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LIMITING BEHAVIOR OF U-STATISTICS FOR STRONGLY MIXING SEQUENCES

By

KEN-ICHI YOSHIHARA

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Summary. Limit theorems for some non-degenerate U-statistics are established when the underlying processes satisfy some strongly mixing condition.

1. Introduction

Let $(X, \mathcal{A})$ be a countably generated measurable space. Let $\{\xi_i, i\geqq 0\}$ be a strictly stationary X-valued stochastic sequence defined on a probability space $(\Omega, \mathcal{F}, P)$. Denote by F the distribution of ξ_1 and by $\mathcal{M}_a^b$ the σ-algebra generated by $\xi_a, \cdots, \xi_b$. The sequence is called a *-mixing sequence if

$$\psi(n)=\sup_{k\geq 0}\sup_{A\in\mathcal{M}_0^k,\, B\in\mathcal{M}_{k+n}^\infty}\frac{|P(AB)-P(A)P(B)|}{P(A)P(B)}\longrightarrow 0 \quad (n\to\infty) \tag{1.1}$$

and it is called a ϕ-mixing sequence if

$$\phi(n)=\sup_{k\geq 0}\sup_{A\in\mathcal{M}_0^k,\, B\in\mathcal{M}_{k+n}^\infty}\frac{|P(AB)-P(A)P(B)|}{P(A)}\longrightarrow 0 \quad (n\to\infty). \tag{1.2}$$

Finally, the sequence is called a strongly mixing sequence if

$$\alpha(n)=\sup_{k\geq 0}\sup_{A\in\mathcal{M}_0^k,\, B\in\mathcal{M}_{k+n}^\infty}|P(AB)-P(A)P(B)|\longrightarrow 0 \quad (n\to\infty). \tag{1.3}$$

Next, let H be a Borel measurable kernel defined on X^m. Therefore, H: $X^m\to R$ be a measurable function such that

$$\theta(F)=\int\cdots\int H(x_1, \cdots, x_m)\prod_{i=1}^m dF(x_i)<\infty \tag{1.4}$$

and H is symmetric in its m arguments. A U-statistic U_n is then given by

$$U_n=\frac{1}{\binom{n}{m}}\sum_{1\leq i_1<\cdots<i_m\leq n}H(\xi_{i_1}, \cdots, \xi_{i_m}) \quad (n\geqq m). \tag{1.5}$$

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A kernel H is called degenerate (for the distribution F) if for all choices of $a_i \in X (1 \leqq i \leqq m)$ and all $j (1 \leqq j \leqq m)$

$$EH(a_1, \cdots, a_{j-1}, \xi_1, a_{j+1}, \cdots, a_m)=0 \tag{1.6}$$

and a U-statistic is called degenerate if the corresponding kernel has this property. For every $r (r=0, 1, \cdots, m)$ let

$$H_r(x_1, \cdots, x_r)=\int \cdots \int H(x_1, \cdots, x_m) \prod_{i=r+1}^{m} dF(x_i) \tag{1.7}$$

so that $H_0=\theta(F)$ and $H_m=H$. It is known that every U-statistic can be written as a finite weighted sums of degenerate ones, namely,

$$Y_n=\sum_{r=0}^{m} \binom{m}{r} U_n^{(r)} \tag{1.8}$$

where $U_n^{(r)}$ denotes the U-statistic obtained from the degenerate kernel

$$h_r(x_1, \cdots, x_r)=\sum_{j=0}^{r} \binom{r}{j} (-1)^{r-j} H_j(x_1, \cdots, x_j). \tag{1.9}$$

Now, let $L^2(X, F)$ be the Hilbert space of square-integrable functions with respect to F. Let $\{g_i\}$ be an orthonormal basis of $L^2(X, F)$ such that $g_0=1$. For each $r (1 \leqq r \leqq m)$ put

$$g_{i_1, \cdots, i_r}(x_1, \cdots, x_r)=\prod_{j=1}^{r} g_{i_j}(x_j). \tag{1.10}$$

It is well known that for each $r (1 \leqq r \leqq m)$ the system $\{g_{i_1, \cdots, i_r} : 0 \leqq i_1 < i_2 < \cdots < i_r < \infty\}$ is a basis of the Hilbert space $L^2(X^r, F^r)$. Let $\lambda_r(i_1, \cdots, i_r)$ be the Fourier coefficient of the function h_r, i.e.,

$$\lambda_r(i_1, \cdots, i_r)=\int \cdots \int h_r(x_1, \cdots, x_r) \prod_{j=1}^{r} g_{i_j}(x_j) \prod_{j'=1}^{r} dF(x_{j'}). \tag{1.11}$$

