YOKOHAMA MATHEMATICAL
JOURNAL VOL. 39, 1992

ROTATIONAL HYPERSURFACES IN S^n AND H^n WITH CONSTANT SCALAR CURVATURE

By

HIROSHI MORI

(Received June 11, 1991)

1. Introduction

In a recent work [4], M. L. Leite classified all complete, rotational, (spherical) hypersurfaces in $\boldsymbol{R}^n$ and H^n with constant scalar curvature. She also presented partial results for S^n.

In this paper we shall classify all complete, rotational, (hyperbolic and parabolic) hypersurfaces in H^n with constant scalar curvature with which M. L. Leite does not treat. In particular, we shall exhibit a collection of new complete hypersurfaces in H^n with S ranging in the closed interval $[-1, 0]$. And we shall accomplish Leite's result on classification of complete, rotational hypersurfaces in S^n with constant scalar curvature S with the exception of that $n\geqq 11$ and $S\neq$ one number $(>n/(n-1))$.

We refer the readers to Section 2 and [7] for the terminology.

Theorem 1 (*Classification of hyperbolic, rotational hypersurfaces in* H^n).

(i) *There is no complete, rotational hypersurface with constant scalar curvature* S, *for* $S<-1$, *or* $S>0$.

(ii) *Up to isometry in* H^n, *the complete, rotational hypersurfaces with constant scalar curvature* $S\in[-1, 0)$ *form a one-parameter family of examples.*

(iii) *There exists a one-parameter family of complete, rotational hypersurfaces with scalar curvature zero, any of which is the product of a circle and* $\boldsymbol{R}$ (*resp. an* $(n-2)$*-dimensional hyperbolic space with constant sectional curvature*) *provided* $n=3$ (*resp.* $n\geqq 4$), *given in the Corollary of Proposition* 3.1 *below.*

Theorem 2 (*Classification of parabolic, rotational hypersurfaces in* H^n).

(i) *There is no complete, rotational hypersurface with constant scalar curvature* S, *for* $S<-1$, *or* $S>0$.

(ii) *Up to isometry in* H^n, *the complete, rotational hypersurfaces with constant scalar curvature* $S\in[-1, 0)$ *form a one-parameter family of examples.*

(iii) *There is a one-parameter family of complete, rotational hypersurfaces with constant scalar curvature* 0, *any of which is a horosphere, given in the*

1990 *Mathematics Subject Classification.* Primary 53A10; Secondary 53C42.
Key words and phrases. rotational hypersurface, constant scalar curvature.

Corollary of Proposition 3.1 *below.*

Theorem 3 (*Classification of rotational hypersurfaces in* S^n).

(i) *Up to isometry in* S^n, *there exists a one-parameter family of complete, immersed, rotational hypersurfaces in* S^n *with constant scalar curvature* $S \in ((n-3)/(n-1), 1)$, *converging to the embedded cylinder of Corollary* 2.3 *in* [4]. *There exists an infinite countable subfamily of the family consisting of compact hypersurfaces, which contains an embedded hypersurface provided* $S \in ((n-2)/(n-1), 1)$.

(ii) *For* $S \geqq 1$, *there exists a one-parameter family of complete, immersed, rotational hypersurfaces in* S^n *with constant scalar curvature* S, *converging on one side to the cylinder of Corollary* 2.3 *in* [4]. *An infinite countable subfamily consisting of compact hypersurfaces which converges to a sequence of isometrically embedded spheres of radius* $1/\sqrt{S}$, *with exception of that* $n \geqq 11$ *and* $S \neq$ *one number*$(>n/(n-1))$.

(iii) *There are no complete hypersurfaces in* S^n *with constant scalar curvature* $S \leqq (n-3)/(n-1)$.

