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UNIVERSAL DILATIONS OF A COMPLETELY NONUNITARY NORMAL OPERATOR

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Abstract. Suppose $\mathcal{H}$ is a separable, infinite dimensional, complex Hilbert space. Let $\{\lambda_i : 1 \leqq i \leqq n\}$ be a set of distinct elements in the open unit disc of the complex plane $\boldsymbol{C}$ and let $T \in \boldsymbol{A}_n(\mathcal{H})$ (to be defined below). In this paper, we show that if N is a normal operator on an n-dimensional Hilbert space whose matrix to some orthonormal basis $\{e_i : 1 \leqq i \leqq n\}$ is the diagonal matrix $\mathrm{Diag}(\{\lambda_i : 1 \leqq i \leqq n\})$, then there exist invariant subspaces $\mathcal{M}$ and $\mathcal{N}$ for T with $\mathcal{M} \subset \mathcal{N}$ such that the compression $T_{\mathcal{M} \ominus \mathcal{N}}$ of T to $\mathcal{M} \ominus \mathcal{N}$ is unitarily equivalent to N.

1. Introduction

Let $\mathcal{H}$ be a separable, infinite dimensional, complex Hilbert space and let $\mathcal{L}(\mathcal{H})$ be the algebra of all bounded linear operators on $\mathcal{H}$. A *dual algebra* is a subalgebra of $\mathcal{L}(\mathcal{H})$ that contains the identity operator $I_{\mathcal{H}}$ and is closed in the ultraweak operator topology on $\mathcal{L}(\mathcal{H})$. The theory of dual algebras is deeply related to the study of the classes $\boldsymbol{A}_n$ (to be defined below), where n is any cardinal number such that $1 \leqq n \leqq \aleph_0$ (cf. [1], [5], and [6]). The structures of the classes $\boldsymbol{A}_n$ have been applied to the topics of invariant subspaces, dilation theory, and reflexivity (cf. [6], [13]). In particular, the study of the classes $\boldsymbol{A}_n$ appearing in the theory of dual algebras has been focused in the last five years on sufficient conditions that a contraction $T \in \mathcal{L}(\mathcal{H})$ belongs to the classes $\boldsymbol{A}_n$. An abstract geometric criterion for membership in $\boldsymbol{A}_{\aleph_0}$ was first given in [1]. Brown-Chevreau-Exner-Pearcy [7][8][9][10] obtained some relationship between dual algebras and Fredholm theory, and established topological criteria for membership in $\boldsymbol{A}_{\aleph_0}$. Recently many functional analysists have studied structures of operators in the class $\boldsymbol{A}_n$, $\boldsymbol{A}_{\aleph_0}$, or $\boldsymbol{A}$ (cf. [3], [4], [11], and [12]). As a sequel to this study, we define in section 3 new classes $\mathcal{C}(A_n)$ which give a good motivation for attacking the main work of this paper. In section 4, we

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obtain some results concerning the classes $C(\boldsymbol{A}_n)$ and some dilation theorems of operators in the classes $\boldsymbol{A}_n$.

