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AUTOMORPHISMS ON THE JAMES SUM OF BANACH ALGEBRAS

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Introduction

S. F. Bellenot [2] defined and studied the James sum (in short, J-sum) of a sequence $\langle X_n\rangle$ of Banach spaces, as a generalization of the classical James space J [3], and gave concrete examples of the motivating James-Lindenstrauss result ([4], [8]) that every separable Banach space E can be realized as the quotient X^{**}/X for a suitable Banach space X with shrinking basis. A. D. Andrew and W. L. Green [1] studied J as a Banach algebra and characterized the automorphisms of J in terms of certain permutations in N. In an earlier paper [6] we defined and studied the J-sum $J(X_n)$ of an increasing sequence $\langle X_n\rangle$ of Banach algebras and their operator theory. The purpose of this paper is to study automorphisms on $J(X_n)$ and the associated algebra $J(X_n)^{LIM}$.

There are two basic types of automorphisms on $J(X_n)$: one which leave each X_n invariant, i.e. the so-called invariant automorphisms, and the other obtained by permuting X_n's amongst themselves whenever permissible. In the case of J, each $X_n=\boldsymbol{R}$ is one dimensional, and the only way to obtain automorphisms on J is through permutations of basis elements. Since the basis of J is conditional, not every permutation on N gives rise to an automorphism. The permutations which do give rise to an automorphism on J have been characterized in [1], Theorem 4.7. It is easy to see that if each $\boldsymbol{R}$ is replaced by a general Banach algebra X then the same characterization holds for such automorphisms of the algebra $J(X_n)$ with $X_n=X$ for each n, which may also be denoted by $J(X)$ in view of [10]. Further the same characterization can be readily modified to be applicable to this type of automorphisms of $J(X_n)$ when there is $r\in\boldsymbol{N}$ such that $X_n=X_r$ for all $n\geqq r$.

In this paper we study the invariant automorphisms on $J(X_n)$ and $J(X_n)^{LIM}$ with emphasis on the case when each X_n has a basis $\{e_j^n : j\in\Lambda_n\}$ $\Lambda_n\subset\Lambda_{n+1}$ and $e_j^n=e_j^{n+1}$ if $j\in\Lambda_n$. An operator T on $J(X_n)$ is called invariant if it leaves each X_n invariant. Invariant isometries on $J(X_n)$ and $J(X_n)^{LIM}$ were characterized

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in the more general setting of Banach spaces with certain compatibility conditions in [7]. The theory of invariant operators developed in [6] (see also [5]) can be used to advantage to study automorphisms on $J(X_n)$ and $J(X_n)^{LIM}$. In this sense this paper may well be regarded as continuation of [6] and we freely borrow the notation, terminology and results from there, e. g. for $\tilde{J}(Y_n)$ see § 2.4 where we confine ourselves to the case when $Y_n=B(X_n)$ for each n, see also ([9], [5], Remark I.6.3), for $|||\cdot|||$ and spaces like Inv $B(J(X_n)^{LIM})$, $J_T(X_n)$ and $\tilde{J}_T(Y_n)$ see § 2.5 and for operators β and θ see § 1.6. In a forthcoming paper we note that $J_T(X_n)$ is isomorphic to $c_0(X_n)$, the space of null sequences $\langle x_n\rangle$ with x_n in X_n for each n; however, $J_T(X_n)$ has much fewer isometries compared with $c_0(X_n)$ (Presented at Conference of Society of Mathematical Sciences at South Delhi Campus, Delhi, July 1992).

As in [6], the J-sum of an increasing sequence $\langle X_n\rangle$ of closed subalgebras of a Banach algebra Z with $X_1\neq\{0\}$ is defined as follows:

For $x=\langle x_n\rangle$, $x_n\in X_n$ for each n, and for $P=\{p_1, \cdots, p_k\}\in\mathcal{P}$ (here $\mathcal{P}$ is the set of finite increasing sequences of non-negative integers), let

$$\|x\|_P^2=\sum_{i=1}^{k-1}\|x_{p_i}-x_{p_{i+1}}\|^2+\|x_{p_k}\|^2$$

where, for notational convenience, we set $X_0=\{0\}$. Define

$$2\|x\|_J^2=\sup_{P\in\mathcal{P}}\|x\|_P^2 .$$

Then $\|\cdot\|_J$ is a norm on the space

$\phi(X_n)=\{x=\langle x_n\rangle:\ x_n\in X_n$ for each n, x_n is zero for all but finitely many n's$\}$

and its completion is called the J-sum of X_n's and is denoted by $J(X_n)$.