2. Preliminaries

We shall denote by L^{n+1} the vector space of $(n+1)$-tuples $x=(x_1, \cdots, x_{n+1})$ with the Lorentzian metric $\langle x, y\rangle = -x_1y_1 + x_2y_2 + \cdots + x_{n+1}y_{n+1}$, where $y=(y_1, \cdots, y_{n+1})$, and shall consider the hyperbolic n-space $H^n(c)$ with constant sectional curvature c, $c<0$, as a hypersurface of L^{n+1}, namely,

$$H^n(c) = \{x \in L^{n+1}; \langle x, x\rangle = 1/c,\ x_1 > 0\}.$$

We shall set $H^n = H^n(-1)$ for simplicity. An orthogonal transformation of L^{n+1} is a linear map which preserves the Lorentzian metric $\langle\ ,\ \rangle$; the orthogonal transformations induce, by restriction, all the isometries of H^n. We shall denote by P^k a k-dimensional linear subspace of L^{n+1} and by $O(P^k)$ the set of orthogonal transformations of L^{n+1} with positive determinant which leave P^k pointwise fixed. We shall say that P^k is Lorentzian (resp. Riemannian, resp. degenerate) if the restriction $\langle\ ,\ \rangle|_{P^k}$ is a Lorentzian metric (resp. Riemannian metric, resp. a degenerate quadratic form).

Definition 2.1. Choose P^2 and $P^3 \supset P^2$, and let C be a regular C^2-curve in $P^3 \cap H^n$ which does not meet P^2. The orbit of C under the action of $O(P^2)$ is said to be a rotational, spherical (resp. hyperbolic, resp. parabolic) hypersurface in H^n if P^2 is Lorentzian (resp. Riemannian, resp. degenerate).

We shall write down the parametrization of rotational hypersurfaces explicitly. It is easily shown that we can choose a basis e_k of L^{n+1} satisfying the following conditions:

(1) P^2 is the plane generated by e_n and e_{n+1};

(2) P^3 is the 3-subspace generated by e_1 and P^2;

(3) for two vectors $x=\sum x_i e_i$ and $y=\sum y_i e_i$, we have that the inner product $\langle x, y\rangle$ is equal to

$$x_1y_1+\cdots+x_ny_n-x_{n+1}y_{n+1} \qquad \text{(spherical case)},$$

$$-x_1y_1+x_2y_2+\cdots+x_{n+1}y_{n+1} \qquad \text{(hyperbolic case)},$$

$$-x_1y_n+x_2y_2+\cdots+x_{n-1}y_{n-1}-x_ny_1+x_{n+1}y_{n+1} \qquad \text{(parabolic case)}.$$

When $\langle\ ,\ \rangle|_{P^2}$ is a nondegenerate (resp. degenerate) quadratic form, let $x_1=x(s)$, $x_n=y(s)$ (resp. $x_1=y(s)$, $x_n=x(s)$) and $x_{n+1}=z(s)$, $s\in J$ be an equation of the curve C which is parametrized by arc length and whose domain of definition J is an open interval of $\boldsymbol{R}$ containing zero. Using the profile curve C there exists a C^2-mapping f from the product space $J\times S^{n-2}$ (resp. $J\times H^{n-2}$, resp. $J\times R^{n-2}$) into H^n whose local parametrization is as follows:

$$f(s,\theta_1,\cdots,\theta_{n-2})=x(s)\sin\theta_1 e_1+x(s)\cos\theta_1 \times\Theta(\theta_2,\cdots,\theta_{n-2})+y(s)e_n+z(s)e_{n+1},$$
$$s\in J,\ -\pi/2<\theta_1<\pi/2, \qquad \text{(spherical case)}, \tag{2.1}$$

$$f(s,\theta_1,\cdots,\theta_{n-2})=x(s)\cosh\theta_1 e_1+x(s)\sinh\theta_1 \times\Theta(\theta_2,\cdots\cdots\theta_{n-2})+y(s)e_n+z(s)e_{n+1},$$
$$s\in J,\ 0<\theta_1<\infty, \qquad \text{(hyperbolic case)}, \tag{2.2}$$

$$f(s,\theta_1,\cdots,\theta_{n-2})=\left[-y(s)+\frac{1}{2}\theta_1^2x(s)\right]e_1+x(s)\theta_1 \times\Theta(\theta_2,\cdots,\theta_{n-2})+x(s)e_n+z(s)e_{n+1},$$
$$s\in J,\ 0<\theta_1<\infty, \qquad \text{(parabolic case)}, \tag{2.3}$$

where $\Theta(\theta_2,\cdots,\theta_{n-2})$ is a local parametrization of the unit $(n-3)$-sphere S^{n-3} in the Euclidean $(n-2)$-space generated by the vectors $e_2,\cdots,e_{n-1}$:

$$\Theta(\theta_2,\cdots,\theta_{n-2})=\sin\theta_2e_2+\cos\theta_2\sin\theta_3e_3+\cdots+\cos\theta_2\cdots \times\cos\theta_{n-3}\sin\theta_{n-2}e_{n-2}+\cos\theta_2\cdots\cos\theta_{n-2}e_{n-1},$$

where $-\pi/2<\theta_i<\pi/2$ $(i=2,\cdots,n-3)$, $-\pi<\theta_{n-2}<\pi$ (cf. [3], [5]).