2. Preliminaries and notation

The notation and terminology employed herein agree with those in [2], [6], and [15]. We shall denote by $\boldsymbol{D}$ the open unit disc in the complex plane $\boldsymbol{C}$, and we write $\boldsymbol{T}$ for the boundary of $\boldsymbol{D}$. For $1 \leqq p < \infty$, we denote by $L^p = L^p(\boldsymbol{T})$ the Banach space of complex valued, Lebesgue measurable functions f on $\boldsymbol{T}$ such that $|f|^p$ is Lebesgue integrable, and by $L^\infty = L^\infty(\boldsymbol{T})$ the Banach algebra of all complex valued Lebesgue measurable, essentially bounded functions on $\boldsymbol{T}$. If for $1 \leqq p \leqq \infty$ we denote by $H^p = H^p(\boldsymbol{T})$ the subspace of L^p consisting of those functions whose negative Fourier coefficients vanish, then one knows that the preannihilator ${}^\perp(H^\infty)$ of H^∞ in L^1 is the subspace H_0^1 consisting of those functions g in H^1 whose analytic extension $\tilde{g}$ to $\boldsymbol{D}$ satisfies $\tilde{g}(0)=0$. It is well known that H^∞ is the dual space of L^1/H_0^1, where the duality is given by the pairing $\langle f, [g]\rangle = \frac{1}{2\pi}\int_0^{2\pi} f(e^{it})g(e^{it})dt$, $f \in H^\infty$, $[g] \in L^1/H_0^1$. For $T \in \mathcal{L}(\mathcal{H})$, let $\mathcal{A}_T$ denote the smallest subalgebra of $\mathcal{L}(\mathcal{H})$ that contains T and $I_{\mathcal{H}}$ and is closed in the ultraweak operator topology. Moreover, let Q_T denote the quotient space $C_1/{}^\perp\mathcal{A}_T$, where C_1 is the trace-class ideal in $\mathcal{L}(\mathcal{H})$ under the trace norm, and ${}^\perp\mathcal{A}_T$ denotes the preannihilator of $\mathcal{A}_T$ in C_1. One knows that $\mathcal{A}_T$ is the dual space of Q_T and that the duality is given by $\langle A, [L]\rangle = tr(AL)$, $A \in \mathcal{A}_T$, $[L] \in Q_T$. For vectors x and y in $\mathcal{H}$, we write, as usual, $x \otimes y$ for the rank one operator in C_1 defined by $(x\otimes y)(u) = (u, y)x$, $u \in \mathcal{H}$. Recall that any contraction T can be written as a direct sum $T = T_1 \oplus T_2$, where T_1 is a completely nonunitary contraction and T_2 is a unitary operator. If T_2 is absolutely continuous or acts on the space (0), T will be called an *absolutely continuous contraction*.

Let T be an absolutely continuous contraction in $\mathcal{L}(\mathcal{H})$. Then it follows from Fois-Nagy functional calculus [6, Theorem 4.1] that there is an algebra homomorphism $\Phi_T : H^\infty \to \mathcal{A}_T$ defined by $\Phi_T(f) = f(T)$ such that (a) $\Phi_T(1) = 1_{\mathcal{H}}$, $\Phi_T(\xi) = T$, (b) $\|\Phi_T(f)\| \leqq \|f\|_\infty$, $f \in H^\infty$, (c) Φ_T is continuous if both H^∞ and $\mathcal{A}_T$ are given their weak* topologies, (d) the range of Φ_T is weak* dense in $\mathcal{A}_T$, (e) there exists a bounded, linear, one-to-one map $\phi_T : Q_T \to L_1/H_0^1$ such that $\phi_T^* = \Phi_T$, and (f) if Φ_T is an isometry, then Φ_T is a weak* homeomorphism of H^∞ onto $\mathcal{A}_T$ and ϕ_T is an isometry of Q_T onto L^1/H_0^1. Let $\mathcal{A} \subset \mathcal{L}(\mathcal{H})$ be a dual algebra and let n be any cardinal number such that $1 \leqq n \leqq \aleph_0$. Then $\mathcal{A}$ will be said to have *property* $(\boldsymbol{A}_n)$ provided every $n \times n$ system of simultaneous equations of the form $[L_{ij}] = [x_i \otimes y_j]$, $0 \leqq i, j < n$ (which the $[L_{ij}]$ are arbitrary but fixed elements from $Q_{\mathcal{A}}$) has a solution $\{x_i\}_{0 \leqq i < n}$, $\{y_j\}_{0 \leqq j < n}$ consisting of a pair of sequences of vectors from $\mathcal{H}$ (cf. [5]). The class $A(\mathcal{H})$ consists of all those absolutely continuous contraction T in $\mathcal{L}(\mathcal{H})$ for which the functional calculus $\Phi_T: H^\infty \to \mathcal{A}_T$ is an isometry. Furthermore, if n is any cardinal number such that $1 \leqq n \leqq \aleph_0$, we denote by $A_n(\mathcal{H})$ the set of all T in $A(\mathcal{H})$ such that the algebra $\mathcal{A}_T$ has property (A_n).

