{1\leqq i\leqq n} E\xi_i^2 > 0$. *Then, for the normalized sum* $n^{-1/2}S_n$, *the following inequalities hold when* n *is sufficiently large and* m $(\leqq n)$ *is an arbitrary positive integer*:

If $0<t<(\sigma_0^2/M_0)n^{1/2}$,

$$P(n^{-1/2}|S_n| \geqq t) \leqq 2m \exp\left\{-\frac{(m^{-1/2}t)}{2\sigma_0^2}\left(1-\frac{M_0 t}{\sigma_0^2(mn)^{1/2}}\right)\right\} + 4m\beta(m). \tag{3.22}$$

Let $a_n = n^{-2/5}\log n$ and put

$$A_n = \{\omega | \max_{k\in I_n(a,b)} |\hat{\theta}_{n,k}-\bar{\theta}_{n,k}| > a_n\} \quad (n \geqq 1). \tag{3.23}$$

Further, fix $C_1(>b/f(x_0))$ and write

$$B_n = \{\omega | Y_{n,[rb]} \leqq C_1 n^{-1/5}\} \quad (n \geqq 1). \tag{3.24}$$

Lemma 7. *Suppose* $\vartheta_{n,k}$*'s are* $\mathcal{M}_{-\infty}^{\infty}(X)$*-measurable random variables with*

$$|\vartheta_{n,k}-\bar{\theta}_{n,k}| \leqq C_2 n^{-2/5}\log n = \varepsilon_n(C_2). \tag{3.25}$$

Then, there exists C_3 *such that*

$$\begin{aligned}\sum_{n=1}^{\infty} n \max_{k\in I_n(a,b)} P(|\{\hat{G}_{n,k}(\vartheta_{n,k})-\hat{G}_{n,k}(\bar{\theta}_{n,k})\} \\ -\{\bar{G}_{n,k}(\vartheta_{n,k})-\bar{G}_{n,k}(\theta_{n,k})\}| > C_3 n^{-3/5}\log n) < \infty.\end{aligned} \tag{3.26}$$

Proof. Put

$$V_{n,k,i} = I(\bar{\theta}_{n,k} < Z_{n,i} \leqq \vartheta_{n,k})$$

and

$$\mu_{n,k,i}=G_{n,i}(\vartheta_{n,k})-G_{n,i}(\bar{\theta}_{n,k})=E(V_{n,k,i}\,|\,\mathcal{M}_{-\infty}^{\infty}(X)).$$

Then

$$\{\hat{G}_{nk}(\vartheta_{n,k})-\hat{G}_{n,k}(\bar{\theta}_{n,k})\}-\{\bar{G}_{n,k}(\vartheta_{n,k})-\bar{G}_{n,k}(\bar{\theta}_{n,k})\}$$

(3.27)
$$=k^{-1}\sum_{i=1}^{k}(V_{n,k,i}-\mu_{n,k,i}).$$

Since

$$|\vartheta_{n,k}-\bar{\theta}_{n,k}|\leqq\varepsilon_n(C_2)=C_2 n^{-2/5}\log n,$$

Lemma 4 implies that there exists C_4 and N such that for $n\geqq N$ and for z lying between $\vartheta_{n,k}$ and $\bar{\theta}_{n,k}$,

$$|z-\theta|\leqq C_2 n^{-2/5}\log n+C_4 n^{-2/5}\leqq 2\varepsilon_n(C_2)$$

holds on the set B_n. Using the Bhattacharya and Ganpopadhyay method [2] we conclude that

$$\max_{1\leq i\leq[rb]}|\mu_{n,k,i}|\leqq 2g(\theta)\varepsilon_n(C_2)=\varepsilon_n^*, \qquad \text{(say)},$$

which implies

(3.28)
$$\max_{1\leq i\leq[rb]}\operatorname{var}V_{n,k,i}\leqq\max_{1\leq i\leq[rb]}|\mu_{n,k,i}|\leqq\varepsilon_n^*.$$