We assume that for each n and a, $b\in X_n$, $\|ab\|\leqq 1/2\|a\|\|b\|$, or introduce an equivalent norm in each X_n in the beginning, if the need be, and then $(J(X_n), \|\cdot\|_J)$ is a Banach algebra. The Banach algebra $(J(X_n)^{LIM}, \|\cdot\|_J)$ is defined as

$$J(X_n)^{LIM}=\{x=\langle x_n\rangle:\ x_n\in X_n \text{ for each } n,\ \|x\|_J<\infty\} .$$

Invariant Operators on $J(X_n)$ and $J(X_n)^{LIM}$

1. Let $\langle T_n\rangle$ be a sequence with $T_n\in B(X_n)=Y_n$ for each n.

(a) We say $\langle T_n\rangle$ is eventually compatible (cf. [7], p. 116; [2], p. 100) if there exists $m\in N$: $T_n|X_{n-1}=T_{n-1}$ for all $n\geqq m$. Obviously such a $\langle T_n\rangle$ with $\sup_n\|T_n\|<\infty$ is in $\tilde{J}(Y_n)^{LIM}$.

(b) We next consider $\langle T_n\rangle$ such that there is a family $\{S_\alpha:\alpha\in\Lambda\}$ of operators on Z such that for each n, $T_n=S_\alpha|X_n$ for some $\alpha\in\Lambda$. If Λ is finite say $\Lambda=\{1, 2, \cdots, l\}$ then $\langle T_n\rangle\in\tilde{J}(Y_n)^{LIM}$ if and only if $\langle T_n\rangle$ is eventually compatible. Sufficiency is obvious being only a special case of (a) above. To see the necessity we observe that eventual compatibility amounts to saying that eventually only one of $S_1, \cdots, S_l$ survives. Suppose, on the other hand, we assume that two of them, say S_1, S_2 occur infinitely often. Let $i \in \boldsymbol{N}$: $\|(S_1-S_2)|X_i\| = k>0$. We can choose sequences $\langle n_j \rangle$, $\langle n'_j \rangle$ in $\boldsymbol{N}$: $i \leqq n_1 < n'_1 < \cdots < n_j < n'_j < n_{j+1} < \cdots$, $T_{n_j} = S_1 | X_{n_j}$ and $T_{n'_j} = S_2 | X_{n'_j}$ for each j. Let, for each $r \in \boldsymbol{N}$ $P^{(r)} = \{n_1, n'_1, \cdots, n_r, n'_r\}$. Then $\|\langle T_n \rangle\tilde{\|}_{P^{(r)}} > \sqrt{r}\,k$. This shows that $\|\langle T_n \rangle\tilde{\|}$ is not finite.

On the other hand for any non-zero $S \in B(Z)$ which satisfies $SX_n \subset X_n$ for each n, let, for each n, $S_n = (1/n)S$ and $T_n = S_n | X_n$. Then $\langle T_n \rangle \in \tilde{J}(Y_n)^{LIM}$. This shows that eventual compatibility is not necessary for $\langle T_n \rangle$ to be in $\tilde{J}(Y_n)^{LIM}$ when Λ is infinite.

However, it is so when X_n's have bases satisfying certain conditions and T_n's are automorphisms as the following result shows:

2. **Proposition.** *Let, for each n, $\Lambda_n \subset \Lambda_{n+1} \subset \boldsymbol{N}$ and X_n have a symmetric and orthogonal basis $\{e^n_j : j \in \Lambda_n\}$, $e^n_j = e^{n+1}_j$ if $j \in \Lambda_n$, and let, $\Lambda_n = \boldsymbol{N}$ $\forall\, n \geqq r$ for some $r \in \boldsymbol{N}$. Suppose there exists K such that for each n and for each $j \in \Lambda_n$, $\|e^n_j\| \leqq K$. Let for each $n \geqq r$, $k_n = \inf_{i \neq j \in \boldsymbol{N}} \|e^n_i - e^n_j\|$. If $k = \liminf k_n > 0$, then $\langle T_n \rangle \in \tilde{J}(Y_n)^{LIM}$ if and only if $\langle T_n \rangle$ is eventually compatible.*