We write simply A_n for $A_n(\mathcal{H})$ when there is no confusion. If $T \in \mathcal{L}(\mathcal{H})$ and $\mathcal{M} \subset \mathcal{H}$ is a semi-invariant subspace for T (i.e., there exist invariant subspaces $\mathcal{N}_1$ and $\mathcal{N}_2$ for T with $\mathcal{N}_1 \supset \mathcal{N}_2$ such that $\mathcal{M} = \mathcal{N}_1 \ominus \mathcal{N}_2 = \mathcal{N}_1 \cap \mathcal{N}_2^\perp$), we write $T_{\mathcal{M}}$ for the compression of T to $\mathcal{M}$. In other words, $T_{\mathcal{M}} = P_{\mathcal{M}} T | \mathcal{M}$, where $P_{\mathcal{M}}$ is the orthogonal projection whose range is $\mathcal{M}$. Let n be any cardinal number such that $1 \leqq n \leqq \aleph_0$ and let $(\mathcal{SC})_n$ denote the class of strict contractions A acting on Hilbert space of dimension n (i.e., $\|A\| < 1$). Throughout this paper, we write $\boldsymbol{N}$ for the set of natural numbers. For a Hilbert space $\mathcal{K}$ and any operators $T_i \in \mathcal{L}(\mathcal{K})$, $i = 1, 2$, we write $T_1 \cong T_2$ if T_1 is unitarily equivalent to T_2.

3. Universal A_n-compressions

We start this section as the following definition. It should be compared with [5, Definition 4.9].

Definition 3.1. Suppose n is any cardinal number such that $1 \leqq n \leqq \aleph_0$ If A is an operator on a Hilbert space of dimension $\leqq n$ and every operator T in $A_n(\mathcal{H})$ has the property that some compression of T to a semi-invariant subspace is unitarily equivalent to A, then we call A a *universal A_n-compression*, and we denote the set of all universal A_n-compressions by $\mathcal{C}(A_n)$.

For a contraction operator $T \in \mathcal{L}(\mathcal{H})$, we recall that $T \in C_{00}$ if $\|T^n x\| \to 0$ and $\|T^{*n} x\| \to 0$ $(n \to \infty)$ for all x in $\mathcal{H}$. It is obvious that every A in $\mathcal{C}(A_n)$ is a completely non-unitary contraction since $\mathcal{C}(A_n) \subset C_{00}$.

Proposition 3.2. *Let n be any cardinal number such that $1 \leqq n \leqq \aleph_0$. Then the class $\mathcal{C}(A_n)$ is self-adjoint.*

Proof. Let $A \in \mathcal{C}(A_n)$. Then for an operator $T \in A_n$, there exists a semi-invariant subspace $\mathcal{K}$ for T such that A is unitarily equivalent to $T_{\mathcal{K}}$. Hence there exist invariant subspaces $\mathcal{M}$ and $\mathcal{N}$ for T with $\mathcal{M} \supset \mathcal{N}$ such that $\mathcal{K} = \mathcal{M} \ominus \mathcal{N}$ and T is the operator matrix form

$$\begin{pmatrix} T_{11} & T_{12} & T_{13} \\ 0 & T_{\mathcal{K}} & T_{23} \\ 0 & 0 & T_{33} \end{pmatrix} \tag{3.1}$$

relative to $\mathcal{N}\oplus\mathcal{K}\oplus\mathcal{M}^{\perp}$. Since $\mathcal{M}\ominus\mathcal{N}=\mathcal{M}\cap\mathcal{N}^{\perp}=\mathcal{N}^{\perp}\ominus\mathcal{M}^{\perp}$, T^* is the operator matrix form

$$\begin{pmatrix} T_{33}^* & T_{23}^* & T_{13}^* \\ 0 & (T_{\mathcal{K}})^* & T_{12}^* \\ 0 & 0 & T_{11}^* \end{pmatrix} \tag{3.2}$$

relative to $\mathcal{M}^{\perp}\oplus\mathcal{K}\oplus\mathcal{N}$. Hence A^* is unitarily equivalent to $(T_{\mathcal{K}})^*=(T^*)_{\mathcal{K}}$. Since it is well-known that $\boldsymbol{A}_n$ is self-adjoint, we have $A^*\in\mathcal{C}(\boldsymbol{A}_n)$. Hence $\mathcal{C}(\boldsymbol{A}_n)$ is self-adjoint. The proof is complete.