Therefore, by condition III (iii), Lemma 6 and (3.28) we have that for large n

$$\max_{k\in I_n(a,b)}P\left(\left|k^{-1}\sum_{i=1}^{k}(V_{n,k,i}-\mu_{n,k,i})\right|>C_3 n^{-3/5}\log n\right)$$

(3.29)
$$=\max_{k\in I_n(a,b)}EP\left(\left|k^{-1}\sum_{i=1}^{k}(V_{n,k,i}-\mu_{n,k,i})\right|>C_3 n^{-3/5}\log n\,\middle|\,\mathcal{M}_{-\infty}^{\infty}(X)\right)$$

$$\leqq 2m\exp\left\{-\frac{(m^{-1/2}t)}{2\varepsilon_n^*}\left(1-\frac{2t}{\varepsilon_n^*(mn)^{1/2}}\right)\right\}+4m\beta(m)$$

where $m=n^{1/5}$ and $t=n^{-1/5}\log n$.

Now, (3.25) follows from Lemma 1 and (3.29). □

Lemma 8.

(3.30)
$$P(A_n\quad i.o.)=0.$$

Proof. Let

$$\eta_{n,k,i}=I(Z_{n,i}\leqq\bar{\theta}_{n,k}-a_n)-G_{n,i}(\bar{\theta}_{n,k}-a_n).$$

Then, $\hat{\theta}_{n,k}\leqq\bar{\theta}_{n,k}-a_n$ implies

$$k^{-1}\sum_{i=1}^{k}\eta_{n,k,i}\geqq\frac{[kp]}{k}-\bar{G}_{n,k}(\bar{\theta}_{n,k}-a_n).$$

It follows from Lemma 3 that on the set B_n

$$\min_{k\in I_n(a,b)}\left\{\frac{[kp]}{k}-\bar{G}_{n,k}(\bar{\theta}_{n,k}-a_n)\right\}\geqq\frac{1}{2}g(\theta)a_n$$

when n is large. Hence, by the above method we get

$$\sum_{n=1}^{\infty}P(\min_{k\in I_n(a,b)}(\hat{\theta}_{n,k}-\bar{\theta}_{n,k})\leqq -a_n)<\infty$$

which implies

$$P(\min_{k\in I_n(a,b)}(\hat{\theta}_{n,k}-\bar{\theta}_{n,k})\leqq -a_n\quad i.o.)=0.$$

In the same way,

$$P(\max_{k\in I_n(a,b)}(\hat{\theta}_{n,k}-\bar{\theta}_{n,k})\geqq a_n\quad i.o.)=0,$$

and the proof is completed. □

Next, let $b_n=n^{1/5}$ and divide the interval

$$[\bar{\theta}_{n,k}-a_n,\ \bar{\theta}_{n,k}+a_n]$$

into $2[b_n]$ equal intervals:

$$\begin{aligned}J_{n,k,j}&=\left[\bar{\theta}_{n,k}+\frac{ja_n}{b_n},\ \bar{\theta}_{n,k}+\frac{(j+1)a_n}{b_n}\right]\\&=[d_{n,k,j},\ d_{n,k,j+1}]\qquad (j=-b_n,\ \cdots,\ -1,\ 0,\ 1,\ \cdots,\ b_n-1),\end{aligned}$$

each of length $a_n/b_n=n^{-3/5}\log n$. Let

$$\begin{aligned}&H_{n,k}(z)=\{\hat{G}_{n,k}(z)-\hat{G}_{n,k}(\bar{\theta}_{n,k})\}-\{(\bar{G}_{n,k}(z)-\bar{G}_{n,k}(\bar{\theta}_{n,k})\},\\ &H^*_{n,k}=\sup_{|z-\bar{\theta}_{n,k}|\leq a_n}|H_{n,k}(z)|=\sup_{z\in J_{n,k,j}}|H_{n,k}(z)|,\\ &H^*_n=\max_{k\in I_n(a,b)}|H^*_{n,k}|.\end{aligned}\tag{3.31}$$

