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HOLOMORPHIC TANGENT BUNDLES AND NORMAL BUNDLES OF COMPLEX SUBMANIFOLDS OF COMPLEX PROJECTIVE SPACES

Dedicated to Professor Kisuke Tsuchida on his 70th birthday

By

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Summary. It is well known that the holomorphic tangent bundle and normal bundle of a complex submanifold of the complex Euclidean space are holomorphically isomorphic to pullbacks of the universal subbundle and quotient bundle, respectively, over the complex Grassmannian by means of its Gauss mapping. For a complex submanifold of the complex projective space, we shall prove a result (Theorem 5.1) corresponding to this fact.

1. Preliminaries

1.1. Notations. Let $\boldsymbol{C}^n$ be the complex vector space of n-dimensional complex column vectors and $M=M(n, m; \boldsymbol{C})$ the complex vector space of complex matrices of type (n, m). Set $gl(n, \boldsymbol{C})=M(n, n; \boldsymbol{C})$. Denote the n-dimensional unitary group by $U(n)$. For $A\in M(n, m; \boldsymbol{C})$, A^* is a transposed-conjugate matrix of A. For $A=(a_{ij})\in M(m, p; \boldsymbol{C})$ and $B=(b_{\mu\nu})\in M(n, q; \boldsymbol{C})$, $A\otimes B$ means a matrix

$$\begin{pmatrix} Ab_{11} & \cdots & \cdots & Ab_{1q} \\ \vdots & & & \vdots \\ \vdots & \cdots & Ab_{\mu\nu} \cdots & \vdots \\ \vdots & & & \vdots \\ Ab_{n1} & \cdots & \cdots & Ab_{nq} \end{pmatrix} \in M(mn, pq; \boldsymbol{C}),$$

where $a_{ij}b_{\mu\nu}$ is the $(m(\mu-1)+i,\ p(\nu-1)+j)$-component of $A\otimes B$. In particular, we have $b\otimes a\in \boldsymbol{C}^{nm}$ for $b\in \boldsymbol{C}^n$ and $a\in \boldsymbol{C}^m$. Hence we may identify the tensor product $\boldsymbol{C}^n\otimes \boldsymbol{C}^m$ of complex vector spaces $\boldsymbol{C}^n$ and $\boldsymbol{C}^m$ with the complex vector

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space $\boldsymbol{C}^{nm}$. Let $u_1, \cdots, u_m$ be column vectors in $\boldsymbol{C}^n$. Then the correspondence

$$(u_1, \cdots, u_m) \longmapsto \begin{pmatrix} u_1 \\ \vdots \\ u_m \end{pmatrix}$$

define a linear $\boldsymbol{C}$-isomorphism $\chi: M(n, m; \boldsymbol{C}) \to \boldsymbol{C}^{nm}$. For $B \in M(q, n; \boldsymbol{C})$, $A \in M(p, m; \boldsymbol{C})$ and $Z \in M(n, m; \boldsymbol{C})$, the equality

$$(B \otimes A)\chi(Z) = \chi(BZ\,{}^tA) \tag{1.1}$$

holds. In particular, we have the equality $b \otimes a = \chi(b\,{}^ta)$ for $b \in \boldsymbol{C}^n$ and $a \in \boldsymbol{C}^m$.

1.2. Complex derivatives. Let D be an open set in $\boldsymbol{C}^m$. For a C^∞-mapping $f: D \to \boldsymbol{C}^n$, the derivative $(df)_z: \boldsymbol{C}^m \to \boldsymbol{C}^n$ of f at $z \in D$ is a real linear mapping defined by

$$(df)_z(a) = \frac{d}{dt} f(z+ta)\Big|_{t=0}.$$

Now for the above mapping f, at $z \in D$, we define a complex linear mapping $(\partial f)_z: \boldsymbol{C}^m \to \boldsymbol{C}^n$, a conjugate linear mapping $(\bar\partial f)_z: \boldsymbol{C}^m \to \boldsymbol{C}^n$ by

$$(\partial f)_z(a) = \frac{1}{2}\{(df)_z(a) - \sqrt{-1}(df)_z(\sqrt{-1}a)\}, \tag{1.2}$$

$$(\bar\partial f)_z(a) = \frac{1}{2}\{(df)_z(a) + \sqrt{-1}(df)_z(\sqrt{-1}a)\}, \tag{1.3}$$

respectively. Then we have

$$(df)_z = (\partial f)_z + (\bar\partial f)_z, \tag{1.4}$$

$$(\partial f)_z(e_i) = \frac{\partial f}{\partial z^i}, \quad (\bar\partial f)_z(e_i) = \frac{\partial f}{\partial \bar z^i} \quad (1 \leqq i \leqq m) \tag{1.5}$$

for $z = {}^t(z^1, \cdots, z^m) \in D$, where e_i is the i-th member of the canonical basis of $\boldsymbol{C}^m$. It follows from these equalities that f is holomorphic in D if and only if $(\bar\partial f)_z = 0$ or $(df)_z = (\partial f)_z$ for each point $z \in D$. The following proposition is easily shown.

Proposition 1.1. (Chain rule of complex derivatives) *Let D, D' be open sets in $\boldsymbol{C}^m$, $\boldsymbol{C}^n$, respectively and $f: D \to \boldsymbol{C}^n$, $g: D' \to \boldsymbol{C}^p$ C^∞-mappings such that $f(D) \subset D'$, then the composed mapping $g \circ f: D \to \boldsymbol{C}^p$ is a C^∞-mapping and equalities:*

(i) $$(\partial(g \circ f))_z = (\partial g)_{f(z)} \circ (\partial f)_z + (\bar\partial g)_{f(z)} \circ (\bar\partial f)_z,$$

(ii) $$(\bar\partial(g \circ f))_z = (\bar\partial g)_{f(z)} \circ (\partial f)_z + (\partial g)_{f(z)} \circ (\bar\partial f)_z$$

for $z \in D$ hold.

2. Grassmannians and vector bundles over them

2.1. Grassmannians. Let $G_{N,k}$ be the complex Grassmannian of all linear subspaces of complex dimension $k+1$ in C^{N+1} $(0 \leqq k < N)$. The Grassmannian $G_{N,0}$ is equal to the N-dimensional complex projective space P_N. We may think of $G_{N,k}$ as the set of k-dimensional projective subspaces of P_N. Set

$$V'_{n,m} = \{X \in M(m+m,\, m\,;\, C) \mid X^*X \in GL(m,\, C)\}.$$

The space $V'_{n,m}$ is called the complex Stiefel manifold of all m-frames in C^{m+n}. For each $(k+1)$-frame $X \in V'_{N-k,k+1}$, we denote a linear subspace of complex dimension $k+1$ in C^{N+1} spanned by all column vectors of the matrix X by $[X] \in G_{N,k}$. Set

$$M_{n,m} = \{P \in gl(m+n,\, C) \mid P^* = P,\ P^2 = P \text{ and trace } P = m\}.$$