Recall that a completely nonunitary contraction $T\in\mathcal{L}(\mathcal{H})$ is said to be of class C_0 if there exists $u\in H^{\infty}$, $u\not\equiv 0$, such that the functional calculus $u(T)=0$ (cf. [2]).

Proposition 3.3. $(\mathcal{SC})_1=\mathcal{C}(\boldsymbol{A}_1)\subset\mathcal{C}(\boldsymbol{A}_2)\subset\cdots\subset C_0\cap\mathcal{C}(\boldsymbol{A}_{\aleph_0})$.

Proof. Since $\boldsymbol{A}_1\supset\boldsymbol{A}_2\supset\cdots\supset\boldsymbol{A}_{\aleph_0}$, it is obvious that $\mathcal{C}(\boldsymbol{A}_1)\subset\mathcal{C}(\boldsymbol{A}_2)\subset\cdots\subset\mathcal{C}(\boldsymbol{A}_{\aleph_0})$. To show $\mathcal{C}(\boldsymbol{A}_n)\subset C_0$ for any $n\in\boldsymbol{N}$, let $A\in\mathcal{C}(\boldsymbol{A}_n)$. Because the unilateral shift $S^{(n)}$ of multiplicity n belongs to the class $\boldsymbol{A}_n$ (cf. [5, Theorem 3.7]), there exist semi-invariant subspaces $\mathcal{M}$ and $\mathcal{N}$ for $S^{(n)}$ with $\mathcal{M}\supset\mathcal{N}$ such that $A\cong S^{(n)}_{\mathcal{M}\ominus\mathcal{N}}$. If we write $\tilde{S}=S^{(n)}_{\mathcal{M}\ominus\mathcal{N}}$, then we can say

$$S^{(n)}\cong\begin{pmatrix} R & * & * \\ 0 & \tilde{S} & * \\ 0 & 0 & * \end{pmatrix} \tag{3.3}$$

relative to a decomposition $\mathcal{N}\oplus(\mathcal{M}\ominus\mathcal{N})\oplus\mathcal{M}^{\perp}$. Hence there exists $k\in\boldsymbol{N}$ with $1\leqq k\leqq n$ such that

$$S^{(k)}\cong\begin{pmatrix} R & * \\ 0 & \tilde{S} \end{pmatrix} \tag{3.4}$$

relative to a decomposition $\mathcal{N}\oplus(\mathcal{M}\ominus\mathcal{N})$. It is obvious that $R\cong S^{(k)}$ since the dimension of $\mathcal{M}\ominus\mathcal{N}$ is finite. According to [14, Corollary 2.22], we have $\tilde{S}\in C_0$ and $A\in C_0$. Hence $\mathcal{C}(\boldsymbol{A}_n)\subset C_0$. Let A be a strictly contraction acting on one dimensional Hilbert space $\mathcal{H}_1$. Then there exists $\lambda\in\boldsymbol{D}$ such that $Ax=\lambda x$ for all $x\in\mathcal{H}_1$ Let $T\in\boldsymbol{A}_1(\mathcal{H})$. Then it follows from [5, Corollary 3.6] that there exist invariant subspaces $\mathcal{M}$ and $\mathcal{N}$ for T with $\mathcal{M}\supset\mathcal{N}$ such that $\dim(\mathcal{M}\ominus\mathcal{N})=1$ and $T_{\mathcal{M}\ominus\mathcal{N}}=\lambda I$. Hence $A\in\mathcal{C}(\boldsymbol{A}_1)$. Conversely, let $A\in\mathcal{C}(\boldsymbol{A}_1)$. Let $\mathcal{H}_1$ be the acting Hilbert space of A. Then there exists $\lambda\in\boldsymbol{C}$ such that $Ax=\lambda x$ for all $x\in\mathcal{H}_1$. Since $A\in C_0\subset C_{00}$, we have $\|A^n e\|=|\lambda|^n\to 0$ $(n\to\infty)$, where e is a unit vector in $\mathcal{H}_1$. Hence $|\lambda|<1$. Therefore $A\in(\mathcal{SC})_1$. Hence the proof is complete.