Proof. We need prove only necessity. We first note that the condition $\liminf k_n > 0$ is equivalent to the condition $\sum_{n \in M} k_n^2 = \infty$ for $M \subseteqq \boldsymbol{N}$, M infinite. If $\langle T_n \rangle$ is not eventually compatible, then there exist sequences $\langle n_j \rangle$, $\langle n'_j \rangle$ in $\boldsymbol{N}$: $r \leqq n_1 < n'_1 < \cdots < n_j < n'_j < n_{j+1} < \cdots$ and $T_{n'_j} | X_{n_j} \neq T_{n_j}$ $\forall\, j \in \boldsymbol{N}$. Since automorphisms take basis elements into basis elements, there exist for each j, indices i_j, $l_j \neq m_j$: $(T_{n'_j} - T_{n_j})(e^{n_j}_{i_j}) = e^{n_j}_{l_j} - e^{n_j}_{m_j}$. So $\|T_{n'_j} | X_{n_j} - T_{n_j}\|^2 \geqq \|e^{n_j}_{l_j} - e^{n_j}_{m_j}\| / \|e^{n_j}_{i_j}\| \geqq k/K$. Since $\|\langle T_n \rangle\tilde{\|}^2 \geqq \sup_{q \in \boldsymbol{N}} \sum_{j=1}^{q} \|T_{n'_j} - T_{n_j}\|^2$ we have $\|\langle T_n \rangle\tilde{\|} = \infty$.

3. **Remarks.** (a) A common illustration is provided by taking $X_n = l_2$ $\forall\, n$. In this case the basis is orthogonal, symmetric and normalized and $\|e^n_i - e^n_j\| = \sqrt{2}$ $\forall\, i \neq j \in \boldsymbol{N}$, $\forall\, n$.

(b) If $T = \beta(\langle T_n \rangle)$ is in $B(J(X_n))$ (respectively $B(J(X_n)^{LIM})$ and is onto $J(X_n)$ (respectively $J(X_n)^{LIM}$) then each T_n is onto.

(c) If $T = \beta(\langle T_n \rangle)$ is in $B(J(X_n))$ and has closed range then ontoness of each T_n implies that of T for the simple reason that $\phi(X_n)$ is dense in $J(X_n)$.

(d) If $T = \beta(\langle T_n \rangle)$ is in $B(J(X_n)^{LIM})$ and has closed range then ontoness of each T_n and eventual compatiblity of $\langle T_n \rangle$ implies that T is onto. For this we observe that the eventual compatiblity together with ontoness of each T_n implies that $T | \Omega(X_n)$ is onto $\Omega(X_n)$. Ontoness of T now follows from the density of

$\Omega(X_n)$ in $J(X_n)^{LIM}$ and the hypothesis that the range T is closed.

(e) The eventual compatibility in (d) above can in fact be replaced by the following weaker condition: for each $j\in N$ and each $x_j\in X_j$ there exists $m\in N$: $T_n^{-1}(x_j)=T_m^{-1}(x_j)$ for all $n\geqq m$ because this together with ontoness of each T_n also gives that $T|\Omega(X_n)$ is onto $\Omega(X_n)$. For this we consider some $x\in\Omega(X_n)$. Then there exists $j\in N$: $x_n=x_j$ for $n\geqq j$. Let $m\geqq j$ be such that $T_n^{-1}x_j=T_m^{-1}x_j=y_m$ say for $n\geqq m$. Also by ontoness of T_n's, for each $n<m$ there exists $y_n\in X_n$ such that $T_n y_n$ equals x_n for $n\leqq j$ and equals x_j for $j\leqq n<m$. Then $y=\langle y_n\rangle\in\Omega(X_n)$ and $\beta(\langle T_n\rangle)(y)=x$.

4. Proposition. *Let $\langle T_n\rangle$ be such that T_n's are invertible.*

(i) *Suppose that $\langle T_n^{-1}\rangle\in\tilde{J}(Y_n)^{LIM}$.*

(a) *If $T=\beta(\langle T_n\rangle)\in B(J(X_n)^{LIM})$, then $T(J(X_n)^{LIM})=J(X_n)^{LIM}$.*

(b) *If $T=\beta(\langle T_n\rangle)\in B(J(X_n))$, then $T(J(X_n))=J(X_n)$.*

(ii) *If $T=\beta(\langle T_n\rangle)\in B(J(X_n)^{LIM})$ and it is onto $J(X_n)^{LIM}$, then $\langle T_n^{-1}\rangle\in\tilde{J}_T^{LIM}(Y_n)$.*