We define a mapping $\rho : V'_{n,m} \to M_{n,m}$ by $\rho(X) = X(X^*X)^{-1}X^*$. Moreover we define a mapping $\tilde{\rho} : G_{N,k} \to M_{N-k,k+1}$ by

$$\tilde{\rho}([X]) = \rho(X) = X(X^*X)^{-1}X^* \tag{2.1}$$

for each $X \in V'_{N-k,k+1}$. Then it is easily shown that $\tilde{\rho}$ is bijective. We identify $G_{N,k}$ with $M_{N-k,k+1}$ by $\tilde{\rho}$. Now we shall introduce a holomorphic structure to $M_{n,m}$. Let $P_0 \in M_{n,m}$ be an arbitrary point. There exists $A \in U(m+n)$ such that

$$P_0 = A\begin{pmatrix} E_m & O \\ O & O \end{pmatrix} A^{-1},$$

where E_m is the unit matrix of degree m. We define an into-homeomorphism $\psi_A : M(n,\, m\,;\, C) \to M_{n,m}$ by

$$\psi_A(Z) = \rho\left(A\begin{pmatrix} E_m \\ Z \end{pmatrix}\right) = A\begin{pmatrix} E_m \\ Z \end{pmatrix} L_Z^{-1}(E_m\ Z^*)A^{-1}, \tag{2.2}$$

where $L_Z = E_m + Z^*Z$ for $Z \in M(n,\, m\,;\, C)$. Set

$$U_A = \psi_A(M(n,\, m\,;\, C)), \qquad \phi_A = \psi_A^{-1} : U_A \longrightarrow M(n,\, m\,;\, C).$$

Then we have

$$U_A = \left\{P \in M_{n,m} \mid (E_m\ \ O)A^{-1}PA\begin{pmatrix} E_m \\ O \end{pmatrix} \in GL(m,\, C)\right\},$$

$$\phi_A(P) = (O\ \ E_n)A^{-1}PA\begin{pmatrix} E_m \\ O \end{pmatrix}\left\{(E_m\ \ O)A^{-1}PA\begin{pmatrix} E_m \\ O \end{pmatrix}\right\}^{-1}.$$

For $A,\, B \in U(m+n)$, homeomorphims $\phi_B \circ \phi_A^{-1} : \phi_A(U_A \cap U_B) \to \phi_B(U_A \cap U_B)$, $\phi_A \circ \phi_B^{-1}$:

$\phi_B(U_A \cap U_B) \to \phi_A(U_A \cap U_B)$ are described as

$$
(2.3) \qquad \begin{aligned} W &= \phi_B \circ \phi_A^{-1}(Z) = (b+dZ)(a+cZ)^{-1} \\ &= -(d^* - Zb^*)^{-1}(c^* - Za^*), \\ Z &= \phi_A \circ \phi_B^{-1}(W) = (d^* + c^*W)(a^* + b^*W)^{-1} \\ &= -(d - Wc)^{-1}(b - Wa), \end{aligned}
$$

respectively, where

$$
\begin{pmatrix} a & c \\ b & d \end{pmatrix} = B^{-1}A \in U(m+m); \quad a \in gl(m, \boldsymbol{C})
$$

Hence $\phi_B \circ \phi_A^{-1}$, $\phi_A \circ \phi_B^{-1}$ are holomorphic. Therefore we regard $\boldsymbol{M}_{n,m}$ as a complex manifold with a holomorphic structure

$$
\{(U_A, \phi_A) \mid A \in U(m+n)\}.
$$

For $A \in U(m+n)$, the above mapping ψ_A is a local parametrization near $P_0 = \phi_A(O)$ in $\boldsymbol{M}_{n,m}$. Hereafter, as a complex manifold, we identify $\boldsymbol{G}_{N,k}$ with $\boldsymbol{M}_{N-k,k+1}$ by $\tilde{\rho}$.

2.2. Universal bundles. Set

$$
(2.4) \qquad \begin{aligned} \gamma_{n,m} &= \{(P, a) \in \boldsymbol{M}_{n,m} \times \boldsymbol{C}^{m+n} \mid Pa = a\}, \\ \gamma_{n,m}^{\perp} &= \{(P, a) \in \boldsymbol{M}_{n,m} \times \boldsymbol{C}^{m+n} \mid Pa = 0\}, \\ \gamma_{n,m}^{*} &= \{(P, a) \in \boldsymbol{M}_{n,m} \times \boldsymbol{C}^{m+n} \mid {}^tPa = a\}. \end{aligned}
$$

Define mappings $\pi : \gamma_{n,m} \to \boldsymbol{M}_{n,m}$, $\pi^{\perp} : \gamma_{n,m}^{\perp} \to \boldsymbol{M}_{n,m}$, $\pi^* : \gamma_{n,m}^* \to \boldsymbol{M}_{n,m}$ by $\pi(P, a) = P$, $\pi^{\perp}(P, a) = P$, $\pi^*(P, a) = P$, respectively. Then $\gamma_{n,m}$, $\gamma_{n,m}^{\perp}$, $\gamma_{n,m}^*$ are holomorphic vector bundles over $\boldsymbol{M}_{n,m}$ of rank m, n, m with the following local trivialities $\omega_A : U_A \times \boldsymbol{C}^m \to \pi^{-1}(U_A)$, $\omega_A^{\perp} : U_A \times \boldsymbol{C}^n \to \pi^{\perp -1}(U_A)$, $\omega_A^* : U_A \times \boldsymbol{C}^m \to \pi^*(U_A)$ by

$$
(2.5) \qquad \begin{aligned} \omega_A(P, u) &= \left(P,\ A\begin{pmatrix} E_m \\ Z \end{pmatrix} u\right), \\ \omega_A^{\perp}(P, v) &= \left(P,\ A\begin{pmatrix} -Z^* \\ E_n \end{pmatrix} M_Z^{-1} v\right), \\ \omega_A^*(P, u) &= \left(P,\ \bar{A}\begin{pmatrix} E_m \\ \bar{Z} \end{pmatrix} {}^tL_Z^{-1} u\right), \end{aligned}
$$

respectively, where $Z = \phi_A(P)$, $L_Z = E_m + Z^*Z$ and $M_Z = E_n + ZZ^*$. $\gamma_{n,m}$ and $\gamma_{n,m}^{\perp}$ are called the universal subbundle and the universal quotient bundle, respectively.

