YOKOHAMA MATHEMATICAL
JOURNAL VOL. 38, 1990

FREE INVOLUTIONS ON CERTAIN 3-MANIFOLDS

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(Received April 24, 1989)

1.

The orbit spaces of a free involution on $S^1\times S^2$ were first classified by Tao [T]. Tollefson [To] classified orbit spaces of connected sums of 3-manifolds where each factor is irreducible. In [K-T] a general structure theorem was found for nonprime 3-manifolds admitting involutions, with applications to nonprime manifolds with no 2-sphere bundle summands. In this paper we describe the orbit spaces of free involutions on connected sums of 2-sphere bundles (Theorem 4) and on connected sums of 3-manifolds where each summand is a 2-sphere bundle or irreducible with finite fundamental group (Theorem 5).

Let $T: N\rightarrow N$ be a free involution. The orbit space N/T is denoted by N^*. A 2-sphere S in N is *equivariant* if $T(S)=S$ or $T(S)\cap S=\emptyset$; and S is *invariant* if $T(S)=S$. The complement of the interior of a regular neighborhood of S in N is the manifold *N cut along S*. A *punctured* 3-*cell* is obtained from the 3-cell B^3 by removing open cells from $\mathrm{Int}\,B^3$. By P^n we denote real projective n-space (for $n=2, 3$). By H we denote an S^2-bundle over S^1.

Given a 3-manifold M, denote by M' (resp. M'') the 3-manifold obtained by deleting one (resp. two) open 3-balls from $Int\,M$, and call the resulting boundary spheres of M the distinguished 2-spheres. Recall that $M_1\#M_2=M_1'\cup M_2'$ where the union is along a sphere of $\partial M_i'$ and that $M\#H$ is obtained from M'' by identifying its distinguished spheres (see e.g. [He]). Note that if the free involution $T: M''\rightarrow M''$ interchanges the two distinguished spheres then T can be extended to a free involution $M\rightarrow M$ and $(M'')^*=(M^*)'$.

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Lemma 1. *Let N be a 3-manifold that contains a 2-sphere not bounding a punctured 3-cell in N. Let T be a free involution. Then N contains an equivariant 2-sphere S not bounding a punctured 3-cell. Furthermore, if N contains a nonseparating 2-sphere then N contains an equivariant nonseparating 2-sphere.*

This is a generalization of Lemma 1 of [To]. The proof is similar to the proof in [To] and the proof of Lemma 4 of [H].

Proposition 2. *Let N be a 3-manifold that contains a nonseparating 2-sphere and let $T: N \to N$ be a free involution. Let H denote a S^2-bundle over S^1. Then N and N^* admit one of the structures* (a)-(e).

(a) $N = M \# H$ *and* $N^* = M^* \# P^3$.

(b) $N = M \# M \# H$ *and* $N^* = M \# H$.

(c) $N = M_1 \# M_2 \# H$ *and* $N^* = M_1^* \# M_2^*$

(d) $N = M \# H \# H$ *and* $N^* = M^* \# H$

(e) $N = M \# H$, *the two distinguished boundary spheres of M'' are invariant under T, and N^* is obtained from $(M'')^*$ by identifying the two projective planes of $\partial(M'')^*$.*

Proof. By Lemma 1 there is a nonseparating equivariant 2-sphere S.

Case (1). $S \cap T(S) = \emptyset$.

(i) Suppose $S \cup T(S)$ bounds a submanifold $Q \approx S^2 \times I$ in N. Let $M'' = N - \operatorname{Int} Q$. Then $N \approx M \# H$. If $T(M'') = M''$ then $(M'')^* = (M^*)'$ and by filling in the boundary spheres of Q with 3-balls we can extend T to a free involution on S^3. Hence $Q^* \approx (P^3)'$ and $N^* = (M^*)' \cup (P^3)' = M^* \# P^3$. This is case (a) of the Proposition. If T interchanges Q and M'' then $N \approx H$ and N^* is obtained from Q by identifying S and $T(S)$. Thus $N^* \approx H$, which is case (b) with $M = S^3$.

(ii) Suppose S is not parallel to $T(S)$ and $S \cup T(S)$ separates N into M_1'' and M_2''. Identifying M_1'' and M_2'' along S we obtain $N_1'' \approx M_1' \# M_2'$ and identifying M_1' and M_2' along $T(S)$ we obtain $N \approx N_1 \# H \approx M_1 \# M_2 \# H$. If $T(M_i'') = M_i''$ then $N^* = (M_1'')^* \cup (M_2'')^* = (M_1^*)' \cup (M_2^*)' = M_1^* \# M_2^*$ and we get case (c). If $T(M_1'') = M_2''$ then N^* is obtained from M_1'' by identifying the two boundary spheres S and $T(S)$ of M_1''. Hence $N^* \approx M_1 \# H$, which is case (b).

