YOKOHAMA MATHEMATICAL
JOURNAL VOL. 38, 1991

PLANAR BRANCHED COVERINGS OF GRAPHS

By

SHIGERU KITAKUBO

(Received April 9, 1990)

Abstract: A graph $\tilde{G}$ is called a *branched covering* of G with a *projection* $p:\tilde{G}\rightarrow G$ if there is a surjection $p: V(\tilde{G})\rightarrow V(G)$ such that $p|_{N(\tilde{v})}: N(\tilde{v})\rightarrow N(v)$ is surjective for any vertex $v\in V(G)$ and $\tilde{v}\in p^{-1}(v)$. It is said to be *regular* if there is a subgroup A of $Aut(\tilde{G})$ such that for any two vertices $u, v\in V(\tilde{G})$, $p(u)=p(v)$ if and only if $\tau(u)=v$ for some $\tau\in A$. In this paper we show that G has a planar and finite regular branched covering if and only if G is either planar or projective-planar.

1. Introduction

Let G be a simple, connected, finite graph and let $V(G)$ and $E(G)$ denote the vertex set and the edge set of G respectively. We call the set of vertices adjacent to a vertex $v\in V(G)$ the *neighborhood* of v and denote it by $N(v)$.

A graph $\tilde{G}$ is called a *branched covering* of G with a *projection* $p:\tilde{G}\rightarrow G$ if there is a surjection $p: V(\tilde{G})\rightarrow V(G)$ such that $p|_{N(\tilde{v})}: N(\tilde{v})\rightarrow N(v)$ is a surjection for any vertex $v\in V(G)$ and $\tilde{v}\in p^{-1}(v)$. $\tilde{G}$ is called a *covering* (or an *unbranched covering*) of G if $p|_{N(\tilde{v})}: N(\tilde{v})\rightarrow N(v)$ is in particular a bijection. A branched covering is possibly infinite but is assumed to be connected throughout this paper.

A branched covering $p:\tilde{G}\rightarrow G$ is called *regular* if there is a subgroup A of $Aut(\tilde{G})$, the automorphism group of $\tilde{G}$, such that two vertices $u, v\in V(\tilde{G})$ project to the same vertex $p(u)(=p(v))$ of G if and only if some automorphism $\tau\in A$ carries u to v ($\tau(u)=v$). This group A is called the *covering translation group* of $p:\tilde{G}\rightarrow G$ and each element of A is called a *covering translation.* Note that the covering translations of a regular unbranched covering, except the identical translation, have no fixed point in $V(\tilde{G})$.

Our main result is the following.

Theorem 1.1. *A graph G has a regular branched covering $\tilde{G}$ which is a finite planar graph if and only if G is either planar or projective-planar.*

This is motivated by a theorem recently proved by Negami [3].

Theorem 1.2 (Negami [3]). *A graph G has a regular covering $\tilde{G}$ which is a fiinte planar graph if and only if G is either planar or projective-planar.*

If a graph G has a regular covering $\tilde{G}$, $\tilde{G}$ is also a regular branched covering of G by definition. So Theorem 1.1 is an extension of Theorem 1.2. Combining this theorem with Corollary 2 in [2], we have the following.

Corollary 1.3. *If a graph G has a regular branched covering $\tilde{G}$ which is a finite and planar graph, then G has a 2-fold planar unbranched covering.*

After some preliminaries in the next section, we prove the theorem, using Whitney's theorem [5] and the property of 2-orbifolds which can be found in Thurstom [4, Chap. 13].

2. Lemmas

Denote the minimal degree of a graph $\tilde{G}$ by $\delta(\tilde{G})$. Assuming $\delta(\tilde{G})\geqq 3$, we discuss the situation for a while. Let τ be any automorphism which belongs to a subset A of $Aut(\tilde{G})$. Suppose G is n-connected and U is an n-cut of G. A subset S of $V(G)$ is called a *fragment* for U if the induced subgraph $\langle S\rangle$ of G is a component of $G-U$. A fragment S is called an *A-equivariant fragment* if either $\tau(S)\cap S=\emptyset$ or $\tau(S)=S$.