Transition functions $g_{BA}: U_A \cap U_B \to GL(m, \boldsymbol{C})$, $g^{\perp}_{BA}: U_A \cap U_B \to GL(n, \boldsymbol{C})$, $g^{*}_{BA}: U_A \cap U_B \to GL(m, \boldsymbol{C})$ of $\gamma_{n,m}$, $\gamma^{\perp}_{n,m}$, $\gamma^{*}_{n,m}$ are given by

$$
\begin{aligned}
&g_{BA}(P) = a + cZ\,, \\
(2.6)\qquad &g^{\perp}_{BA}(P) = (d^* - Zb^*)^{-1}\,, \\
&g^{*}_{BA}(P) = {}^t(a + cZ)^{-1}\,,
\end{aligned}
$$

respectively, where $Z = \phi_A(P)$ and

$$\begin{pmatrix} a & c \\ b & d \end{pmatrix} = B^{-1}A \in U(m+n)\,;\ a \in gl(m, \boldsymbol{C})\,.$$

Moreover we have the following holomorphic short exact sequence:

$$(2.7)\qquad 0 \longrightarrow \gamma_{n,m} \xrightarrow{\alpha} \hat{\boldsymbol{C}}^{m+n} \xrightarrow{\beta} \gamma^{\perp}_{n,m} \longrightarrow 0\,,$$

where $\hat{\boldsymbol{C}}^{m+n} = \boldsymbol{M}_{n,m} \times \boldsymbol{C}^{m+n}$ (product bundle), α is a natural inclusion and β is written as $\beta(P, a) = (P, (E_{m+n} - P)a)$. $\gamma^{*}_{n,m}$ is a dual bundle of $\gamma_{n,m}$ and the dual bundle of $\gamma^{\perp}_{n,m}$ is described as $\gamma^{\perp *}_{n,m} = \{(P, a) \in \boldsymbol{M}_{n,m} \times \boldsymbol{C}^{m+n} \mid {}^tPa = 0\}$. Furthermore, from (2.7), the following holomorphic short exact sequence:

$$(2.8)\qquad 0 \longleftarrow \gamma^{*}_{n,m} \xleftarrow{\alpha^*} \hat{\boldsymbol{C}}^{m+n} \xleftarrow{\beta^*} \gamma^{\perp *}_{n,m} \longleftarrow 0$$

is obtained, where β^* is a naturol inclusion and α^* is given by $\alpha^*(P, a) = (P, {}^tPa)$. Now we define a biholomorphic mapping $\Delta: \boldsymbol{M}_{n,m} \to \boldsymbol{M}_{m,n}$ by

$$(2.9)\qquad \Delta(P) = E_{m+n} - {}^tP\,.$$

Then the local expression of Δ is an follows:

$$(2.10)\qquad \phi_{\hat{A}} \circ \Delta \circ \phi_{A}^{-1}(Z) = {}^tZ\,;\ \ \hat{A} = \bar{A}\begin{pmatrix} O & -E_m \\ E_n & O \end{pmatrix},$$

where (U_A, ϕ_A), $(U_{\hat{A}}, \phi_{\hat{A}})$ are local charts of $\boldsymbol{M}_{n,m}$, $\boldsymbol{M}_{m,n}$, respectively such that $\Delta(U_A) = U_{\hat{A}}$. Then the following proposition is easily shown.

Proposition 2.1. *The dual bundles* $\gamma^{*}_{n,m}$, $\gamma^{\perp *}_{n,m}$ *of* $\gamma_{n,m}$, $\gamma^{\perp}_{n,m}$ *are holomorphically isomorphic to pull-backs* $\Delta^{-1}\gamma^{\perp}_{m,n}$, $\Delta^{-1}\gamma_{m,n}$ *of* $\gamma^{\perp}_{m,n}$, $\gamma_{m,n}$, *respectively and by the isomorphism, the following diagram:*

$$
\begin{array}{ccccccccc}
0 \longleftarrow & \gamma^{*}_{n,m} & \xleftarrow{\ \alpha^*\ } & \hat{\boldsymbol{C}}^{m+n} & \xleftarrow{\ \beta^*\ } & \gamma^{\perp *}_{n,m} & \longleftarrow 0 \\
& \downarrow & & \downarrow & & \downarrow & \\
0 \longleftarrow & \Delta^{-1}\gamma^{\perp}_{m,n} & \xleftarrow{\ \Delta^{-1}\beta\ } & \hat{\boldsymbol{C}}^{m+n} & \xleftarrow{\ \Delta^{-1}\alpha\ } & \Delta^{-1}\gamma_{m,n} & \longleftarrow 0
\end{array}
$$

is commutative, where $\Delta^{-1}\alpha$, $\Delta^{-1}\beta$ *are homomorphisms induced by* $\alpha: \gamma_{m,n}\to\hat{C}^{m+n}$, $\beta: \hat{C}^{m+n}\to\gamma_{m,n}^{\perp}$.

2.3. Holomorphic tangent bundles of Grassmannians. Let $P\in \boldsymbol{M}_{n,m}$ be arbitrary point and (U_A, ϕ_A) the local chart defined in 2.1 such that $\phi_A(P)=O\in M(n, m; \boldsymbol{C})$. Let $\psi_A=\phi_A^{-1}$. Then we have

$$(d\psi_A)_Z(z)=X_Z(z)+X_Z(z)^* \tag{2.11}$$

$$(\partial\psi_A)_Z(z)=X_Z(z), \qquad (\bar\partial\psi_A)_Z(z)=X_Z(z)^*, \tag{2.12}$$

where $z\in M(n, m; \boldsymbol{C})$,

$$X_Z(z)=A\begin{pmatrix}-Z^*\\ E_n\end{pmatrix}M_Z^{-1}zL_Z^{-1}(E_m \ \ Z^*)A^{-1}, \tag{2.13}$$

$$L_Z=E_m+Z^*Z, \qquad M_Z=E_n+ZZ^*. \tag{2.14}$$

It follows from (2.11) that the real tangent space $T(\boldsymbol{M}_{n,m})_P$ of $\boldsymbol{M}_{n,m}$ at P is written as

$$T(\boldsymbol{M}_{n,m})_P=\{X\in gl(m+n, \boldsymbol{C})\,|\, X^*=X,\ XP+XP=X\}.$$

Moreover the almost complex structure $I_P: T(\boldsymbol{M}_{n,m})_P\to T(\boldsymbol{M}_{n,m})_P$ at $P\in\boldsymbol{M}_{n,m}$ is given by $I_P(X)=\sqrt{-1}[X, P]$, where $[X, P]=XP-PX$. We may regard the complexified tangent space $T(\boldsymbol{M}_{n,m})_P^{\boldsymbol{C}}$ of $\boldsymbol{M}_{n,m}$ at P as

$$T(\boldsymbol{M}_{n,m})_P^{\boldsymbol{C}}=\{X\in gl(m+n, \boldsymbol{C})\,|\, XP+PX=X\}.$$

Therefore the holomorphic tangent space $T(\boldsymbol{M}_{n,m})'_P$, the antiholomorphic tangent space $T(\boldsymbol{M}_{n,m})''_P$ of $\boldsymbol{M}_{n,m}$ at P are given by

$$T(\boldsymbol{M}_{n,m})'_P=\{X\in gl(m+n, \boldsymbol{C})\,|\, XP=X,\ PX=O\},$$

$$T(\boldsymbol{M}_{n,m})''_P=\{X\in gl(m+n, \boldsymbol{C})\,|\, XP=O,\ PX=X\},$$

respectively. Hence we may describe the holomorphic tangent bundle $\tau(\boldsymbol{M}_{n,m})$ of $\boldsymbol{M}_{n,m}$ as

$$\tau(\boldsymbol{M}_{n,m})=\{(P, X)\in \boldsymbol{M}_{n,m}\times gl(m+n, \boldsymbol{C})\,|\, XP=X,\ PX=O\}. \tag{2.15}$$