Corollary 3.4. *Suppose A is a normal operator on an n-dimensional Hilbert space $\mathcal{H}_n$ whose matrix to some orthonormal basis $\{e_i\}_{i=1}^n$ for $\mathcal{H}_n$ is the diagonal matrix* $\mathrm{Diag}(\{\lambda_i\}_{i=1}^n)$. *If $A \in C(\boldsymbol{A}_n)$, then $A \in (\mathcal{S}C)_n$.*

Proof. Assume $A \in C(\boldsymbol{A}_n)$. By Proposition 3.3, we have $A \in C_0 \subset C_{00}$. Then $\|A^k e_i\| = |\lambda_i|^k \to 0$ $(k \to \infty)$, for $i=1, \cdots, n$. Therefore $\lambda_i \in \boldsymbol{D}$, for $i=1, \cdots, n$, and $\|A\| = \max\{|\lambda_1|, \cdots, |\lambda_n|\} < 1$. So $A \in (\mathcal{S}C)_n$. The proof is complete.

4. Dilation theorems of a completely nonunitary normal operator

The following is the main theorem of this paper.

Theorem 4.1. *Let $T \in \boldsymbol{A}_n(\mathcal{H})$, $n \in \boldsymbol{N}$ and let N be a completely nonunitary normal contraction acting on an n-dimensional Hilbert space $\mathcal{H}_n$, $2 \leqq n \in \boldsymbol{N}$, whose matrix relative to some orthonormal basis $\{u_k\}_{k=1}^n$ for $\mathcal{H}_n$ is diagonal matrix* $\mathrm{Diag}(\{\lambda_k\}_{k=1}^n)$. *Suppose λ_i, $i=1, \cdots, n$, are distinct. Then there exist $\mathcal{M}, \mathcal{N} \in$* $\mathrm{Lat}(T)$, *the lattice of invariant subspaces for T, with $\mathcal{M} \supset \mathcal{N}$ such that $T_{\mathcal{M} \ominus \mathcal{N}} \cong N$.*

Proof. Since N is a completely nonunitary contraction operator, we have $\{\lambda_k\}_{k=1}^n \subset \boldsymbol{D}$. Hence we can take ε with $0 < \varepsilon < 1 - \max\{|\lambda_i| : i=1, \cdots, n\}$. Let $m = n(n+1)/2$ and let $\mathcal{H}_m$ be an m-dimensional Hilbert space. Let us consider an operator $\tilde{A}$ on $\mathcal{H}_m$ whose matrix relative to some orthonormal basis $\{e_k\}_{k=1}^m$ for $\mathcal{H}_m$ is

$$\left(\begin{array}{c|c|c}
\begin{matrix}\lambda_1 & \varepsilon & & \\ & \ddots & \ddots & \\ & & \ddots & \varepsilon \\ & & & \lambda_n\end{matrix} & & O \\ \hline
 & \begin{matrix}\lambda_2 & \varepsilon & & \\ & \ddots & \ddots & \\ & & \ddots & \varepsilon \\ & & & \lambda_n\end{matrix} & \\ \hline
O & & \begin{matrix}\ddots & \\ & \lambda_n\end{matrix}
\end{array}\right) \in \mathcal{L}(\mathcal{H}_m) \tag{4.1}$$