Lemma 9. *For large C*

$$P(H^*_n>Cn^{-3/5}\log n\quad i.o.)=0.\tag{3.32}$$

Proof. It follows from the monotonicity of $\hat{G}_{n,k}(\cdot)$ and $\bar{G}_{n,k}(\cdot)$ that for $z\in J=[d_{n,k,j},\ d_{n,k,j+1}]$

$$H_{n,k}(d_{n,k,j})-\alpha_{n,k,j}\leqq H_{n,k}(z)\leqq H_{n,k}(d_{n,k,j+1})+\alpha_{n,k,j},$$

where

$$\alpha_{n,k,j}=\bar{G}_{n,k}(d_{n,k,j+1})-\bar{G}_{n,k}(d_{n,k,j}).$$

Hence,

$$H^*_{n,k}=\sup_{|z-\bar{\theta}_{n,k}|\leq a_n}|H_{n,k}(z)|$$

$$\leqq \max_{-b_n \leq j \leq b_n} |H_{n,k}(d_{n,k,j})| + \max_{-b_n \leq j \leq b_n - 1} \alpha_{n,k,j}.$$

Then, by Lemmas 3 and 4

$$\max_{-b_n \leq j \leq b_n - 1} \alpha_{n,k,j} \leqq 2g(\theta) n^{-3/5} \log n$$

on the set S_n when n is large. Consequently, by the previous method we have that for M sufficiently large

$$\sum_{n=1}^{\infty} P(H_n^* > (M + 2g(\theta)) n^{-3/5} \log n)$$

$$\leqq \sum_{n=1}^{\infty} P(\max_{k \in I_n(a,b)} \max_{-b_n \leq j \leq b_n - 1} |H_{n,k}(d_{n,k,j})| > M n^{-3/5} \log n) + \sum_{n=1} P(B_n^c) < \infty,$$

which implies (3.32). □

Lemma 10.

$$p - \hat{G}_{n,k}(\bar{\theta}_{n,k}) = O(n^{-2/5} \log n) \quad a.s. \tag{3.33}$$

Proof. By Lemmas 8 and 9, we have

$$\begin{aligned} p - \hat{G}_{n,k}(\bar{\theta}_{n,k}) &= \bar{G}_{n,k}(\hat{\theta}_{n,k}) - \bar{G}_{n,k}(\bar{\theta}_{n,k}) + R_{n,k}(1) \\ &= (\hat{\theta}_{n,k} - \bar{\theta}_{n,k}) \bar{g}_{n,k}(\theta_{n,k}^*) + R_{n,k}(1) \end{aligned} \tag{3.34}$$

where $\theta_{n,k}^*$ lies between $\hat{\theta}_{n,k}$ and $\bar{\theta}_{n,k}$, and

$$\max_{k \in I_n(a,b)} |R_{n,k}(1)| = O(n^{-3/5} \log n), \quad \text{a.s.} \tag{3.35}$$

By Corollary to Lemma 4 and Lemma 8

$$\max_{k \in I_n(a,b)} |\theta_{n,k}^* - \theta| = O(n^{-2/5} \log n), \quad \text{a.s.}$$

Consequently, by Lemmas 1 and 3 we have

$$\max_{k \in I_n(a,b)} |\bar{g}_{n,k}(\theta_{n,k}^*) - g(\theta)| = O(n^{-2/5} \log n), \quad \text{a.s.} \tag{3.36}$$

Now, (3.33) follows from (3.34)-(3.36). □

4. Proofs

Using lemmas in the preceding section we can prove Theorems 1 and 2 by the identical methods to those of [2]. We write here only the outlines of those methods.