Then, for each $r (1 \leqq r \leqq m)$ we have

$$h_r(x_1, \cdots, x_r)=\sum_{(i_j)} \lambda_r(i_1, \cdots, i_r) g_{i_1, \cdots, i_r}(x_1, \cdots, x_r) \tag{1.12}$$

in the L^2-sense and by the Parseval inequality

$$\int \cdots \int h_r^2(x_1, \cdots, x_r) \prod_{j=1}^{r} dF(x_j)=\sum |\lambda_r(i_1, \cdots, i_r)|^2 < \infty. \tag{1.13}$$

Put

$$\lambda(i_1, \cdots, i_r, 0, \cdots, 0)=\lambda_r(i_1, \cdots, i_r) \quad (1 \leqq r \leqq m-1), \tag{1.14}$$

and

$$\lambda(i_1, \cdots, i_m)=\lambda_m(i_1, \cdots, i_m).$$

Let

(1.15) $$\sigma_n^2=E\left(\sum_{j=1}^{n} h_1(\xi_j)\right)^2$$

and

(1.16) $$\sigma^2=Eh_1^2(\xi_1)+2\sum_{j=1}^{\infty} Eh_1(\xi_1)h_1(\xi_{j+1}).$$

It is known that under conditions of Theorem 1 (below) the series in (1.16) converges absolutely and

(1.17) $$\sigma_n^2=n\sigma^2(1+o(1)) \quad \text{as} \quad n\to\infty.$$

Denker and Keller [2] considered the limiting behavior of $(n/m\sigma_n)(U_n-\theta(F))$ when $\{\xi_i\}$ is absolutely regular. The obtained results generalized the results in [6]. In this paper, we consider the analogous problems for some strongly mixing sequences when all of degenerate kernels $\{h_2(x_1, x_2), \cdots, h_m(x_1, \cdots, x_m)\}$ can be written in the form of (1.12).

2. Main results

In what follows, we use the following notations:

(i) For any random variables η we put $\|\eta\|_r=\{E|\eta|^r\}^{1/r}(r\geqq 1)$ if $E|\eta|^r<\infty$.

(ii) $[s]$ denotes the largest integer k such that $k\leqq s$.

(iii) c, with or without subscript, denotes an absolute constant which is not always the same.

Let $D[0, 1]$ be the space of functions on $[0, 1]$ that are right-continuous and have left-hand limits. We endow $D[0, 1]$ the J_1-topology. Let $\{\xi_i\}$ be a strictly stationary X-valued sequence. Let $h_1(\cdot)$ and $\{g_{k,r}(\cdot)\}$ be the ones defined in Section 1. For positive numbers a and b we put formally

(2.1) $$K_1(a)=\|h_1(\xi_1)\|_a$$

and

(2.2) $$K_2(b)=\max_{2\leq r\leq m}\sup_{k\geq 1}\|g_{k,r}(\xi_1)\|_b.$$

Define the process $\{Y_n(t): t\geqq 0\}$ by

(2.3) $$Y_n(t)=\frac{n}{m\sigma_n}(U_{[nt]}-\theta(F)) \qquad (t\geqq 0).$$

Theorem 1. *Let $\{\xi_i\}$ be a strictly stationary X-valued sequence. Let H: $X^m\to R$ be a nondegenerate kernel. Suppose that* (1.12) *holds for all $r\,(2\leqq r\leqq m)$ and σ^2, defined by* (1.16), *is positive. Suppose that*

(2.4) $$\sum|\lambda(i_1, \cdots, i_m)|<\infty$$

holds. Suppose further that one of the following sets of conditions is satisfied:

(i) $\{\xi_n\}$ *is* *-*mixing*, $K_1(2)<\infty$, $K_2(4)<\infty$ *and*

$$\sum_{n=1}^{\infty} n^m \psi(n) < \infty . \tag{2.5}$$

(ii) $\{\xi_n\}$ *is* ϕ-*mixing*, $K_1(2)<\infty$, $K_2(2m)<\infty$ *and*

$$\sum_{n=1}^{\infty} n^m \phi^{1/2m}(n) < \infty . \tag{2.6}$$

(iii) $\{\xi_n\}$ *is strongly mixing and there exists a* $\delta(>0)$ *such that* $K_1(2+\delta/m)<\infty$, $K_1(2m+\delta)<\infty$ *and*

$$\sum_{n=1}^{\infty} n^m \alpha^{\delta/(2m+\delta)}(n) < \infty . \tag{2.7}$$

Then, as $n\to\infty$

$$Y_n=\{Y_n(t): 0\leq t\leq 1\} \xrightarrow{D} W=\{W(t): 0\leq t\leq 1\} \quad in \quad D[0, 1] \tag{2.8}$$

where W *is a Wiener process on* $[0, 1]$.