Remark. When $n=3$, the term $\Theta(e_2,\cdots,e_{n-2})$ in (2.1)-(2.3) is replaced by e_2; and the range of θ_1 in (2.2) and (2.3) can be replaced by the one "$0<|\theta_1|<\infty$".

We see, with respect to the parametrization that the first fundamental form of the C^2-mapping f is

$$I = ds^2 + x(s)^2 d\theta_1^2 + x(s)^2 \cos^2\theta_1 d\Theta^2 \tag{2.4}$$

$$[\text{resp.}\quad ds^2 + x(s)^2 d\theta_1^2 + x(s)^2 \theta_1^2 d\Theta^2,$$

$$\text{resp.}\quad ds^2 + x(s)^2 d\theta_1^2 + x(s)^2 \sinh^2\theta_1 d\Theta^2],$$

in spherical (resp. parabolic, resp. hyperbolic) case, where $d\Theta^2$ is the canonical Riemannian metric of the unit $(n-3)$-spere S^{n-3}.

From (2.1)-(2.4) it follows that the mapping f is an immersion if and only if the following condition is satisfied on the interval J,

$$x(s) > 0, \quad \text{(spherical and parabolic cases)}. \tag{2.5}$$

$$x(s) \geqq 1, \quad \text{(hyperbolic case)}.$$

It will sometimes be convenient to use the notation M_δ, $\delta = 1$, 0 or -1, to denote a rotational hypersurface in H^n, where $\delta = 1$ (resp. $\delta = 0$, resp. $\delta = -1$) means M_δ is a spherical (resp. parabolic, resp. hyperbolic) hypersurface. The following result is obtained easily (cf. [3]).

Unless otherwise stated, all manifolds are connected and, we are in the C^∞ category.

Proposition 2.1. *Let M_δ be a rotational hypersurface in H^n defined by the immersion f. Assume that $\delta + x(s)^2 - x'(s)^2 > 0$ on J. Then the tangential directions of the parameters $\theta_1, \cdots, \theta_{n-2}$ and s are principal directions; the principal curvatures along the coordinates curves θ_i are all equal and given by*

$$\lambda = \sqrt{\delta + x^2 - x'^2}/x,$$

and the principal curvature along the coordinate curve s is

$$\mu = -(x'' - x)/\sqrt{\delta + x^2 - x'^2}.$$

3. Rotational hypersurfaces in H^n with constant scalar curvature

From Proposition 2.1 and (2.5) it can be shown (cf. [3], [5]) under the assumption

$$\delta + x(s)^2 - x'(s)^2 > 0 \quad \text{on } J, \tag{3.1}$$

that the mapping f is of constant scalar curvature S if and only if, on the interval J, the following relations hold:

$$2xx'' - (n-3)(\delta - x'^2) + (n-1)Sx^2 = 0; \tag{3.2}$$

(3.3) $$y=(x^2+1)^{1/2}\sinh\varphi(s),\qquad z=(x^2+1)^{1/2}\cosh\varphi(s),$$
$$\varphi(s)=\int_0^s(1+x^2-x'^2)^{1/2}(x^2+1)^{-1}d\sigma,\quad\text{and}\quad x>0$$
(spherical case),

(3.4) $$y=(x^2-1)^{1/2}\sin\varphi(s),\qquad z=(x^2-1)^{1/2}\cos\varphi(s),$$
$$\varphi(s)=\int_0^s(-1+x^2-x'^2)^{1/2}(x^2-1)^{-1}d\sigma,\quad\text{and}\quad x>1$$
(hyperbolic case),

(3.5) $$y=-(z^2+1)/2x,\qquad z=x\int_0^s(x^2-x'^2)^{1/2}x^{-2}d\sigma,$$
and $x>0$ (parabolic case).