Complex derivatives

$$(\partial\psi_A)_Z: M\longrightarrow T(\boldsymbol{M}_{n,m})'_Q, \qquad (\bar\partial\psi_A)_Z: M\longrightarrow T(\boldsymbol{M}_{n,m})''_Q$$

are a complex linear isomorphism, a conjugate linear isomorphism, respectively, where $M=M(n, m; \boldsymbol{C})$ and $Q=\psi_A(Z)$. By (1.1), the equality

$$\chi((\partial\psi_A)_Z(z))=\chi(X_Z(z))=J(\psi_A)_Z\cdot\chi(z) \tag{2.16}$$

follows from (2.12) and (2.13), where

$$(2.17)\qquad J(\psi_A)_Z = A\begin{pmatrix} -Z^* \\ E_n \end{pmatrix} M_Z^{-1} \otimes \bar{A}\begin{pmatrix} E_m \\ \bar{Z} \end{pmatrix} {}^t L_{\bar{Z}}^{-1} \quad (\text{cf. } (2.5))$$

$$= (A\otimes\bar{A})\cdot\left(\begin{pmatrix} -Z^* \\ E_n \end{pmatrix}\otimes\begin{pmatrix} E_m \\ \bar{Z} \end{pmatrix}\right)\cdot(M_Z\otimes {}^tL_Z)^{-1}.$$

We think of the mapping $J(\psi_A)$: $M(n, m; \boldsymbol{C})=\phi_A(U_A)\to V'_{(m+n)^2-mn, mn}$ as a local frame field on U_A of the holomorphic tangent bundle $\tau(\boldsymbol{M}_{n,m})$. The complex derivative $\partial(\phi_B\circ\phi_A^{-1})_Z$: $M(n, m; \boldsymbol{C})\to M(n, m; \boldsymbol{C})$ of the holomorphic coordinate transformation $\phi_B\circ\phi_A^{-1}$ (cf. (2.3)) at $Z\in M(n, m; \boldsymbol{C})$ is given by

$$(\partial(\phi_B\circ\phi_A^{-1}))_Z(z)=(d^*-Zb^*)^{-1}z(a+cZ)^{-1}.$$

From this, by (1.1), the equality

$$(2.18)\qquad \chi((\partial(\phi_B\circ\phi_A^{-1}))_Z(z))=J(\phi_B\circ\phi_A^{-1})_Z\cdot\chi(z)$$

follows, where

$$(2.19)\qquad J(\phi_B\circ\phi_A^{-1})_Z=(d^*-Zb^*)^{-1}\otimes{}^t(a+cZ)^{-1}\in GL(mn, \boldsymbol{C}).$$

Since $\phi_B\circ\phi_A^{-1}$ is holomorphic and $\psi_A=\psi_B\circ(\phi_B\circ\phi_A^{-1})$, by (i) of Proposition 1.1, we obtain $(\partial\psi_A)_Z=(\partial\psi_B)_W\circ(\partial(\phi_B\circ\phi_A^{-1}))_Z$, where $W=\phi_A\circ\phi_A^{-1}(Z)$. Hence, by (2.16) and (2.18), the equality

$$(2.20)\qquad J(\psi_A)_Z=J(\psi_B)_W\cdot J(\phi_B\circ\phi_A^{-1})_Z$$

holds. From (2.6) and (2.19), the equality

$$(2.21)\qquad J(\phi_B\circ\phi_A^{-1})_Z=g_{BA}^{\perp}(\psi_A(Z))\otimes g_{BA}^{*}(\psi_A(Z))$$

follows. Since the holomorphic mapping $J(\phi_B\circ\phi_A^{-1})$: $\phi_A(U_A\cap U_B)\to GL(mn, \boldsymbol{C})$ is the transition function of the holomorphic tangent bundle $\tau(\boldsymbol{M}_{n,m})$, we have the following proposition.

Propossition 2.2. *The holomorphic tangent bundle $\tau(\boldsymbol{M}_{n,m})$ is holomorphically isomorphic to the tensor product $\gamma_{n,m}^{\perp}\otimes\gamma_{n,m}^{*}$ by the mapping χ: $T(\boldsymbol{M}_{n,m})'_P\to E_P^{\perp}\otimes E_P^{*}$ $(gl(m+n, \boldsymbol{C})\to \boldsymbol{C}^{m+n}\otimes\boldsymbol{C}^{m+n}=\boldsymbol{C}^{(m+n)^2})$, where $P\in\boldsymbol{M}_{n,m}$ be an arbitrary point and*

$$E_P^{\perp}=\{b\in\boldsymbol{C}^{m+n}\mid Pb=0\},$$

$$E_P^{*}=\{a\in\boldsymbol{C}^{m+n}\mid {}^tPa=a\}.$$

3. Pointed Grassmannians and vector bundles over them

3.1. Pointed Grassmannians. Set

$$\begin{aligned} \boldsymbol{F}_{N,m} &= \{([x], [X]) \in \boldsymbol{P}_N \times \boldsymbol{G}_{N,m} \mid x \in \boldsymbol{C}^{N+1} - \{0\}, \\ &\qquad\qquad\qquad X \in \boldsymbol{V}'_{N-m,m+1};\ [x] \subset [X]\} \\ &= \{(P, Q) \in \boldsymbol{M}_{N,1} \times \boldsymbol{M}_{N-m,m+1} \mid PQ = P\}. \end{aligned}$$

Then $\boldsymbol{F}_{N,m}$ is C^ω-diffeomorphic to the homogeneous space $U(N+1)/(U(1)\times U(m) \times U(N-m))$. $\boldsymbol{F}_{N,m}$ is called the pointed Grassmannian (cf. [2]). Now we shall introduce a holomorphic structure to $\boldsymbol{F}_{N,m}$. For any point $([x_0], [X_0]) \in \boldsymbol{F}_{N,m}$, we define a local parametrization $\hat{\phi}_A: \boldsymbol{C}^N \times M(N-m, m; \boldsymbol{C}) \to \boldsymbol{F}_{N,m}$ near $([x_0], [X_0])$ by

$$\hat{\phi}_A\left(\begin{pmatrix} z \\ \xi \end{pmatrix}, J\right) = \left(\left[A\begin{pmatrix} 1 \\ z \\ \xi \end{pmatrix}\right], \left[A\begin{pmatrix} 1 & 0 \\ z & E_n \\ \xi & J \end{pmatrix}\right]\right), \tag{3.1}$$

where $z \in \boldsymbol{C}^m$, $\xi \in \boldsymbol{C}^{N-m}$, $J \in M(N-m, m; \boldsymbol{C})$ and $A \in U(N+1)$ such that $\hat{\phi}_A\left(\begin{pmatrix} 0 \\ 0 \end{pmatrix}, O\right) = ([x_0], [X_0])$. Set $\hat{U}_A = \hat{\phi}_A(\boldsymbol{C}^N \times M(N-m, m; \boldsymbol{C}))$ and $\hat{\phi}_A = \hat{\phi}_A^{-1}$. Then the coordinate transformation $\hat{\phi}_B \circ \hat{\phi}_A^{-1}: \hat{\phi}_A(\hat{U}_A \cap \hat{U}_B) \to \hat{\phi}_B(\hat{U}_A \cap \hat{U}_B)$ is given by