Proof of Theorem 1. From (3.34)-(3.36), we have

$$\max_{k\in I_n(a,b)} |(\hat{\theta}_{n,k}-\bar{\theta}_{n,k})-\{g(\theta)\}^{-1}[p-\hat{G}_{n,k}(\bar{\theta}_{n,k})]|=O(n^{-3/5}\log n), \quad \text{a. s.}$$

Since

$$p-\hat{G}_{n,k}(\bar{\theta}_{n,k})=k^{-1}\sum_1^k[I(Z_{n,i}>\bar{\theta}_{n,k})-\{1-G_{n,i}(\bar{\theta}_{n,k})\}],$$

we have the following representation:

$$\hat{\theta}_{n,k}=\bar{\theta}_{n,k}+\{kg(\theta)\}^{-1}\sum_1^k\{I(Z_{n,i}>\bar{\theta}_{n,k})-\{1-G_{n,i}(\bar{\theta}_{n,k})\}]+R_{n,k} \tag{4.1}$$

and

$$\max_{k\in I_n(a,b)} |R_{n,k}|=O(n^{-3/5}\log n), \quad \text{a. s.} \tag{4.2}$$

This representation can be easily modified to two other slightly different forms, viz.

$$\hat{\theta}_{n,k}=\bar{\theta}_{n,k}+\{kg(\theta)\}^{-1}\sum_1^k[I(Z_{n,i}>\theta)-(1-G_{n,i}(\theta))]+R_{n,k} \tag{4.3}$$

and

$$\hat{\theta}_{n,k}=\bar{\theta}_{n,k}+\{kg(\theta)\}^{-1}\sum_1^k[I(Z^*_{n,i}>\theta)-\{1-G(\theta)\}]+R_{n,k}, \tag{4.4}$$

where in both (4.3) and (4.4)

$$\max_{k\in I_n(a,b)} |R_{n,k}|=O(n^{-3/5}\log n) \quad \text{a. s.,}$$

$$Z^*_{n,i}=G^{-1}\circ G_{n,i}(Z_{n,i})$$

and $G(\cdot)=G(\cdot|x_0)$ is the conditional distribution function of Z given $X=x_0$. Combine Lemma 5 with (4.4) and note that $G(\theta)=p$ to complete the proof of Theorem 1.

Proof of Theorem 2. The kernel estimator $\tilde{\theta}_{n,h}$ can be regarded as the NN estimator $\hat{\theta}_{n,K_n(h)}$ in which $K_n(h)$ is a random integer given by (2.8). A formal substitution for k by $K_n(h)$ in the representation given in Theorem 1 leads to

$$\tilde{\theta}_{n,h}-\theta=\rho(\theta)\left\{\frac{K_n(h)}{n}\right\}^2+\frac{1}{K_n(h)g(\theta)}\sum_1^{K_n(h)}[I(Z^*_{n,i}>\theta)-(1-p)]+R_{n,K_n(h)}. \tag{4.5}$$

Thus, it is enough to show that

(a) $\sup_{h\in J_n(c,d)} |R_{n,K_n(h)}|$ converges at a fast rate,

where $J_n=[n^{-1/5}c, n^{-1/5}d]$, $0<c<d$, and

(b) in the first two terms of the right hand side of (4.5), $K_n(h)$ can be replaced by the leading term of its deterministic component without slowing down the rate of convergence of the remainder term.

(a) and (b) can be shown by Condition IV and the following lemma.

Lemma 11. *Let* $\Delta_n(h)=K_n(h)-nhf(x_0)$. *Then*

(4.6) $$\sup_{h\in J_n(c,d)}|\Delta_n(h)|=O(n^{2/5}\log n) \qquad a.s.$$

To prove Lemma 11 we need the following lemma which is essentially due to Lemma 2.1 in [4]. (See also Lemmas 2.1.2 and 2.1.3 in [8].)