Remark. If the condition (3.1) breaks down (i.e. the condition is replaced by the following

(3.1)′ $$\delta+x(s)^2-x'(s)^2\geqq 0\quad\text{on } J),$$

we cannot use the formulae (3.3)-(3.5) directly. But the condition that our hypersurfaces are rotational guarantees the existence of profile curve C in such an extreme case. We shall, in detail, explain it in the proof of Theorem 1.

Leite studied the rotational, spherical hypersurfaces in H^n as well as for those ones in S^n and $\boldsymbol{R}^n$.

In what follows, we consider only rotational hyperbolic and parabolic hypersurfaces in H^n. Multiplying by $x'x^{n-4}$ on the both sides of (3.2) and then integrating we have the following.

Proposition 3.1. *Equation* (3.2) *is equivalent to the following first order DE*

(3.6) $$x^{n-3}(-\delta+Sx^2+x'^2)=K,$$

where K is a constant; moreover, for a constant solution, which we may put as $(a^2-\delta)^{1/2}$ (a : positive constant), we have

$$S=(n-3)\delta[(a^2-\delta)(n-1)]^{-1},$$
$$K=-2\delta[(a^2-\delta)^{(n-3)/2}](n-1)^{-1}.$$

Corollary. *The hyperfurface in H^n corresponding to the constant solution in Proposition* 3.1 *is, for a positive constant a, the product $H^{n-2}(-1/(a^2+1))\times S^1(a)$ of a circle and a hyperbolic $(n-2)$-space with constant sectional curvature (resp. a horosphere) provided $\delta=-1$ (resp. $\delta=0$).*

4. The existence theorem of ODE (3.6)

Equation (3.6) tells us a local solution $x(s)$ of (3.1) paired with its first derivative is a subset, denoted by (x, x'), of a level curve for the function $H(u, v)$ defined by

$$H(u, v)\equiv u^{n-3}(-\delta+Su^2+v^2)=K, \tag{4.1}$$

where K is a constant.

Definition 4.1. We say that a solution $x\geqq-\delta$ of (3.6) is complete if x is defined for all s in $\boldsymbol{R}$ and satisfies the following condition:

$$\delta+x(s)^2-x'(s)^2\geqq 0, \qquad s\in\boldsymbol{R}. \tag{3.1$''$}$$

Lemma 4.1. *Let S be a negative constant, n an integer, $n\geqq 4$ and, $\delta=-1$ or 0. Then, there exists a unique C^∞-function $u=u(t, K)$ defined on $\boldsymbol{R}\times(-\infty, 0)$ which is a solution of the following ordinary differential equation*

$$(du/dt)^2=\delta-Su^2+Ku^{3-n}\equiv\varphi(u, K) \tag{4.2}$$

with the initial condition $\varphi(u(0, K), K)=0$, where K is regarded as parameter.

Proof. First, recall that the following sublemma of local existence and uniqueness for a normal ODE (see [6]).

Sublemma. Let D (resp. D') be an open subset of $\boldsymbol{R}^n$ (resp. $\boldsymbol{R}^m$), and I an open interval of $\boldsymbol{R}$. Denote by f a C^∞-mapping from the product space $D\times I\times D'$ into $\boldsymbol{R}^n$. Then, for a given point x_0 in D and a given compact subset K' of D', there exist an open subinterval $I_0=(-\varepsilon, \varepsilon)$ of I and a unique C^∞-mapping $x(t, \alpha)$ from the product $I_0\times\mathrm{Int}(K')$ into $\boldsymbol{R}^n$ such that for fixed α in K and λ in $\mathrm{Int}(K')$

$$\frac{d}{dt}x(t, \alpha)=f(x(t, \alpha), t, \lambda), \qquad t\in I_0,$$

$$x(0, \alpha)=x_0,$$

where $\mathrm{Int}(K')$ is the interior of the set K'.