$$\hat{\phi}_B \circ \hat{\phi}_A^{-1}\left(\begin{pmatrix} z \\ \xi \end{pmatrix}, J\right) = \left(\begin{pmatrix} w \\ \eta \end{pmatrix}, K\right);$$

$$\begin{aligned} & w = \frac{a + Sz + D^*\xi}{\alpha + b^*z + d^*\xi}, \qquad \eta = \frac{c + Cz + T\xi}{\alpha + b^*z + d^*\xi}, \\ & K = \{C + TJ - \eta(b^* + d^*J)\}\{S + D^*J - w(b^* + d^*J)\}^{-1} \\ & \quad = -\left\{T^* - (\xi - Jz, J)\begin{pmatrix} c^* \\ C^* \end{pmatrix}\right\}^{-1}\left\{D - (\xi - Jz, J)\begin{pmatrix} a^* \\ S^* \end{pmatrix}\right\}, \end{aligned} \tag{3.2}$$

where $z, w \in \boldsymbol{C}^m$; $\xi, \eta \in \boldsymbol{C}^{N-m}$; $J, K \in M(N-m, m; \boldsymbol{C})$ and $B^{-1}A = \begin{pmatrix} \alpha & b^* & d^* \\ a & S & D^* \\ c & C & T \end{pmatrix} \in U(N+1)$; $\alpha \in \boldsymbol{C}$; $a, b \in \boldsymbol{C}^m$, $c, d \in \boldsymbol{C}^{N-m}$; $C, D \in M(N-m, m; \boldsymbol{C})$; $S \in gl(m, \boldsymbol{C})$; $T \in gl(N-m, \boldsymbol{C})$. We think of $\boldsymbol{F}_{N,m}$ as a complex manifold with a holomorphic structure

$$\{(\hat{U}_A, \hat{\phi}) \mid A \in U(N+1)\}.$$

For $\left(\begin{pmatrix} z \\ \xi \end{pmatrix}, J\right) \in \hat{\phi}_A(\hat{U}_A \cap \hat{U}_B)$, we set

$$\begin{aligned} & h_{BA} = \alpha + b^*z + d^*\xi \in GL(1, \boldsymbol{C}) = \boldsymbol{C} - \{0\}, \\ & P_{BA} = S + D^*J - w(b^* + d^*J) \in GL(m, \boldsymbol{C}), \\ & Q_{BA} = \left\{T^* - (\xi - Jz, J)\begin{pmatrix} c^* \\ C^* \end{pmatrix}\right\}^{-1} \in GL(N-m, \boldsymbol{C}), \end{aligned} \tag{3.3}$$

$$G_{BA}=\begin{pmatrix} h_{BA} & b^*+d^*J \\ 0 & P_{BA} \end{pmatrix}\in GL(m+1, \boldsymbol{C}),$$

$$H_{BA}=\begin{pmatrix} P_{BA} & D^*-wd^* \\ O & Q_{BA} \end{pmatrix}\in GL(N, \boldsymbol{C}).$$

3.2. Vector bundles over $\boldsymbol{F}_{N,m}$. We define holomorphic mappings $p: \boldsymbol{F}_{N,m} \to \boldsymbol{P}_N=\boldsymbol{M}_{N,1}$, $q: \boldsymbol{F}_{N,m}\to\boldsymbol{G}_{N,m}=\boldsymbol{M}_{N-m,m+1}$ by

(3.4) $$p(P, Q)=P, \qquad q(P, Q)=Q .$$

Then local expressions of p, q are as follows:

(3.5) $$\phi_A\circ p_A\circ \hat{\phi}_A^{-1}\left(\begin{pmatrix} z \\ \xi \end{pmatrix}, J\right)=\begin{pmatrix} z \\ \xi \end{pmatrix},\ \phi_A\circ q\circ \hat{\phi}_A^{-1}\left(\begin{pmatrix} z \\ \xi \end{pmatrix}, J\right)=(\xi-Jz, J).$$

Set

(3.6) $$\xi_F=p^{-1}\gamma_{N,1}, \qquad \xi_F^{\perp}=p^{-1}\gamma_{N,1}^{\perp},$$
$$\gamma_F=q^{-1}\gamma_{N-m,m+1}, \qquad \gamma_F^{\perp}=q^{-1}\gamma_{N-m,m+1}^{\perp}.$$

Then ξ_F, $\xi_F^{\perp}$, γ_F, $\gamma_F^{\perp}$ are holomorphic vector bundles over $\boldsymbol{F}_{N,m}$ of rank 1, N, $m+1$, $N-m$, respectively. Moreover they are described as

(3.7)
$$\xi_F=\{(P, Q\,;\, a)\in\boldsymbol{F}_{N,m}\times\boldsymbol{C}^{N+1}\,|\,Pa=a\},$$
$$\xi_F^{\perp}=\{(P, Q\,;\, a)\in\boldsymbol{F}_{N,m}\times\boldsymbol{C}^{N+1}\,|\,Pa=0\},$$
$$\gamma_F=\{(P, Q\,;\, a)\in\boldsymbol{F}_{N,m}\times\boldsymbol{C}^{N+1}\,|\,Qa=a\},$$
$$\gamma_F^{\perp}=\{(P, Q\,;\, a)\in\boldsymbol{F}_{N,m}\times\boldsymbol{C}^{N+1}\,|\,Qa=0\}.$$

Now we define a vector bcndle η_F over $\boldsymbol{F}_{N,m}$ of rank m by

(3.8) $$\eta_F=\{(P, Q\,;\, a)\in\boldsymbol{F}_{N,m}\times\boldsymbol{C}^{N+1}\,|\,Pa=0,\ Qa=a\}$$

and its projection $\pi_\eta: \eta_F\to\boldsymbol{F}_{N,m}$ by $\pi_\eta(P, Q\,;\, a)=(P, Q)$. Then its local triviality $\theta_A: \hat{U}_A\times\boldsymbol{C}^m\to\pi_\eta^{-1}(\hat{U}_A)$ is given by

(3.9) $$\theta_A((P, Q), u)=(P, Q\,;\, Y_A(z, \xi\,;\, J)u),$$

where $\left(\begin{pmatrix} z \\ \xi \end{pmatrix}, J\right)=\hat{\phi}_A(P, Q)$ and

(3.10)
$$Y_A=Y_A(z, \xi\,;\, J)=A\begin{pmatrix} -z^* & -\xi^* \\ & E_N \end{pmatrix}\Lambda_{z\xi}^{-1}\begin{pmatrix} E_m \\ J \end{pmatrix}$$
$$=X_A(X_A^*X_A)^{-1}\begin{pmatrix} 0 \\ \hat{L} \end{pmatrix};\ \Lambda_{z\xi}=E_N+\begin{pmatrix} z \\ \xi \end{pmatrix}(z^*\,\xi^*)=M_{\binom{z}{\xi}},$$

$$X_A=X_A(z,\xi;J)=A\begin{pmatrix}1&0\\z&E_m\\\xi&J\end{pmatrix},\quad \hat{L}=(E_m\ \ J^*)\Lambda_{z\xi}^{-1}\begin{pmatrix}E_m\\J\end{pmatrix}.$$