(iii) Suppose S is not parallel to $T(S)$ and $S \cup T(S)$ does not separate N. Let M'''' be N cut along $S \cup T(S)$. Then $N \approx M \# H \# H$ and N^* is obtained from $(M^*)''$ by identifying the two copies of $S^* = p(S \cup T(S))$ in $\partial(M^*)''$. Hence $N^* = M^* \# H$ which gives (d) of the Proposition.

Case (2). $S = T(S)$.

Let U be a regular invariant neighborhood of S and let $M'' = N - \mathit{Int}\, U$. If T interchanges the components of ∂U we get case 1(i). Otherwise $U^* \approx P^2 \times I$ and we get N^* as in (e).

Remark. In case (b) the S^2-bundles H need not be the same, e.g. it could mean the orientable one in N and the nonorientable one in N^*.

Applying this proposition to the connected sum of S^2-bundles we obtain the following lemma.

Lemma 3. *Let M_n be a connected sum of n S^2-bundles over S^1 and let $T: M_n \to M_n$ be a free involution. Let H denote a S^2-bundle over S^1. Then for $n \geqq 2$ one of* (a)-(d) *below holds:*

(a) $M_n = M_{n-1} \# H$ *and* $M_n^* = M_{n-1}^* \# P^3$.

(b) $M_n = M_{n-1} \# H$ *and* $M_n^* = M_{k+1}$, *where* $2k = n-1$.

(c) $M_n = M_i \# M_j \# H$ *and* $M_n^* = M_i^* \# M_j^*$ *with* $i+j = n-1$.

(d) $M_n = M_{n-2} \# H \# H$ *and* $M_n^* = M_{n-2}^* \# H$.

Proof. By uniqueness of the number of S^2-bundle factors of M_n, cases (a)-(d) of Prop. 2 yield (a)-(d) of the lemma. In case (e) of Prop. 2 the manifold M_n is obtained from M''_{n-1} by identifying the two boundary spheres and $(M''_{n-1})^*$ by identifying the two projective plane boundaries. Since $n \geqq 2$ there is an equivariant nonseparating 2-sphere S in M''_{n-1}, by Lemma 1. If $T(S) \cap S = \emptyset$ or if S is invariant and interchanges the boundary components of a regular neighborhood U of S then cases (a)-(d) of Prop. 2 (and hence of the lemma) apply. Thus assume $T(S) = S$ and S does not interchange ∂U. Then M''_{n-1} cut along S is $(M''_{n-2})''$ which is invariant under T. Proceeding in this way we either obtain cases (a)-(d) or we end up with an invariant submanifold M_0 which is obtained by cutting M_n along n mutually disjoint non-separating spheres and $\hat{M}_0 \approx S^3$. Since each of the $2n$ boundary spheres of M_0 is invariant, M_0 covers a nonorientable 3-manifold with fundamental group Z_2 and $2n$ projective planes as boundary. This cannot happen for $n > 1$, by [E].

We now adopt the following notational convention. K denotes either an S^2-bundle over S^1 or $P^2 \times S^1$. The symbol $\underset{m}{\#} P^3 \underset{n}{\#} K$ denotes a connected sum of m factors of P^3 and n factors each of which is a S^2-bundle or $P^2 \times S^1$.

Theorem 4. *Let M_n be a connected sum of n S^2-bundles over S^1 and let $T: M_n \to M_n$ be a free involution. Then $M_n^* = \underset{n+1-2k}{\#} P^3 \underset{k}{\#} K$ for some k, with $0 \leqq k \leqq \frac{n}{2}$ for n even and $0 \leqq k \leqq \frac{n+1}{2}$ for n odd.*

Proof. For $n=1$, M_1^* is $\underset{2}{\#} P^3$ or K, by [T]. For $n=2$ we apply Lemma 3 to obtain $M_2^* = M_1^* \# P^3$ (hence $M_2^* = \underset{3}{\#} P^3$ or $M_2^* = P^3 \# K$).

The general case follows from Lemma 3 by straight forward induction. We illustrate the case when $n+1=m$ is even and Lemma 3(c) applies: $M_m^* = M_i^* \# M_j^*$ with $i+j=n$ and we can assume that i is odd, $0 < i \leqq n$, and j is even, $0 \leqq j < n$. By induction $M_i^* = \underset{i+1-2k}{\#} P^3 \underset{k}{\#} K$ for some k with $0 \leqq k \leqq \frac{i+1}{2}$

and $M_j^* = \underset{j+1-2l}{\#} P^3 \underset{l}{\#} K$ for some l with $0 \leqq l \leqq \frac{j}{2}$. Thus $M_m^* = \underset{i+j+2-2k-2l}{\#} P^3 \underset{k+l}{\#} K = \underset{m+1-2s}{\#} P^3 \underset{s}{\#} K$ for s with $0 \leqq s \leqq \frac{m}{2}$.