We shall now proceed to prove Lemma 4.1. Consider the function $\varphi(u, K)$ gvien in Lemma 4.1. It can be easily seen that for a fixed $K<0$ there exists a unique number $u^*=u^*(n, S, K, \delta)>-\delta$ such that

$$\varphi(u, K)>0 \quad \text{for} \quad u>u^*, \qquad \varphi(u^*, K)=0.$$

It then follows from sublemma that for each $u_0>u^*$ and each $K_0<0$, there exist positive numbers ε and η, and a C^{∞}-function $v(t, K)$ on $(-\varepsilon, \varepsilon)\times(K_0-\eta, K_0+\eta)$ such that

$$\frac{d}{dt}v(t, K)=\sqrt{\varphi(v(t, K), K)}, \qquad |t|<\varepsilon, \quad |K-K_0|<\eta, \tag{4.3}$$

$$v(0, K)=u_0, \qquad |K-K_0|<\eta. \tag{4.4}$$

Here and in what follows, K is regarded as a parameter, unless otherwise stated. Note that an open set (u^*, ∞) of $\boldsymbol{R}$ is a Lindelöf space. Using this fact and applying sublemma to the ODE (4.2) to glue such local solutions. From this method we can show that there exists a unique C^{∞}-function $u(t, K)$ defined on $(0, \infty)\times(-\infty, 0)$ which satisfies, for each fixed $K<0$, that

$$\frac{d}{dt}u(t, K)=\sqrt{\varphi(u(t, K), K)}, \qquad t>0, \tag{4.5}$$

$$u(+0, K)=u^*, \qquad u(+\infty, K)=+\infty. \tag{4.6}$$

We extend the function $u(t, K)$ to a C^{∞}-function defined on $(\boldsymbol{R}\setminus\{0\})\times(-\infty, 0)$, by

$$u(t, K)=u(-t, K), \qquad t, K<0.$$

Then we have, for each fixed $K<0$, that

$$\frac{d}{dt}u(t, K)=\eta\sqrt{\varphi(u(t, K), K)}, \qquad \eta=\operatorname{sign} t, \tag{4.7}$$

$t\in\boldsymbol{R}\setminus\{0\}$ and that $u(t, K)\to u^*$ (resp. $+\infty$) as $t\to 0$ (resp. $|t|\to+\infty$). From this together with (4.7) it follows that $(d/dt)u(t, K)\to 0$ as $t\to 0$.

If we define $u(0, K)=u^*$, $(d/dt)u(0, K)=0$, we see that $u(t, K)$ is a C^2-function on $\boldsymbol{R}\times(-\infty, 0)$. Differentiating (4.7) on the both sides we have, for each fixed $K<0$, that

$$\frac{d^2}{dt^2}u(t, K)=-Su(t, K)+\frac{(3-n)K}{2}u(t, K)^{2-n}. \tag{4.8}$$

It follows from (4.8) that $d^2u(t, K)/dt^2$ may be extended to a C^2-function on $\boldsymbol{R}\times(-\infty, 0)$, which implies in turn that $u(t, K)$ may be regarded as C^4-function on $\boldsymbol{R}\times(-\infty, 0)$. Repeating this argument we see that $u(t, K)$ may, in fact, be a C^{∞}-function on $\boldsymbol{R}\times(-\infty, 0)$. This completes the proof.

By using Lemma 3.2 in [4] and Lemma 7.2 in [2] together with Lemma 4.1, we can show the (global) existence theorem of (3.6) and the completeness of our rotational hypersurfaces.

Lemma 4.2. *When $S>0$ or $S<-1$, there exist no solutions of* (3.6) *or a solution of* (3.6) (*if there exists*) *cannot be extended to a complete one. When $S=0$ and $n=3$* (*resp. $n\geqq 4$, $\delta=-1$, resp. $n\geqq 4$, $\delta=0$*), *a complete solution of* (3.6) *is the constant one provided $K=-\delta$* (*resp. $K>-\delta$, resp. $K=-\delta$*). *When $-1\leqq$*

$S<0$ and $n\geqq 4$, a solution of (3.6) *can* (*resp. cannot*) *be extended to a complete one provided $K\leqq 0$* (*resp. $K>0$*). *When $-1\leqq S<0$ and $n=3$, a solution of* (3.6) *can* (*resp. cannot*) *be extended to a complete one provided $K\leqq-\delta(1+S)$* (*resp. $K>-\delta(1+S)$*).

Lemma 4.3. *Suppose that the profile curve C in Section 2 is C^∞ and is defined on $\boldsymbol{R}$. If the function $x(t)$ satisfies for all $t\in\boldsymbol{R}$, that*

$$x(t)>0, \qquad (\textit{parabolic and spherical cases}),$$

$$x(t)\geqq 1, \qquad (\textit{hyperbolic case}),$$

then the hypersurface in H^n given by the immersion f is complete.