$Y_A=Y_A(z,\xi;J)$ gives a local frame field on $\hat{U}_A$ of η_F and for $\left(\binom{z}{\xi},J\right)\in\hat{\phi}_A(\hat{U}_A\cap\hat{U}_B)$, the equality

$$Y_A(z,\xi;J)=Y_B(w,\eta;K)\cdot P_{BA} \tag{3.11}$$

holds, where $\left(\binom{w}{\eta},K\right)=\hat{\phi}_B\circ\hat{\phi}_A^{-1}\left(\binom{z}{\xi},J\right)$. It follows from this equality that P_{BA} gives a holomorphic transition function

$$\hat{U}_A\cap\hat{U}_A\ni\hat{\phi}_A\left(\begin{pmatrix}z\\\xi\end{pmatrix},J\right)\longrightarrow P_{BA}\in GL(m,\boldsymbol{C})$$

of η_F. Hence η_F is a holomorphic vector bundle of rank m. For each $\left(\binom{z}{\xi},J\right)\in\boldsymbol{C}^N\times M(N-m,m;\boldsymbol{C})$, set

$$\begin{aligned}
&x_A=x_A(z,\xi)=A\begin{pmatrix}1\\z\\\xi\end{pmatrix},\\
&x_A^\perp=x_A^\perp(z,\xi;J)=A\begin{pmatrix}-z^*&-\xi^*\\&E_N\end{pmatrix}\Lambda_{z\xi}^{-1}\begin{pmatrix}E_m&O\\J&E_n\end{pmatrix},\\
&X_A=X_A(z,\xi;J)=A\begin{pmatrix}1&0\\z&E_m\\\xi&J\end{pmatrix},\\
&X_A^\perp=X_A^\perp(z,\xi;J)=A\begin{pmatrix}-Z^*\\E_n\end{pmatrix}M_Z^{-1}\quad\text{(cf. (2.5)),}
\end{aligned}\tag{3.12}$$

where $Z=(\xi-Jz,J)\in M(N-m,m+1;\boldsymbol{C})$ and $n=N-m$. Then x_A, $x_A^\perp$, X_A, $X_A^\perp$ give local frame fields on $\hat{U}_A$ of ξ_F, $\xi_F^\perp$, γ_F, $\gamma_F^\perp$, respectively. Moreover we have the following equalities on $\hat{\phi}_A(\hat{U}_A\cap\hat{U}_B)$

$$\begin{aligned}
&x_A=x_B\cdot h_{BA}, &&x_A^\perp=x_B^\perp\cdot H_{BA},\\
&X_A=X_B\cdot G_{BA}, &&X_A^\perp=X_B^\perp\cdot Q_{BA},
\end{aligned}\tag{3.13}$$

where $x_B=x_B(w,\eta)$, $x_B^\perp=x_B^\perp(w,\eta;K)$, $X_B=X_B(w,\eta;K)$, $X_B^\perp=X_B^\perp(w,\eta;K)$ and $\left(\binom{w}{\eta},K\right)=\hat{\phi}_B\circ\hat{\phi}_A^{-1}\left(\binom{z}{\xi},J\right)$. Hence h_{BA}, H_{BA}, G_{BA}, Q_{BA} give transition functions on $\hat{U}_A \cap \hat{U}_B$ of $\xi_F, \xi_F^{\perp}, \gamma_F, \gamma_F^{\perp}$, respectively. By (2.7), we have the following holomorphic short exact sequences:

$$(3.14)\qquad \begin{gathered} 0 \longrightarrow \xi_F \xrightarrow{\alpha_\xi} \hat{C}^{N+1} \xrightarrow{\beta_\xi} \xi_F^{\perp} \longrightarrow 0 \\ 0 \longrightarrow \gamma_F \xrightarrow{\alpha_\gamma} \hat{C}^{N+1} \xrightarrow{\beta_\gamma} \gamma_F^{\perp} \longrightarrow 0 \end{gathered}$$

where $\hat{\boldsymbol{C}}^{N+1} = \boldsymbol{F}_{N,m} \times \boldsymbol{C}^{N+1}$ (product bundle). Furthermore, we have the following short exact sequence:

$$(3.15)\qquad 0 \longrightarrow \xi_F \xrightarrow{\alpha'} \gamma_F \xrightarrow{\beta'} \eta_F \longrightarrow 0,$$

$$(3.16)\qquad 0 \longrightarrow \eta_F \xrightarrow{\alpha''} \xi_F^{\perp} \xrightarrow{\beta''} \gamma_F^{\perp} \longrightarrow 0,$$

where α', α'' are natural inclusions and

$$(3.17)\qquad \beta'(P, Q\,;\, a) = (P, Q\,;\, (E_{N+1} - P)a),$$

$$(3.18)\qquad \beta''(P, Q\,;\, a) = (P, Q\,;\, (E_{N+1} - Q)a).$$

Local expressions of α', β', α'', β'' are as follows:

$$\alpha'_{(P,Q)}(x_A e) = X_A \begin{pmatrix} 1 \\ 0 \end{pmatrix} e, \qquad \beta'_{(P,Q)}\left(X_A \begin{pmatrix} e \\ u \end{pmatrix} \right) = Y_A u$$

$$\alpha''_{(P,Q)}(Y_A u) = x_A^{\perp} \begin{pmatrix} E_m \\ O \end{pmatrix} u, \qquad \beta''_{(P,Q)}\left(x_A^{\perp} \begin{pmatrix} u \\ v \end{pmatrix} \right) = X_A^{\perp} v,$$

where $P = x_A(x_A^* x_A)^{-1} x_A^*$, $Q = X_A(X_A^* X_A)^{-1} X_A^*$; $e \in \boldsymbol{C}$, $u \in \boldsymbol{C}^m$, $v \in \boldsymbol{C}^n$. Hence short exact sequences (3.15) and (3.16) are holomorphic. Moreover we have the following commutative diagram:

$$(3.19)\qquad \begin{tikzcd} & & \xi_F \arrow[dl, "\alpha'"'] \arrow[d, "\alpha_\xi"] & \\ & \gamma_F \arrow[r, "\alpha_\gamma"] \arrow[dl, "\beta'"'] & \hat{C}^{N+1} \arrow[d, "\beta_\xi"] \arrow[dr, "\beta_\gamma"] & \\ \eta_F \arrow[rr, "\alpha''"] & & \xi_F^{\perp} \arrow[r, "\beta''"] & \gamma_F^{\perp}. \end{tikzcd}$$