The following lemmas are useful to prove the theorem when $\tilde{G}$ is not 3-connected.

Lemma 2.1. *Let $\tilde{G}$ be a connected, but not 3-connected, graph and $\delta(\tilde{G})\geqq 3$. If $p: \tilde{G}\to G$ is a finite regular branched covering with covering translation group A of order $\geqq 2$, then the following three hold:*

(1) *The inclusion minimal fragment S for a 1- or 2-cut U of $\tilde{G}$ is an A-equivariant fragment.*

(2) *If $K=\langle S\cup U\rangle$ is not 3-connected, then U is a 2-cut, say $\{u, v\}$, and $K+uv$ is 3-connected.*

(3) *If $p(U)$ s not a cutset of G, $V(G)=p(S)\cup p(U)$.*

Proof. Choose a 1- or 2-cut U of $\tilde{G}$ and the fragment S for U such that S is minimal with respect to inclusion among all the fragments of $\tilde{G}$. Note that S contains neither a 1-cut nor a vertex of any 2-cut of $\tilde{G}$.

Let $\tau\in A$ be any covering translation. Since τ is an automorphism of $\tilde{G}$, $\tau(U)$ is also a cutset of $\tilde{G}$ and $|U|=|\tau(U)|$. So we consider how U and $\tau(U)$ are placed in $\tilde{G}$. Considering the minimality of S and the property of U, as mentioned above, we can decide the location of the vertices and we get seven cases shown in Fig. 2.1. For example, in the first case, that is, when U is a

1-cut and $\tau(U)\neq U$, $\tau(S)\cap S=\emptyset$ because $\tilde{G}$ is connected.

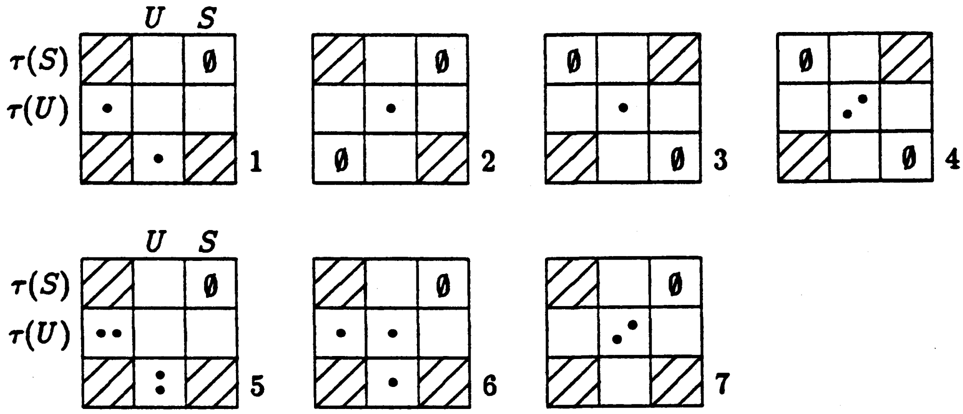

Fig. 2.1.

In each of there cases either $\tau(S)\cap S=\emptyset$ or $\tau(S)=S$. Thus (1) follows. Note that neither of the vertices in $\tau(U)$ lies in S.

Let $K=\langle S\cup U\rangle$ with S being the same minimal fragment as chosen above and suppose K has either a 1- or 2-cut U_K.

If $U=\{u\}$, U_K is also a cutset of $\tilde{G}$. This contradicts the minimality of S. So U must be a 2-cut $\{u, v\}$. Let $U_S=S\cap U_K$. Since $\langle S\rangle$ is connected, $U_S\neq\emptyset$ and U_S is not a cutset of $\tilde{G}$ by the minimality of S. If U_S is a 1-cut u', $\{u, u'\}$ is a 2-cut of $\tilde{G}$. This contradicts the minimality of S, so $U_S=U_K$.