Remark. Conditions (ii) and (iii) in Theorem 1 may be replaced by more flexible conditions (cf. [3], [4]).

Theorem 2. *Suppose conditions of Theorem* 1 *are satisfied. Then, without changing its distribution we can redefine the sequence* $\{\xi_n, n\geq 1\}$ *on a richer probability space on which there exists a Wiener process* $\{W(t): 0\leq t\leq 1\}$ *such that with probability one*

$$|t^{1/2}Y_{[t]}(1)-W(t)|=o(t^{1/2}) \quad as \quad t\to\infty . \tag{2.9}$$

3. Auxiliary results

The following is known.

Lemma 3.1. *Let* $\{\zeta_i\}$ *be some sequence. Let random variables* X *and* Y *be* $\mathcal{M}_0^k$- *and* $\mathcal{M}_{k+n}^{\infty}$-*measurable, respectively.*

(i) *If* $\{\zeta_i\}$ *is* *-*mixing*, $E|X|<\infty$ *and* $E|Y|<\infty$, *then*

$$|EXY-EXEY|\leq\psi(n)E|X|E|Y|, \tag{3.1}$$

which implies

$$|EXY|\leq E|XY|\leq(1+\psi(n))E|X|E|Y| . \tag{3.2}$$

(ii) *Suppose* $\{\zeta_i\}$ *is* ϕ-*mixing. Let* $p(>1)$ *and* $q(>1)$ *be numbers such that* $p^{-1}+q^{-1}=1$. *If* $E|X|^p<\infty$ *and* $E|Y|^q<\infty$, *then*

$$|EXY-EXEY|\leq 2\phi^{1/p}(n)\|X\|_p\|Y\|_q . \tag{3.3}$$

(iii) *Suppose* $\{\zeta_i\}$ *is strongly mixing. Let* $p(>1)$ *and* $q(>1)$ *be numbers such that* $\gamma=p^{-1}+q^{-1}<1$. *If* $E|X|^p<\infty$ *and* $E|Y|^q<\infty$, *then*

$$|EXY-EXEY|\leq 10\alpha^{\gamma}(n)\|X\|_p\|X\|_q. \tag{3.4}$$

Next, we consider the following lemma.

Lemma 3.2. *Let* $\{\zeta_i\}$ *be some mixing sequence of zero-mean random variables. Then, for any positive integer* r

$$E|\sum_{1\leq i_1<\cdots<i_r\leq n}\zeta_{i_1}\cdots\zeta_{i_r}|^2\leq cn^r, \tag{3.5}$$

if one of the following sets of conditions is satisfied:

(i) $\{\zeta_i\}$ *is* *-*mixing*, $\sup_{i\geq 1}\|\zeta_i\|_2<\infty$ *and*

$$\sum n^r\phi(n)<\infty. \tag{3.6}$$

(ii) $\{\zeta_i\}$ *is* ϕ-*mixing*, $\sup_{i\geq 1}\|\zeta_i\|_{2r}<\infty$ *and*

$$\sum n^r\phi^{1/2r}(n)<\infty. \tag{3.7}$$

(iii) $\{\zeta_i\}$ *is strongly mixing and there exists a* $\delta(>0)$ *for which* $\sup_i\|\zeta_i\|_{2r+\delta}<\infty$ *and*

$$\sum n^r\alpha^{\delta/(2r+\delta)}(n)<\infty. \tag{3.8}$$

The proof of Lemma 3.2 can be carried out by using Lemma 3.1 and the method of the proof of Lemma 3 in [6] and so omitted (cf. [8], Lemma 11.4.1).

The following lemma is fundamental.

Lemma 3.3. *Suppose conditions of Theorem* 1 *are satisfied. Then for any* $t(0<t\leq 1)$

$$E|U_{[nt]}^{(r)}|^2\leq c(r)n^{-r} \qquad (r=2,\cdots,m) \tag{3.9}$$

where $c(2),\cdots,c(m)$ *are absolute constants which do not depend on* n.