3.3. Milnor manifolds. Set

$$\begin{aligned} \boldsymbol{H}_{2m+1} = \boldsymbol{H}_{m+1,m+1} &= \{([x], [y]) \in \boldsymbol{P}_{m+1} \times \boldsymbol{P}_{m+1} \mid {}^t x y = 0\} \\ &= \{(P, Q) \in \boldsymbol{M}_{m+1,1} \times \boldsymbol{M}_{m+1,1} \mid {}^t P Q = O\}. \end{aligned}$$

$\boldsymbol{H}_{2m+1}=\boldsymbol{H}_{m+1,m+1}$ is a particular one of Milnor manifolds. Now we shall introduce a holomorphic local chart near each point of $\boldsymbol{H}_{2m+1}$. Let $([u_0],[v_0])\in \boldsymbol{H}_{2m+1}$ be an arbitrary point. For some $A\in U(m+2)$, the mapping $\tilde{\varphi}_A: \boldsymbol{C}^{2m+1}\to \boldsymbol{H}_{2m+1}$ defined by

$$\tilde{\varphi}_A\begin{pmatrix} z\\ \xi\\ \zeta\end{pmatrix}=\left(\left[A\begin{pmatrix}1\\ z\\ \xi\end{pmatrix}\right], \left[\bar{A}U_0\begin{pmatrix}1\\ \zeta\\ x\end{pmatrix}\right]\right); \quad \begin{array}{l} z, \zeta\in \boldsymbol{C}^m,\ \xi\in\boldsymbol{C},\\ x=\xi-{}^t\zeta z\end{array}$$

is a local parametrization near $([u_0],[v_0])=\tilde{\varphi}_A(0)$, where

$$U_0=\begin{pmatrix}0 & 0 & -1\\ 0 & -E_m & 0\\ 1 & 0 & 1\end{pmatrix}\in U(m+2).$$

Set $\tilde{U}_A=\tilde{\varphi}_A(\boldsymbol{C}^{2m+1})$ and $\tilde{\phi}_A=\tilde{\varphi}_A^{-1}$. Then the coordinate transformation $\tilde{\phi}_B\circ\tilde{\phi}_A^{-1}$: $\tilde{\phi}_A(\tilde{U}_A\cap\tilde{U}_B)\to\tilde{\varphi}_B(\tilde{U}_A\cap\tilde{U}_B)$ is given by

$$\tilde{\varphi}_B\circ\tilde{\varphi}_A^{-1}\begin{pmatrix} z\\ \xi\\ \zeta\end{pmatrix}=\begin{pmatrix} w\\ \eta\\ \omega\end{pmatrix}; \quad B^{-1}A=\begin{pmatrix}\alpha & b^* & \gamma\\ a & C & d\\ \beta & c^* & \delta\end{pmatrix}\in U(m+2),$$

$$w=\frac{a+Cz+d\xi}{\alpha+b^*z+\gamma\xi},\quad \eta=\frac{\beta+c^*z+\delta\xi}{\alpha+b^*z+\gamma\xi},$$

$$\omega=\frac{-\bar{d}+\bar{C}\zeta+\bar{a}x}{\bar{\delta}-{}^tc\zeta-\bar{\beta}x};\quad x=\xi-{}^t\zeta z;$$

$\alpha,\beta,\gamma,\delta\in\boldsymbol{C}$; $a, b, c, d\in\boldsymbol{C}^m$; $C\in gl(m, \boldsymbol{C})$. We think of $\boldsymbol{H}_{2m+1}$ as a complex manifold with a holomorphic structure

$$\{(\tilde{U}_A, \tilde{\phi}_A)\mid A\in U(m+2)\}.$$

Define a biholomorphic mapping $\tilde{\Delta}: \boldsymbol{H}_{2m+1}\to\boldsymbol{F}_{m+1,m}$ by

$$\tilde{\Delta}(P, Q)=(P, \Delta(Q))=(P, E_{m+2}-{}^tQ) \tag{3.20}$$

(cf. (2.9)). Moreover we define holomorphic mappings $\tilde{p}, \tilde{q}: \boldsymbol{H}_{2m+1}\to\boldsymbol{P}_{m+1}$ by $\tilde{p}(P, Q)=P$, $\tilde{q}(P, Q)=Q$, respectively. Then we have the following commutative diagrams:

$$\begin{tikzcd} \boldsymbol{H}_{2m+1} \arrow[rr, "\tilde{\Delta}"] \arrow[dr, "\tilde{p}"'] & & \boldsymbol{F}_{m+1,m} \arrow[dl, "p"] \\ & \boldsymbol{P}_{m+1}, & \end{tikzcd} \qquad \begin{tikzcd} \boldsymbol{H}_{2m+1} \arrow[r, "\tilde{\Delta}"] \arrow[d, "\tilde{q}"] & \boldsymbol{F}_{m+1,m} \arrow[d, "q"] \\ \boldsymbol{M}_{m+1}=\boldsymbol{P}_{m+1} \arrow[r, "\Delta"] & \boldsymbol{G}_{m+1,m}=\boldsymbol{M}_{1,m+1} \end{tikzcd} \tag{3.21}$$

3.4. Vector bundles over $\boldsymbol{H}_{2m+1}$. Set $\xi_H=\tilde{p}^{-1}\gamma_{m+1,1}$, $\xi_H^{\perp}=\tilde{p}^{-1}\gamma_{m+1,1}^{\perp}$, $\gamma_H^*=\tilde{q}^{-1}\gamma_{m+1,1}^*$, $\gamma_H^{\perp *}=\tilde{q}^{-1}\gamma_{m+1,1}^{\perp *}$. Then we have

$$\xi_H=\{(P, Q\,;\, a)\in \boldsymbol{H}_{2m+1}\times \boldsymbol{C}^{m+2}\,|\, Pa=a\},$$

$$\xi_H^{\perp}=\{(P, Q\,;\, a)\in \boldsymbol{H}_{2m+1}\times \boldsymbol{C}^{m+2}\,|\, Pa=0\},$$

$$\gamma_H^*=\{(P, Q\,;\, a)\in \boldsymbol{H}_{2m+1}\times \boldsymbol{C}^{m+2}\,|\, {}^tQa=a\},$$

$$\gamma_H^{\perp *}=\{(P, Q\,;\, a)\in \boldsymbol{H}_{2m+1}\times \boldsymbol{C}^{m+2}\,|\, {}^tQa=0\}.$$

Set $\eta_H=\tilde{\Delta}^{-1}\eta_F$. Then we have

$$\eta_H=\{(P, Q\,;\, a)\in \boldsymbol{H}_{2m+1}\times \boldsymbol{C}^{m+2}\,|\, Pa=0={}^tQa\}. \tag{3.22}$$

It follows from Proposition 2.1 that γ_H^*, $\gamma_H^{\perp *}$ are equal to $\tilde{\Delta}^{-1}\gamma_F^{\perp}$, $\tilde{\Delta}^{-1}\gamma_F$, respectively. Moreover we have $\xi_H=\tilde{\Delta}^{-1}\xi_F$, $\xi_H^{\perp}=\tilde{\Delta}^{-1}\xi_F^{\perp}$. Hence, by (3.19), we have the following commutative diagram:

$$\begin{tikzcd}
 & & \xi_H \arrow[dl, "\alpha'"'] \arrow[dd, "\alpha_\xi"] & & \\
 & \gamma_H^{\perp *} \arrow[dr, "\beta^*"] \arrow[ddl, "\beta'"'] & & & \\
 & & \boldsymbol{C}^{m+2} \arrow[d, "\beta_\xi"] \arrow[drr, "\alpha^*"] & & \\
\eta_H \arrow[rr, "\mu"] & & \xi_H^{\perp} \arrow[rr, "\nu"] & & \gamma_H^*,
\end{tikzcd} \tag{3.23}$$

where α', μ are canonical inclusions and β', ν are given by

$$\beta'(P, Q\,;\, a)=(P, Q\,;\,(E_{m+2}-P)a), \quad \nu(P, Q\,;\, a)=(P, Q\,;\, {}^tQa).$$

Moreover (α_ξ, β_ξ), (α^*, β^*), (α', β'), (μ, ν) are holomorphic short exact sequences.