Let the components of $\langle S-U_S\rangle$ be $W_1, W_2, \cdots, W_k$. Then for $i=1, 2, \cdots, k$, W_i has at least one vertex which belongs to either $N(u)$ or $N(v)$ in K because of the minimality of S. So U_S is not a cutset of $K+uv$. Thus (2) follows.

Choose a vertex $\tilde{y}\in S$ and fix it. Set $y=p(\tilde{y})$. Assume that $G-p(U)$ is connected. Then there is $y-x$ path Q_x in $G-p(U)$ for any vertex $x\in V(G)-p(U)$. Starting from $\tilde{y}$ and naturally tracing edges in $\tilde{G}$ which project to the edges on Q_x, we can get the path lift $\tilde{Q}_x$ of Q_x and its terminal vertex $\tilde{x}$ projects to x. Since Q_x does not meet $p(u)$ and $p(v)$, its lift $\tilde{Q}_x$ is a path in $\langle S\rangle$. Therefore, for any vertex $x\in V(G)-p(U)$, there is a vertex $\tilde{x}\in S$ such that $p(\tilde{x})=x$. This implies that $V(G)=p(S)\cup p(U)$. Thus (3) follows. $\square$

Lemma 2.2. *Let $\tilde{G}$, p, and U be as in Lemma* 2.1. *If $\tilde{G}$ is a planar branched covering with $|V(\tilde{G})|$ minimal, $p(U)$ must be a cutset of G.*

Proof. Suppose $p(U)$ is not a conset of G by Lemma 2.1, $\tilde{G}$ has an A-equivariant fragment S and $V(G)=p(S)\cup p(U)$.

Here we consider a subgroup B of A which acts as an automorphism group of $K=\langle S\cup U\rangle$, that is, $A\supset B=\{\tau\,|\,\tau(K)=K\}$.

1) When $U=\{u\}$, let $K'=K$.

2) When $U=\{u, v\}$ and $\tau(u)=v$ for some $\tau\in B$, let $K'=K$.

3) When $U=\{u, v\}$ and $\tau(u)=v$ for some $\tau\in A-B$, let K' be the graph obtained from K by identifying u with v.

4) When $U=\{u, v\}$ and $\tau(u)\neq v$ for all $\tau\in A$ and $uv\notin E(\tilde{G})$ and $p(u)p(v)\in E(G)$, let $K'=K+uv$.

5) When $U=\{u, v\}$ and $\tau(u)\neq v$ for all $\tau\in A$ and $uv\notin E(\tilde{G})$ and $p(u)p(v)\notin E(G)$, let $K'=K$.

6) When $U=\{u, v\}$ and $\tau(u)\neq v$ for all $\tau\in A$ and $uv\in E(\tilde{G})$ and $p(u)p(v)\in E(G)$, let $K'=K$.

In each case of 1)~6), it is clear that we still have the same quotient $K'/B=G$. So K' is a regular branched covering of G, which contradicts the minimality of $|V(\tilde{G})|$. □

3. Proof of the theorem

Suppose that $\tilde{G}$ is a finite regular branched covering of G and that $\tilde{G}$ is planar. Let $p:\tilde{G}\to G$ be the branched covering projection and $A\subset Aut(\tilde{G})$ the covering translation group.

When $\tilde{G}$ is 3-connected, the theorem follows fairly immediately from Whitney's theorem [5]:

Theorem 3.1 (Whitney [5]). *A 3-connected planar graph can be embedded in a sphere such that each automorphism of the graph extends to a homeomorphism of the sphere onto itself.*

We call this embedding a *faithful* embedding [1].

Embed $\tilde{G}$ faithfully in a sphere S^2, and the translation group $A\subset Aut(\tilde{G})$ can be regarded as a group consisting of some self-homeomorphism $h: S^2\to S^2$ such that $h(\tilde{G})=\tilde{G}$. Thus we can embed $\tilde{G}/A$ in S^2/A.