(iii) *If $T=\beta(\langle T_n\rangle)\in B(J(X_n))$ and it is onto $J(X_n)$, then $\langle T_n^{-1}\rangle\in\tilde{J}_T^{LIM}(Y_n)$.*

Proof. (i) It follows from [6, Theorem 2.6 (i)] that $\beta(\langle T_n^{-1}\rangle)\in \mathrm{Inv}_J B(J(X_n)^{LIM})$. So in case (a) $T^{-1}=\beta(\langle T_n^{-1}\rangle)$ exists an operator on $J(X_n)^{LIM}$ to itself and thus T is onto $J(X_n)^{LIM}$. Similarly for the case (b).

(ii) By the open mapping theorem T is open and thus $T^{-1}=\beta(\langle T_n^{-1}\rangle)$ is bounded. We now apply [6, Theorem 2.6 (iii)].

(iii) It follows on the same lines as (ii) by noting that the map θ in [6, Theorem 2.6 (iii)] may also be taken on operators on $J(X_n)$ and then applying this theorem.

5. Proposition. *Let $\langle T_n\rangle$ be a sequence of operators, $T_n\in B(X_n)$ such that $T=\beta(\langle T_n\rangle)$ is in $B(J(X_n)^{LIM})$ or $B(J(X_n))$. T is a homomorphism, a Jordan map or a derivation if and only if each T_n is so.*

6. Theorem. *Let $\langle T_n\rangle$ be a sequence such that $T=\beta(\langle T_n\rangle)$ is in $B(J(X_n)^{LIM})$ (or $B(J(X_n))$).*

(i) *If each T_n is an automorphism and $\langle T_n^{-1}\rangle\in\tilde{J}(Y_n)^{LIM}$ then T is an automorphism.*

(ii) *If T is an automorphism then each T_n is an automorphism and $\langle T_n^{-1}\rangle\in\tilde{J}_T^{LIM}(Y_n)$.*

Proof. (i) By Proposition 4 (i) T is onto $J(X_n)^{LIM}$ (or $J(X_n)$). Further T is a homomorphism by Proposition 5. Thus T is an automorphism.

(ii) By Proposition 5 each T_n is a homomorphism. Since T is onto, by

Proposition 4 (ii) and (iii), $\langle T_n^{-1}\rangle \in \hat{J}_T^{LIM}(Y_n)$. Also by Remark 3 (b) each T_n is onto and so an automorphism.

7. Examples. Let for each n, X_n have a symmetric and orthogonal basis $\{e_i^n, i\in\Lambda_n, \Lambda_n\subset \boldsymbol{N}, \Lambda_n\subset\Lambda_{n+1}$ and $e_j^n=e_j^{n+1}$ if $j\in\Lambda_n\}$. Then a permutation σ on Λ_n gives rise to a permutation operator T_n which is an automorphism on X_n. T_n^{-1} is the automorphism on X_n generated by the permutation σ^{-1} on Λ_n.

(i) By Theorem 6 (i) if $\langle T_n^{-1}\rangle\in\tilde{J}(Y_n)^{LIM}$ then $T=\beta(\langle T_n\rangle)$ is an invariant automorphism on $J(X_n)^{LIM}$ (or $J(X_n)$). Further, if Λ_n is finite for each n, and $\sup\#\Lambda_n<\infty$, by § 1 (b), $\langle T_n\rangle\in\tilde{J}(Y_n)^{LIM}$ if and only if $\langle T_n\rangle$ is eventually compatible.

(ii) Let, for each n, $X_n=\boldsymbol{R}^2$. If $\langle T_n\rangle$ is a sequence of permutation operators on X_n's which is eventually compatible say after m^{th} term for some $m\in\boldsymbol{N}$, then $T=\beta(\langle T_n\rangle)$ is an invariant automorphism on $J(X_n)^{LIM}$ (or $J(X_n)$). The cardinality of these invariant automorphisms is $\sup_{m\in N} 2^m=\aleph_0$, the least infinite cardinal number.

(iii) The cardinality of invariant automorphisms which come from eventually compatible sequences of permutation operators on X_n's is $\aleph_0$ if $\sup\#\Lambda_n<\infty$ and is $\aleph$ (the cardinality of the continuum) if there exists $r\in\boldsymbol{N}$: $\Lambda_n=\boldsymbol{N}$ ($\forall\, n\geqq r$).

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