5. Proof of Theorem 1 and 2

We shall only prove Theorem 1 because the proof of Theorem 2 is similar. It is clear that the assertion (i) is true in virtue of Lemma 4.2. We shall prove the assertion (ii) in case $n\geqq 4$, the case $n=3$ is left to the readers.

The level curve $H(u, v)=K$ reduces the following form

$$v^2=-1-Su^2+Ku^{3-n}. \tag{5.8}$$

From Lemma 4.2 it suffices to consider the case where $K\leqq 0$. We shall first consider the subcase $K=0$. Putting $a=\sqrt{-S}$, we see that a complete solution $u=x(s)$ of (3.6) may be defined, up to translation in parameter s, by

$$x(s)=\frac{1}{a}\cosh(as), \qquad s\in\boldsymbol{R}. \tag{5.9}$$

We see that if $-1<S<0$ (i.e., $0<a<1$), then the function $x(s)$ satisfies the condition (3.1). It then follows that the funtions $y(s)$, $z(s)$ may be defined by

$$y(s)=\sqrt{x(s)^2-1}\cos\theta(s), \qquad z(s)=\sqrt{x(s)^2-1}\sin\theta(s),$$

where

$$\theta(s)=\mathrm{Tan}^{-1}[\sinh(as)/\sqrt{1-a^2}], \qquad s\in\boldsymbol{R}.$$

From this together with Lemma 4.3 it can be shown that for each fixed S, $-1<S<0$, there exists, up to isometry leaving the $x_1, \cdots, x_{n-1}$-plane in L^{n+1} fixed, a complete, rotational hypersurface $M(S, 0)$ in H^n with constant scalar curvature S, $-1<S<0$.

We shall next consider the subcase $S=-1$. Note that the condition (3.1) with $\delta=-1$ breaks down for the function $x(s)$ in (5.9) with $a=1$. If $S=-1$ (i.e., $a=1$), it follows that the functions $y(s)$, $z(s)$ which satisfy (2.6) are given, up to isometry leaving the $x_1, \cdots, x_{n-1}$-subspace in L^{n+1} fixed, by

$$x(s)=\cosh s, \qquad y(s)=\sinh s, \qquad z(s)=0.$$

Thus, the complete rotational hypersurface in H^n corresponding to the profile curve $\alpha(s)=(\cosh s)e_1+(\sinh s)e_n$ is the totally geodesic one $H^{n-1}=\{x\in H^n;\ x_{n+1}=0\}$.

Finally, we consider the subcase $K<0$ and $-1\leqq S<0$. In that case, we see that for each fixed $K<0$ the function $x(s)=u(s, K)$, $s\in\boldsymbol{R}$, given in Lemma 4.1 with $\delta=-1$, satisfies the condition (3.1) with $\delta=-1$. So we can define the functions $y(s)$, $z(s)$ and $\varphi(s)$ by (3.4) and they are C^∞.

Thus, it follows from Lemma 4.3 that there exists a one-parameter family of complete, rotational hypersurfaces $M(S, K)$ in H^n with constant scalar curvature S $(-\infty<K<0)$. This completes the proof of (ii). The assertion (iii) is proved by the similar argument. This completes the proof of Theorem 1.

6. Proof of Theorem 3

We shall briefly review the representation of rotational hypersurfaces in S^n. We fix the rectangular coordinates of $\boldsymbol{R}^{n+1}$ in which S^n is realized as the unit hypersphere. A rotational hypersurface M in S^n is, up to isometry of S^n, defined by the immersion $f: J\times S^{n-2}\to S^n$

$$f(s, u_1, \cdots, u_{n-1})=(x(s)u_1, \cdots, x(s)u_{n-1}, y(s), z(s)), \tag{6.1}$$

where J is an open interval in $\boldsymbol{R}$ containing the zero, and $\sum_{j=1}^{n-1}u_j^2=1$. We may assume that

$$x'(s)^2+y'(s)^2+z'(s)^2\equiv 1. \tag{6.2}$$

As in Section 2 we get, through a local parametrization of S^{n-2}, the fundamental form is (2.4), provided L^{n+1} with the Lorentzian metric is replaced by $\boldsymbol{R}^{n+1}$ with the Euclidean metric. It then follows from this observation that the mapping f is an immersion if and only if the following conditin is satisfied on the interval J:

$$x(s)>0. \tag{6.3}$$

Since $x(s)^2+y(s)^2+z(s)^2\equiv 1$ we may put

$$y=(1-x(s)^2)^{1/2}\cos\theta(s), \qquad z=(1-x(s)^2)^{1/2}\sin\theta(s). \tag{6.4}$$