So the following diagram holds.

$$\begin{tikzcd}
\tilde{G} \arrow[r, "\text{faithful embedding}"] \arrow[d] & S^2 \arrow[d] \\
\tilde{G}/A = G \arrow[r, "\text{embedding}"] & S^2/A
\end{tikzcd}$$

Fig. 3.1.

S^2/A is the quotient of S^2 by the action of A, called an elliptic 2-orbifold in [4], and its underlying space is homeomorphic to S^2, D^2 or P^2. Since $G=\tilde{G}/A$, G can be embedded in a sphere or a projective plane.

This completes the proof when $\tilde{G}$ is 3-connected.

The rest of this paper is for the proof when $\tilde{G}$ is not 3-connected. We use induction on the number of vertices of G.

When $|V(G)|\leqq 6$, G is either planar or projective-planar. For the complete graph K_6 can be embedded in a projective plane, as depicted in Fig. 3.2.

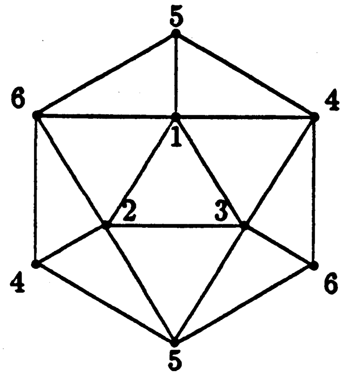

Fig. 3.2.

Next we divide the case into two; when $\delta(\tilde{G})\geqq 3$ and when otherwise. When $\delta(\tilde{G})\geqq 3$, we suppose any graph with finite planar branched covering that has fewer vertices than G is either planar or projective-planar.

First we modify $K=\langle S\cup U\rangle$ and $p(K)$ to prove that $p(K)$ is planar.

(1) When $U=\{u\}$, let $K'=K$ and $G_2=p(K)$. Then $p'=p|_{K'}: K'\to G_2$ is a branched covering with the covering translation group $B=\{\tau|_{K'}: \tau(K)=K, \tau\in A\}$. By Lemma 2.1, K' is 3-connected and so K' can be faithfully embedded in the sphere S^2 by Theorem 3.1. Since $\tau(U)\cap S=\emptyset$ for each $\tau\in B$ (this is obvious when we check the shape of $\tilde{G}$ described in the proof of Lemma 2.1), u is a fixed point of all $\tau\in B$ and then S^2/B is an elliptic 2-orbifold with elliptic points. The underlying space of such 2-orbifolds is homeomorphic to D^2 or S^2 and so G_2 is planar.

(2) When $U=\{u, v\}$ and $\tau(u)\neq v$ for all $\tau\in A$, let $K'=K+uv$ and $G_2=p(K)+p(u)p(v)$. Then $p'=p|_{K'}: K'\to G_2$ is a branched covering with the covering translation group $B=\{\tau|_{K'}: \tau(K)=K, \tau\in A\}$. By Lemma 2.1, K' is 3-connected and so K' can be faithfully embedded in the sphere S^2 by Theorem 3.1. Since $\tau(u)\notin S$ for each $\tau\in B$, uv is a set of fixed points with respect to B and then S^2/B is an elliptic 2-orbifold with a mirror. The underlying space of such 2-orbifolds is homeomorphic to D^2 and so G_2 is planar.

(3) When $U=\{u, v\}$ and $\tau(u)=v$ for some $\tau\in A$ and $\tau(u)\neq v$ for all $\tau\in B$, let $K'=K+uv$ and G_2 be the graph obtained from $p(K)$ by splitting the vertex $p(u)(=p(v))$. (Fig. 3.3 shows an example of splitting a vertex.) Then $p'=p|_{K'}: K'\to G_2$ is a branched covering with the covering translation group $B=\{\tau|_{K'}|\tau(K)=K, \tau\in A\}$. By Lemma 2.1, K' is 3-connected and so K' can be faithfully embedded in the sphere S^2 by Theorem 3.1. Since $\tau(u)\notin S$ for each

$\tau \in B$, uv is a set of fixed points with respect to B and then S^2/B is an elliptic 2-orbifold with a mirror. The underlying space of such 2-orbifolds is homeomorphic to D^2 and so G_2 is planar. Since G_2 was obtained from $p(K)$ by splitting the vertex $p(u)$, $p(K)$ is clearly planar.