Then it is not difficult to show that $\{e_n, e_{n+(n-1)}, e_{n+(n-1)+(n-2)}, \cdots, e_m\}$ is a cyclic set for $\tilde{A}$. Moreover, since $\varepsilon < 1 - \max\{|\lambda_i| : i=1, \cdots, n\}$, it follows from a simple calculation that $\|\tilde{A}\| < 1$. Now applying [5, Theorem 3.7], there exist $\mathcal{M}, \mathcal{N} \in \mathrm{Lat}(T)$ with $\mathcal{M} \supset \mathcal{N}$ such that $T_{\mathcal{M} \ominus \mathcal{N}}$ is similar to $\tilde{A}$. If we define a normal operator $\tilde{N} \in \mathcal{L}(\mathcal{H}_m)$ whose matrix relative to an orthonormal basis $\{u_k^{(i)}\}_{\substack{1 \leq i \leq k \\ 1 \leq k \leq n}}$ for $\mathcal{H}_m$ is the diagonal matrix

$$\mathrm{Diag}(\lambda_1^{(1)}, \underbrace{\lambda_2^{(1)}, \lambda_2^{(2)}}_{(2)}, \underbrace{\lambda_3^{(1)}, \lambda_3^{(2)}, \lambda_3^{(3)}}_{(3)}, \cdots, \underbrace{\lambda_n^{(1)}, \cdots, \lambda_n^{(n)}}_{(n)}), \tag{4.2}$$

where $\lambda_k^{(1)}=\lambda_k^{(2)}=\cdots=\lambda_k^{(k)}=\lambda_k$, for $k=1, 2, \cdots, n$, then it is obvious that $\tilde{N}$ is similar to $T_{\mathcal{M}\ominus\mathcal{N}}$. Let X be an invertible operator with $T_{\mathcal{M}\ominus\mathcal{N}}X=X\tilde{N}$. Note that $\tilde{N}u_k^{(i)}=\lambda_k^{(i)}u_k^{(i)}$, $1\leqq i\leqq k$, $1\leqq k\leqq n$. For a brief notation, we write $\tilde{T}=T_{\mathcal{M}\ominus\mathcal{N}}$. Since X is one-to-one, it is easy to show that there exists a linearly independent set $\{w_k^{(i)}\}_{\substack{1\leq i\leq k\\ 1\leq k\leq n}}$ in $\mathcal{M}\ominus\mathcal{N}$ such that $\|w_k^{(i)}\|=1$ and $\tilde{T}w_k^{(i)}=\lambda_k^{(i)}w_k^{(i)}$, $1\leqq i\leqq k$, $1\leqq k\leqq n$. Taking $f_1=w_1^{(1)}$, we have $\tilde{T}f_1=\lambda_1 f_1$. Assume that there exist $f_1, \cdots, f_k$ in $\mathcal{M}\ominus\mathcal{N}$ $(k<n)$ such that $\tilde{T}f_i=\lambda_i f_i$, $i=1, \cdots, k$. Since $\{w_{k+1}^{(1)}, \cdots, w_{k+1}^{(k+1)}\}$ induces a $(k+1)$-dimensional Hilbert space $\mathcal{R}$, there exists a unit vector $f_{k+1}\in\mathcal{R}$ such that $(f_i, f_{k+1})=0$, for $i=1, 2, \cdots, k$. Let

$$f_{k+1}=\sum_{i=1}^{k+1} a_i w_{k+1}^{(i)}, \tag{4.3}$$

where $a_i\in\boldsymbol{C}$. Then we have

$$\begin{aligned}\tilde{T}f_{k+1}&=\tilde{T}\Big(\sum_{i=1}^{k+1} a_i w_{k+1}^{(i)}\Big)=\sum_{i=1}^{k+1} a_i\lambda_{k+1}^{(i)}w_{k+1}^{(i)}\\ &=\lambda_{k+1}\sum_{i=1}^{k+1} a_i w_{k+1}^{(i)}=\lambda_{k+1}f_{k+1}.\end{aligned} \tag{4.4}$$

Hence by the mathematical induction, there exists an orthonormal set $\{f_k\}_{k=1}^n\subset\mathcal{M}\ominus\mathcal{N}$ such that $\tilde{T}f_k=\lambda_k f_k$, for $k=1, 2, \cdots, n$. Let us denote $\mathcal{K}=\bigvee_{k=1}^n f_k$. If we define a linear map $Y: \mathcal{H}_n\to\mathcal{K}$ with $Yu_k=f_k$, $k=1, 2, \cdots, n$, then it is obvious that Y is onto and isometry. Since $\mathcal{K}$ is an invariant subspace for $\tilde{T}$, $\mathcal{K}$ is a semi-invariant subspace for T. Furthermore, we have $T_{\mathcal{K}}Y=YN$. Hence N is unitarily equivalent to $T_{\mathcal{K}}$ and the proof is complete.