Lemma A. *Let* $\{\zeta_i\}$ *be a strictly stationary sequence of zero-mean Bernoullian random variables with* $\mathrm{Var}(\zeta_i)\leqq B$. *Suppose* $\{\zeta_i\}$ *is* ϕ*-mixing and* $\sum\phi(n)<\infty$. *Let* r_n *be the smallest positive integer such that* $n\phi(i)/i\leqq 1$. *Let* a, λ *be positive numbers and* N *some positive integer obeying the relationship* $a^2NB\leqq\lambda^2$ *and* $ar_N\leqq\lambda$. *Then, for all* $n\leqq N$ *and* $c>0$, *there exist constants* K_1, K_2, K_3 *such that*

(4.7) $$P\left(\left|\sum_{i=1}^{n}\zeta_i\right|\geqq 2K_1\lambda\right)\leqq K_2e^{-a}+K_3Nr_N^{c-1}\left(\frac{\lambda}{a}\right)^{-2c}.$$

The constants K_1, K_2, K_3 *are independent of* λ, a, B *and* N; *however* K_3 *may be depend on* c.

Now, we prove Lemma 11.

Proof of Lemma 11. It is easily shown that if $h\in J_n(c,d)$, then by Condition I

(4.8) $$EI\left(Y_i\leqq\frac{h}{2}\right)-hf(x_0)=\int_{-h/2}^{h/2}\{f(x_0+y)+f(x_0-y)\}\,dy-hf(x_0)$$
$$=O(n^{-3/5}).$$

We note that

$$L_n=\sum_{i=1}^{n}\left\{I\left(Y_i\leqq\frac{h}{2}\right)-EI\left(Y_i\leqq\frac{h}{2}\right)\right\}$$

is a partial sum of a strictly ϕ-mixing sequence

$$\{\zeta_i\}=\left\{I\left(Y_i\leqq\frac{h}{2}\right)-EI\left(Y_i\leqq\frac{h}{2}\right)\right\}$$

of zero-mean Bernoullian random variables, and

$$\mathrm{Var}(\zeta_i)=O(F_Y(h))\leqq C_0h.$$

Hence, by Lemma A with $a=C_1\log n$, $B=dC_0n^{1/5}$, $\lambda=C_2n^{1/5}\log n$ (C_1 and C_2 being suitable positive constants) we can prove that

(4.9) $$L_n\leqq C_3n^{2/5}\log n \qquad \text{a.s.}$$

for some C_3 (>0) which is independent of n. Hence, (4.7) follows from (4.8) and (4.9) □

From Lemma 11 and the fact that

$$\max_{k\in I_n(a,b)} |R_{n,k}|=O(n^{-3/5}\log n) \qquad \text{a. s.}$$

it now follows that

$$\sup_{h\in J_n(c,d)} |R_{n,K_n(h)}|=O(n^{-3/5}\log n), \qquad \text{a. s.}$$

We consider the first two terms on the right hand side of (4.5). Of these,

$$\rho(\theta)\left\{\frac{K_n(h)}{n}\right\}^2=\rho(\theta)f^2(x_0)h^2+R'_{n,h}$$

where

$$R'_{n,h}=\rho(\theta)f^2(x_0)h^2\frac{\Delta_n(h)}{nhf(x_0)}\left\{2+\frac{\Delta_n(h)}{nhf(x_0)}\right\}$$

and by Lemma 11

$$\sup_{h\in J_n(c,d)} |R'_{n,h}|=O(n^{-4/5}\log n) \qquad \text{a. s.}$$

To examine the other term, let

$$U_{n,i}=I(Z^*_{n,i}>\theta)-(1-p), \qquad m_n(h)=[nhf(x_0)].$$

Then

$$\frac{1}{K_n(h)g(\theta)}\sum_{i=1}^{K_n(h)} U_{n,i}=\frac{1}{m_n(h)g(\theta)}\sum_{i=1}^{m_n(h)} U_{n,i}+R''_{n,h}+R'''_{n,h},$$