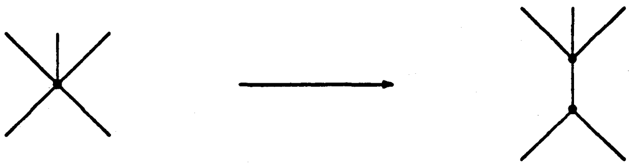

Fig. 3.3.

(4) When $U=\{u, v\}$ and $\tau(u)=v$ for some $\tau \in B$, let $G_2=p(K)$. Then $p'=p|_K: K \to G_2$ is a branched covering with the covering translation group $B=\{\tau|_K: \tau(K)=K, \tau \in A\}$. By Lemma 2.1, $K+uv$ is 3-connected and B is still an automorphism group of $K+uv$, so we embed $K+uv$ faithfully in the sphere S^2. Since $\tau(u) \notin S$ for each $\tau \in B$, the middle point of uv is the only fixed point on uv. In the same manner as the case (1), $(K+uv)/B$ is planar. Then clearly G_2 is planar.

In any case G_2 is planar. Let $G_1=G-p(S)+p(u)p(v)$. If $p(u)=p(v)$, let $G_1=G-p(S)$. Then $G=G_1 \cup G_2$ or $G_1 \cup G_2 - p(u)p(v)$.

By induction hypothesis, G_1 is either planar or projective-planar. Therefore we can easily embed G in a plane or projective plane.

$\boxed{\text{This completes the proof when } \delta(\tilde{G}) \geqq 3.}$

When $\tilde{G}$ is not 3-connected and $\delta(\tilde{G})<3$, we will show that if a planar graph $\tilde{G}$ is a regular branched covering of G, the graph $\tilde{G}'$, which is obtained from $\tilde{G}$ by deleting all the vertices that have degree 1 or 2 in $\tilde{G}$, is also a regular branched covering of G' which is obtained from G by deleting all the images of the deleted vertices of $\tilde{G}$. Then we can reduce this case to the previous case.

Let $p'=p|_{\tilde{G}'}$ and $A'=\{\tau|_{\tilde{G}'}: \tau \in A\}$ in the following arguments (1) and (2), and we show how to get $\tilde{G}'$ and G'.

(1) Suppose that $deg_{\tilde{G}}(u)=1$ for some vertex $u \in V(\tilde{G})$.

Let $\tilde{G}'$ be

$$\tilde{G} - \bigcup_{\tau \in A} \tau(u)$$

and G' be $G-p(u)$. Then $\tilde{G}'$ is planar and is a regular branched covering of $G'=\tilde{G}'/A$ as depicted.

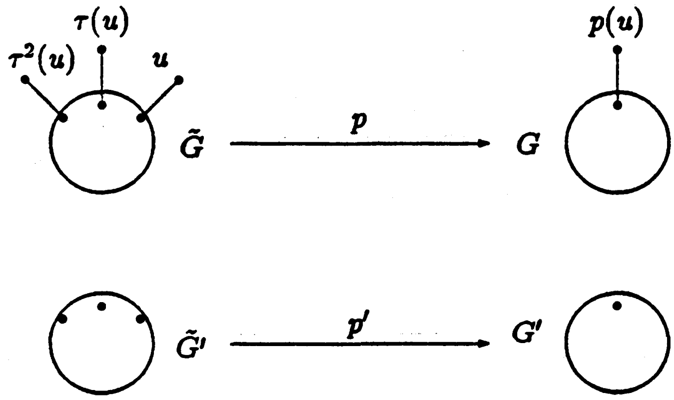

Fig. 3.4.