From the equations of Gauss and Codazzi together with (6.1) and (6.2) we can also have Proposition 2.2 and Corollary 2.3 in [4].

We notice that our functions $x(s)$ and $\theta(s)$ can be identical with the functions $f(s)\equiv\sin r(s)$ and $h(s)$ in Leite's paper respectively (see the proof of Theorem 3.6 in [4]). We can show the following Lemma (cf. Lemma 4.1).

Lemma 6.1. *Let $n\geqq 4$ and $S>(n-3)/(n-1)$, and set $c(S)=\max\{1-S, 0\}$ and $K_0=(2/(n-1))\{(n-3)/(n-1)S\}^{(n-3)/2}$. For each K, $c(S)<K<K_0$, there exist a unique C^∞-function $x=x(s, K)$ defined on $\boldsymbol{R}\times(c(S), K_0)$, and a constant $l=l(K)$ satisfying that*

$$(\partial x/\partial s)^2=1-Sx^2-Kx^{3-n},$$

and for each fixed K, $c(S)<K<K_0$, $x(s, K)$ is an even and periodic function of s with period $2l$, and attains the positive minimum (resp. maximum) at $s=0$ (resp. $s=l$).

Lemma 6.2. *Let n ($\geqq 3$) be an integer and define the function $\varphi(x)$ given in $[1, \infty)$ by*

$$\varphi(x)=\{(n-1)x-(n-3)\}^{-1/2}-\frac{2}{\pi}\mathrm{Tan}^{-1}\{(x-1)^{-1/2}\}.$$

Then the function $\varphi(x)$ has the following properties.

If $3\leqq n\leqq 10$, then $0<\varphi(x)<1/\sqrt{2}$ for all $x\geqq 1$.

If $11\leqq n$, then $0<\varphi(x)<1/\sqrt{2}$ for $1\leqq x<c_n$, $-1/2<\varphi(x)<0$, for $c_n<x$, and $\varphi(x)=0$ for $x=c_n$, where $n/(n-1)<c_n<\infty$.

Proof. It is clear that the sign of the derivative $\varphi'(x)\,(x>1)$ is equal to the one of the function

$$h(x)=\left(x-\frac{n-3}{n-1}\right)^3-\frac{\pi^2}{n-1}x^2(x-1), \qquad x>1. \tag{6.5}$$

We shall next consider the case $11\leqq n$ only, the case $3\leqq n\leqq 10$ is left to the readers. In this case we see that the sign of the coefficients of x^3 on the right hand side of (6.5) is positive and that $h(0)<0$, $h(1)>0$. It can be easily shown that $h(n/(n-1))<0$ and that $h(x)\to\infty$ as $x\to\infty$. From this observation it follows that there exist constants c, d, $1<c<n/(n-1)<d<\infty$ such that $h(x)>0$ (resp. $h(x)<0$) for $1\leqq x<c$ or $x>d$ (resp. for $c<x<d$). This implies that $\varphi'(x)>0$ (resp. $\varphi'(x)<0$) for $1\leqq x<c$ or $x>d$ (resp. for $c<x<d$). Note that $\varphi(x)\to 0$ as $x\to\infty$, and that $\varphi(x)>-(1/\pi)\times\mathrm{Tan}^{-1}\{1/\sqrt{x-1}\}>-1/2$ for all $x>1$. Combining these facts, we see that the assertion of this lemma is true for $n\geqq 11$. This completes the proof of Lemma 6.2.

Suppose that the profile curve C, $\alpha(t)=(x(t), 0, \cdots, y(t), z(t))$, in S^n is extendable to a C^∞-curve defined on $\boldsymbol{R}$. Consider, for a positive constant l, the following systems of conditions.