$$\text{(4.10)} \qquad R''_{n,h}=-\frac{\Delta_n(h)}{K_n(h)}\frac{1}{m_n(h)g(\theta)}\sum_{i=1}^{m_n(h)} U_{n,i},$$

$$R'''_{n,h}=\frac{1-\Delta_n(h)}{K_n(h)}\frac{1}{m_n(h)g(\theta)}\left[\sum_{i=1}^{K_n(h)} U_{n,i}-\sum_{1}^{m_n(h)} U_{n,i}\right]$$

where $\{U_{n,1}, \cdots, U_{n,n}\}$ satisfies the conditional ϕ-mixing condition given $\mathcal{M}_{-\infty}^{\infty}(X)$, with

$$E(U_{n,i}\mid \mathcal{M}_{-\infty}^{\infty}(X))=0.$$

Since the analogous assertion to Lemma A (which, in later, will be called Lemma B) is easily proved for partial sums of the conditional ϕ-mixing sequence, from Lemma 11 we obtain

$$\text{(4.11)} \qquad \sup_{h\in J_n(c,d)} |R''_{n,h}|=O(n^{-3/5}\log n) \qquad \text{a. s.}$$

Next, let $h_{n,0}<h_{n,1}<\cdots<h_{n,\nu_n}$ denote the jump points of $m_n(h)=[nhf(x_0)]$ in $J_n(c, d)$. Since for each j and for all $h_{n,j}\leqq h<h_{n,j+1}$,

$$\left|\sum_{i=1}^{K_n(h)} U_{n,i}-\sum_{i=1}^{m_n(h)} U_{n,i}\right|$$

$$\leqq\left|\sum_{i=1}^{K_n(h_{n,j})} U_{n,i}-\sum_{i=1}^{m_n(h_{n,j})} U_{n,i}\right|+\{K_n(h_{n,j+1})-K_n(h_{n,j})\}$$

we only need to verify

$$\max_{0\leq j\leq \nu_n}\left|\sum_{i=1}^{K_n(h_{n,j})} U_{n,i}-\sum_{i=1}^{m_n(h_{n,j})} U_{n,i}\right|=O(n^{1/5}\log n) \quad \text{a.s.} \tag{4.12}$$

and

$$\max_{0\leq j\leq \nu_n}\{K_n(h_{n,j+1})-K_n(h_{n,j})\}=O(n^{1/5}\log n) \quad \text{a.s.} \tag{4.13}$$

in order to conclude in (4.10)

$$\sup_{h\in J_n(c,d)}|R'''_{n,h}|=O(n^{-3/5}\log n) \quad \text{a.s.}$$

We note here that

$$h_{n,j+1}-h_{n,j}\leqq \{nf(x_0)\}^{-1} \quad \text{and} \quad \nu_n\leqq n^{4/5}(d-c)f(x_0).$$

Hence, using the relations

$$\begin{aligned}K_n(h_{n,j+1})&-K_n(h_{n,j})\\ &=\frac{1}{n}\sum_{i=1}^{n}\left\{I\left(\frac{h_{n,j}}{2}<Y_i\leqq\frac{h_{n,j+1}}{2}\right)-EI\left(\frac{h_{n,j}}{2}<Y_i\leqq\frac{h_{n,j+1}}{2}\right)\right\}\\ &\quad+O(|h_{n,j+1}-h_{n,j}|)\end{aligned}$$

and for some C_4 (>0)

$$\begin{aligned}P(\max_{0\leq k\leq \nu_n}&|K_n(h_{n,j+1})-K_n(h_{n,j})|\geqq C_4n^{1/5}\log n)\\ &=\sum_{j=0}^{\nu_n}P(|K_n(h_{n,j+1})-K_n(h_{n,j})|\geqq C_4n^{1/5}\log n),\end{aligned}$$

from Lemmas B and 11 we have (4.13). Similarly, we can obtain (4.12) from Lemmas B and 11. Thus, we have the desired conclusion. □

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