(2) Suppose that $deg_{\tilde{G}}(u)=2$ and u is adjacent to x and y.

a) If x is not adjacent to y and if $\tau(x)\neq y$ for any $\tau\in A$, let $\tilde{G}'$ be

$$\tilde{G}-\bigcup_{\tau\in A}\tau(u)+\bigcup_{\tau\in A}\tau(x)\tau(y)$$

and G' be $G-p(u)+p(x)p(y)$.

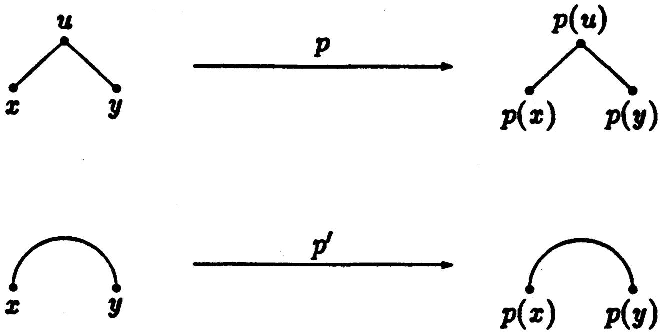

Fig. 3.5.

b) If x is not adjacent to y and if $\tau(x)=y$ for some $\tau\in A$, let $\tilde{G}'$ and G' be the same as (1).

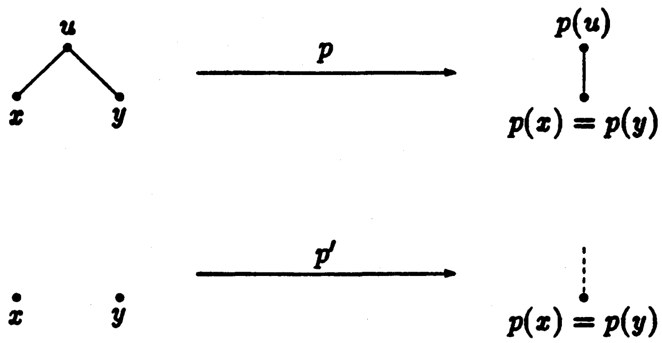

Fig. 3.6.

c) If x is adjacent to y, let $\tilde{G}'$ and G' be also the same as (1).

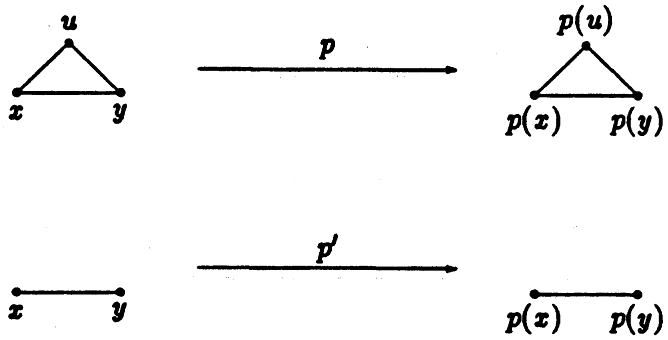

Fig. 3.7.

In each case of a), b), and c), $\hat{G}'$ is a regular branched covering of G'.

By induction hypothesis G' is either planar or projective-planar. Clearly we can get a planar or projective-planar embedding of G by adding the vertex $p(u)$, which has degree at most 2.

This completes the proof when $\delta(\tilde{G})<3$.

References

[1] S. Negami: *Uniqueness and faithfulness of embedding of toroidal graphs*, Discrete Math., **44** (1983), 161-180.
[2] S. Negami: *Enumeration of projective-planar embeddings of a graph*, Discrete Math., **62** (1986), 299-306.
[3] S. Negami: *The spherical genus and virtually planar graphs*, Discrete Math., **70** (1988), 159-168.
[4] W. Thurston: *The geometry and topology of 3-manifolds*. Lecture notes, Dept. of Math., Princeton Univ., Princeton, New Jersey, 1977.
[5] H. Whitney: *Congruent graphs and the connectivity of graphs*, Amer. J. Math., **54** (1932), 150-168.

Department of Information Science
Tokyo Institute of Technology
Oh-Okayama, Meguro-ku,
Tokyo, 152, Japan