(i) $\quad x(t)\equiv -x(-t)\equiv x(t+2l), \qquad x'(0)=1, \qquad x'(l)=-1,$

$\quad 0<x(t) \quad$ for t, $\quad 0<t<l$;

(ii) $\quad x(t)\equiv x(t+2l), \qquad x(t)\geqq x(0)>0 \quad$ for all t;

(iii) $x(t) \equiv$ constant in $(0, 1]$.

Now we can show the following lemma by using Lemma 9.114 in [1] and Lemma 7.2 in [2].

Lemma 6.3. *Assume that the profile curve C, $\alpha(t)=(x(t), 0, \cdots, y(t), z(t))$, is extendable to a C^∞-curve defined on $\boldsymbol{R}$. If the function $x(t)$ satisfies one of the systems* (ii), (iii) (*resp. the system* (i)), *then the hypersurface in S^n given by the immersion f is complete* (*resp. extends to a complete hypersurface in S^n*).

We shall now prove Theorem 3. It is clear that the assertions (i) and (iii) are true (see [4], pp. 300-303). We shall prove the assertion (ii). It can be shown that if $S>(n-3)/(n-1)$ and $c(S)<K<K_0$, then the function $x=x(s, K)$ given in Lemma 6.1 satisfies that

$$1-x^2-(\partial x/\partial s)^2>0 \tag{6.5}$$

for all s in $\boldsymbol{R}$. From this observation we can define the function $\theta(s, K)$ by

$$\theta(s, K)=\int_0^s \{1-x(\sigma, K)^2-(\partial x(\sigma, K)/\partial\sigma)^2\}^{1/2}(1-x(\sigma, K)^2)^{-1}ds.$$

Thus, it follows from Lemma 6.3 that there exists for each fixed S, $S>(n-3)/(n-1)$, a one-parameter family of complete, rotational hypersurfaces $M(S, K)$ $(c(S)<K<K_0)$ in S^n with constant scalar curvature S.

We shall now discuss the compactness of our hypersurfaces $M(S, K)$ is S^n. Putting $P(K)=\theta(l(K), K)$, we have that $P(K)$ is a continuous function of K, $c(S)<K<K_0$. And the following properties hold (cf. [4], pp. 301-303):

$$P(K) \longrightarrow 2\pi/\sqrt{(n-1)S-(n-3)} \quad \text{as} \quad K \uparrow K_0, \tag{6.6}$$

when $S>(n-3)/(n-1)$;

$$P(K) \longrightarrow 2\,\mathrm{Tan}^{-1} 1/\sqrt{S-1} \quad \text{as} \quad K \downarrow 0, \tag{6.7}$$

when $S \geqq 1$.

On the other hand, we see that a rotational hypersurface defined by the immersion f is compact if and only if the profile curve $\alpha(s)=(x(s), 0, \cdots, 0, y(s), z(s))$ is a closed curve, which is, in turn, equivalent to the value $P(K)$ satisfies that

$$P(K)=2\pi r,$$

where r is a positive rational number.

Using this observation and Lemma 6.3 together with (6.6) and (6.7), we see that the assertion (ii) is true. This completes the proof of Theorem 3.

References

[1] A.L. Besse, *Einstein Manifolds*, Springer-Verlag, 1987.
[2] R.L. Bishop and B. O'Neill, Manifolds of negative curvature, *Trans. Amer. Math. Soc.*, **145** (1969), 1-49.
[3] M. do Carmo and M. Dajczer, Rotation hypersurfaces in spaces of constant curvature, *Trans. Amer. Math. Soc.*, **277** (1983), 685-709.
[4] M.L. Leite, Rotational hypersurfaces of space forms with constant scalar curvature, *Manuscripta Math.*, **67** (1990), 284-304.
[5] H. Mori, Stable complete constant mean curvature surfaces in R^3 and H^3, *Trans. Amer. Math. Soc.*, **278** (1983), 671-687.
[6] R. Narasimhan, *Analysis on real and complex manifolds*, North-Holland Publ. Company, Amsterdam, 1968.
[7] M. Spivak, *A comprehensive introduction to Differential Geometry*, vol. **4**, 2nd edn, Publish or Perish, Inc. Berkeley, 1979.

Department of Mathematics
Joetsu University of Education
Joetsu 943, Niigata Pref.
Japan