Proof. Let $r(2\leq r\leq m)$ be arbitrary. To prove (3.9) it is enough to show that

$$E|\sum_{1\leq j_1<\cdots<j_r\leq[nt]}h_r(\xi_{j_1},\cdots,\xi_{j_r})|^2\leq c(r)n^r. \tag{3.10}$$

By virture of (1.12), (3.10) is proved if we show that

$$E|\sum_{1\leq j_1<\cdots<j_r\leq[nt]}\sum_{(i)}\lambda_r(i_1,\cdots,i_r)\prod_{l=1}^{r}g_{i_l}(\xi_{j_l})|^2\leq c(r)n^r. \tag{3.11}$$

For brevity, put $p=[nt]$,

(3.12) $$\lambda_r(\{i\})=\lambda_r(i_1, \cdots, i_r)$$

and

(3.13) $$G_p(\{i\})=\sum_{1\leq j_1<\cdots<j_r\leq p}\prod_{l=1}^{r} g_{i_l}(\xi_{j_l}).$$

Since for each $\{i\}$ fixed $\{g_{i_l}(\xi_j)\}$ is a mixing sequence of zero-mean random variables such that $\{g_{i_l}(\xi_j)\}$ satisfies the same mixing condition as that of $\{\xi_j\}$, by the Schwarz inequality and Lemma 3.2,

(3.14)
$$\begin{aligned}\text{LHS of (3.11)}&\leqq\textstyle\sum_{(i)}\sum_{(i')}|\lambda(\{i\})|\cdot|\lambda(\{i'\})|\cdot E|G_p(\{i\})G_p(\{i'\})|\\&\leqq\textstyle\sum_{(i)}\sum_{(i')}|\lambda(\{i\})|\cdot|\lambda(\{i'\})|\cdot\|G_p(\{i\})\|_2\|G_p(\{i'\})\|_2\\&\leqq cp^r\textstyle\sum_{(i)}|\lambda(\{i\})|\sum_{(i')}|\lambda(\{i'\})|.\end{aligned}$$

Now, (3.10) follows from (3.14) and (1.13). □

4. Proofs

Proof of Theorem 1. It is easily shown that under conditions of Theorem 1

(4.1) $$\{\sigma_n^{-1}\textstyle\sum_{j=1}^{[nt]}(h_1(\xi_j)-\theta(F)):\ 0\leqq t\leqq 1\}\xrightarrow{D} W=\{W(t):\ 0\leqq t\leqq 1\}$$

(cf. [3], [4]). Hence, (2.9) is obtained from (4.1) and Lemma 3.3. □

Proof of Theorem 2. By results in [5], we can show that if condition of Theorem 1 are satisfied, then without changing its distribution we can redefine the sequence $\{\xi_n,\ n\geqq 1\}$ on a richer probability space such that for some γ $(0<\gamma<1/2)$

(4.2) $$|\sigma_n^{-1}\textstyle\sum_{j=1}^{[t]}(h_1(\xi_j)-\theta(F))-W(t)|=O(t^{1/2-\gamma})\quad \text{a.s.}$$

as $t\to\infty$. On the other hand, by Lemma 3.3 we can easily show that

(4.3) $$\left|\textstyle\sum_{r=2}^{m}\binom{m}{2}U_{[t]}^{(r)}\right|=o(t^{-1/8})\quad \text{a.s.}$$

as $t\to\infty$. Hence, (2.10) is obtained from (4.2) and (4.3). □

References

[1] H. Dehling, M. Denker and W. Philipp, Invariance principles for von Mises and U-statistics, *Z. Wahrsch. Verw. Gebiete*, **67** (1984), 139-167.

[2] M. Denker and G. Keller, On U-statistics and v. Mises' statistics for weakly dependent processes, *Z. Wahrsch. Verw. Gebiete*, **64** (1983), 505-522.

[3] N. Herrndorf, The invariance principle for ϕ-mixing sequences, *Z. Wahrsch. Verw. Gebiete*, **63** (1983), 97-109.

[4] N. Herrndorf, A functional central limit theorem for strongly mixing sequences of random variables, *Z. Wahrsch. Verw. Gebiete*, **69** (1985), 541-550.

[5] W. Philipp and W. Stout, Almost sure invariance principles for partial sums of weakly dependent random variables, *Mem. AMS*, **161** (1975), Providence, R.I.

[6] K. Yoshihara, Limiting behavior of U-statistics for stationary absolutely regular processes, *Z. Wahrsch. Verw. Gebiete*, **35** (1976), 237-252.

[7] K. Yoshihara, Weak convergence to some processes defined by multiple Wiener integrals, *Yokohama Math. J.*, **38** (1990), 57-76.

[8] K. Yoshihara, *Summation theory for weakly dependent sequences*, Sanseido, Tokyo, 1992.

Department of Mathematics
Faculty of Engineering
Yokohama National University
156, Tokiwadai, Hodogaya, Yokohama 240,
Japan