Remark 4.2. Note from Theorem 4.1 that if N is a normal completely nonunitary contraction operator acting on an n-dimensional Hilbert space with distinct eigenvalues, then $N\in\mathcal{C}(\boldsymbol{A}_n)$.

Proposition 4.3. *Let N be a normal completely nonunitary contraction operator acting on an n-dimensional Hilbert space $\mathcal{H}_n$, $2\leqq n\in\boldsymbol{N}$, whose matrix relative to some orthonormal basis $\{u_k\}_{k=1}^n$ for $\mathcal{H}_n$ is the diagonal matrix* $\mathrm{Diag}(\{\lambda_k\}_{k=1}^n)$. *Then there exists $m\in\boldsymbol{N}$ with $m<(n+1)n/2$ such that $N\in\mathcal{C}(\boldsymbol{A}_m)$.*

Proof. Because of Remark 4.2, we can assume $\lambda_i=\lambda_j$ for some i, j. Hence without loss of generality we can say $\lambda_1=\lambda_2=\lambda\in\boldsymbol{D}$. Put

$$\tilde{A}=\mathrm{Diag}(\lambda, \lambda, \underbrace{\lambda_3^{(1)}, \lambda_3^{(2)}, \lambda_3^{(3)}}_{(3)}, \cdots, \underbrace{\lambda_n^{(1)}, \cdots, \lambda_n^{(n)}}_{(n)})\in\mathcal{L}(\mathcal{H}_m), \tag{4.5}$$

where $m=(n+2)(n-1)/2<(n+1)n/2$ (cf. (4.2)). Let $T\in\boldsymbol{A}_m$. Since $\tilde{A}$ is a completely nonunitary contraction, by [5, Corollary 3.5], there exist $\mathcal{M}, \mathcal{N}\in\mathrm{Lat}(T)$ with $\mathcal{M}\supset\mathcal{N}$ such that $\dim(\mathcal{M}\ominus\mathcal{N})=m$ and $T_{\mathcal{M}\ominus\mathcal{N}}$ is similar to $\tilde{A}$. For a brief notation, we denote $\tilde{T}=T_{\mathcal{M}\ominus\mathcal{N}}$. Repeating the method of proof of Theorem 4.1, we have a linear independent set

$$\{u_1^{(1)}, u_2^{(1)}, \underbrace{u_3^{(1)}, u_3^{(2)}, u_3^{(3)}}_{(3)}, \cdots, \underbrace{u_n^{(1)}, \cdots, u_n^{(n)}}_{(n)}\} \tag{4.6}$$

in $\mathcal{M}\ominus\mathcal{N}$ such that $\tilde{T}u_k^{(1)}=\lambda u_k^{(1)}$, $k=1, 2$, and $\tilde{T}u_k^{(i)}=\lambda_k u_k^{(i)}$, $1\leqq i\leqq k$, $k=3, \cdots, n$. Take an orthonormal set $\{f_1, f_2\}$ in $\bigvee_{k=1}^2 u_k^{(1)}$. Then it is easy to show that $\tilde{T}f_k=\lambda f_k$, $k=1, 2$. Hence by the proof of Theorem 4.1, there exists an orthonormal set $\{f_k\}_{k=1}^n$ in $\mathcal{M}\ominus\mathcal{N}$ such that $\tilde{T}f_k=\lambda_k f_k$, $k=1, \cdots, n$. Put $\mathcal{K}=\bigvee_{i=1}^n f_i$. Then $\tilde{T}|\mathcal{K}=T_{\mathcal{K}}\cong N$. Hence $N\in\mathcal{C}(A_m)$ and the proof is complete.

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