It is clear that conversely any (orientable) 2-fold covering of the manifold M_n^* given by the Theorem is homeomorphic to M_n.

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Now let $\underset{n}{\#} H$ denote a connected sum of n factors, each homeomorphic to an S^2-bundle over S^1.

Theorem 5. *Let N be a closed 3-manifold that contains no fake 3-cells and such that every irreducible factor of the prime decomposition of N has finite fundamental group. Let $T: N \to N$ be a free involution. Then there are prime manifolds A_i, B_j such that*

$$N \approx (A_1 \# \cdots \# A_r) \# (B_1 \# \cdots \# B_s \underset{s-1}{\#} H) \# (A_1 \# \cdots \# A_r) \text{ and}$$

$$N^* \approx (A_1 \# \cdots \# A_r) \# (B_1^* \# \cdots \# B_s^*)$$

Remark. Some of the B_i's may be homeomorphic to S^3 (in which case $B_i^* \approx P^3$).

Proof. Let k be the number of 2-spheres of a complete system of pairwise disjoint incompressible 2-spheres in N (see [Ha]). If N contains no nonseparating 2-sphere then the Theorem follows from the Theorem of [To] (with $s=1$). Thus we assume that N contains nonseparating 2-spheres and proceed by induction on k. (For $k=0$ we have $r=0$ and $s=1$). Denote $A_1 \# \cdots \# A_r$ by $A(r)$, $B_1 \# \cdots \# B_s$ by $B(s)$ and $B_1^* \# \cdots \# B_s^*$ by $B_*(s)$. Consider the cases of Prop. 2:

(a) $N \approx M \# H$, $N^* \approx M^* \# P^3$. Applying induction to M and M^* we obtain $N \approx A(r) \# (B(s) \# S^3 \underset{s}{\#} H) \# A(r)$ and $N^* \approx A(r) \# B_*(s+1)$.

(b) $N \approx M \# M \# H$, $N^* = M \# H$. Write $N \approx A(r) \# H \# A(r)$ and $N^* \approx A(r) \# H^*$.

(c) and (d) follow similarly.

(e) N is obtained from a manifold M_1 by identifying its two invariant boundary spheres S_{11}, S_{12} and N^* is obtained by identifying the two boundary projection planes of M_1^*. If $\hat{M}_1$ is irreducible then since $\pi_1(\hat{M}_1)$ is finite, it follows from [E] that $M_1^* \approx P^2 \times I$ and $N \approx H$. If $\hat{M}_1$ is not irreducible there is by Lemma 1 an equivariant 2-sphere S that does not bound a punctured 3-cell in M_1.

(i) S separates M_1 and $T(S) \cap S = \emptyset$. M_1 cut along $S \cup T(S)$ consists of 3 components Q_1, Q_2, Q_3 with $S\cup T(S)$ in Q_3. Then S_{11}, S_{12} are in Q_3, T leaves Q_3 invariant and interchanges Q_1 and Q_2. Thus $N\approx N_1\#N_2\#N_1$, where $N_1\approx\hat{Q}_1$, $N_2\approx\hat{Q}_3\#H$, and $N^*\approx N_1\#N_2^*$. Every irreducible factor of N_2 has finite fundamental group and the Theorem follows by induction applied to N_2.

(ii) S separates M_1, $T(S)=S$, and T interchanges the boundary components of an invariant neighborhood of S. This case cannot occur since M_1 contains invariant spheres S_{11}, S_{12}.

(iii) $T(S)=S$ and either S does not separate M_1 or S separates M_1 and T does not interchange sides of S. Let M_2 denote either component of M_1 cut along S. If $\hat{M}_2$ is irreducible then since $\pi_1(\hat{M}_2)$ is finite, it follows from [E] that $M_2\approx S\times I$ hence S separates M_1 and bounds a punctured 3-cell in M_1 which is not true. Thus $\hat{M}_2$ is not irreducible. Continuing this process of cutting along equivariant 2-spheres we eventually must get case (i) for M_n which is a component of N cut along n essential 2-spheres. Thus there is a separating S in M_n, $S\cap T(S)=\emptyset$, and all boundary spheres of M_n are invariant. So $S\cup T(S)$ separates M_n into Q_1, Q_2, Q_3 where Q_3 is invariant, T interchanges Q_1 and Q_2, and $\partial M_n\subset Q_3$. Thus $N\approx N_1\#N_2\#N_1$ with $N_1\approx Q_1$ and N_2 is obtained from Q_3 and other components of N cut along 2-spheres by identifying invariant boundary components in pairs; and $N^*\approx N_1\#N_2^*$. As before the Theorem follows by induction.

As an example note that Theorem 5 applies to a connected sum of lens spaces (including $S^1\times S^2$). In [M] it was shown that the orbit space of a free involution T on a lens space (different from $S^1\times S^2$) is a Seifert fiber space.

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