4. Gauss mappings and pointed Gauss mappings

4.1. Local graph charts. Let M be a complex submanifold of the N-dimensional complex projective space $\boldsymbol{P}_N$ of complex dimension m. Let $p_0\in M$ be an arbitrary point. There exist a non-empty open set D_0 of $\boldsymbol{C}^m$, a holomorphic mapping $g\colon D_0\to \boldsymbol{C}^N$ and some $A\in U(N+1)$ such that the mapping

$$D_0\ni w={}^t(w^1, \cdots, w^m)\longmapsto \left[A\begin{pmatrix}1\\ g(w)\end{pmatrix}\right]\in M$$

is a holomorphic local parametrization of M near p_0 and

$$p_0=\left[A\begin{pmatrix}1\\ g(c)\end{pmatrix}\right], \quad \left(\left(\frac{\partial g^i}{\partial w^j}\right)_c\right)_{1\le i,\,j\le m}\in GL(m, \boldsymbol{C})\,;\; g={}^t(g^1, \cdots, g^N).$$

Set ${}^t g(c)=({}^t a, {}^t b)$; $a\in \boldsymbol{C}^m$, $b\in \boldsymbol{C}^{N-m}$. Then, by Inverse mapping theorem, there exist an open neighborhood D of a in $\boldsymbol{C}^m$ and a holomorphic mapping $f: D\to \boldsymbol{C}^{N-m}$ such that the mapping $\psi_{Af}: D\to M$ defined by

$$\psi_{Af}(z)=\left[A\begin{pmatrix}1\\ z\\ f(z)\end{pmatrix}\right]=\psi_A\begin{pmatrix}z\\ f(z)\end{pmatrix} \tag{4.1}$$

is a holomorphic local parametrization of M near p_0, where $p_0=\psi_{Af}(a)$, $b=f(a)$. Set $U_{Af}=\psi_{Af}(D)$ and $\phi_{Af}=\psi_{Af}^{-1}: U_{Af}\to D$. Then (U_{Af}, ϕ_{Af}) is a holomorphic local chart of M near p_0. In this note, we call (U_{Af}, ϕ_{Af}), ψ_{Af} the *local graph chart, local graph parametrization*, respectively.

4.2. Tangent projective spaces and Gauss mappings. Let M be an m-dimensional complex submanifold of $\boldsymbol{P}_N$ and (U_{Af}, ϕ_{Af}), $\psi_{Af}=\phi_{Af}^{-1}: D\to U_{Af}\subset M$ the local graph chart, local graph parametrization, respectively, where $A\in U(N+1)$, D is an open subset in $\boldsymbol{C}^m$ and $f: D\to \boldsymbol{C}^{N-m}$ is a holomorphic mapping. Set

$$J_f:=\left(\frac{\partial f}{\partial z}\right) \quad \text{(Jacobian matrix of } f).$$

We may regard J_f as an $M(N-m, m; \boldsymbol{C})$-valued holomorphic function on D or U_{Af}. Then the tangent projective space $P(M)_p$ of M at p is written as

$$P(M)_p=\left\{\left[A\begin{pmatrix}1 & 0\\ z & E_m\\ f & J_f\end{pmatrix}x\right]\in \boldsymbol{P}_N \,\middle|\, x\in \boldsymbol{C}^{m+1}-\{0\}\right\}, \tag{4.2}$$

where $p=\psi_{Af}(z)$, $f=f(z)$, $J_f=J_f(z)$ for $z\in D$. Hence the Gauss mapping Γ_M: $M\to \boldsymbol{G}_{N,m}=\boldsymbol{M}_{N-m,m+1}$ is given by

$$\Gamma_M(p)=\left[A\begin{pmatrix}1 & 0\\ z & E_m\\ f & J_f\end{pmatrix}\right]=\psi_A(f(z)-J_f(z)\cdot z, J_f(z)), \tag{4.3}$$

where we think of $\boldsymbol{G}_{N,m}$ as a set of all m-dimensional projective subspaces of $\boldsymbol{P}_N$ and ψ_A is the local parametrization defined in 2.1 (cf. (2.9)). Since $\boldsymbol{P}_{m+1}=\boldsymbol{M}_{m+1,1}$ is biholomorphic to $\boldsymbol{G}_{m+1,m}=\boldsymbol{M}_{1,m+1}$ by $\varDelta$, if $M=V$ is a non-singular complex hypersurface of $\boldsymbol{P}_{m+1}$, we may think of the Gauss mapping Γ_V as a mapping of V into $\boldsymbol{P}_{m+1}$. Then it is given by

$$\Gamma_V(p)=\left[\bar{A}U_0\begin{pmatrix}1\\ {}^t J_f\\ f-J_f\cdot z\end{pmatrix}\right]=\psi_{\bar{A}U_0}\begin{pmatrix}{}^t J_f\\ f-J_f\cdot z\end{pmatrix}, \tag{4.4}$$

where $p=\psi_{Af}(z)$, $f=f(z)\in C$, $J_f=J_f(z)=((\partial f/\partial z^1)_z, \cdots, ((\partial f/\partial z^m)_z)\in M(1, m; C)$ for $z\in D$ and $U_0\in U(m+2)$ is defined in **3.3**. For the above non-singular complex hypersurface V, there exists a homogeneous polynomial $\Psi(x)=\Psi(x_0, x_1, \cdots, x_m, x_{m+1})$ of degree d such that

$$V=V_m(d)=\{[x]\in P_{m+1} \mid x\in C^{m+2}-\{0\}; \Psi(x)=0\};$$

$$\left(\frac{\partial\Psi}{\partial x}\right)_x=(\Psi_0(x), \Psi_1(x), \cdots, \Psi_m(x), \Psi_{m+1}(x))\neq 0\in M(1, m+2; C)$$

for $x={}^t(x_0, x_1, \cdots, x_m, x_{m+1})\in C^{m+2}-\{0\}$, where

$$\Psi_i(x)=\left(\frac{\partial\Psi}{\partial x_i}\right)_x \quad \text{(homogeneous polynomial of degree } d-1,\ 0\leq i\leq m+1).$$

Then $V=V_m(d)$ is called of degree d. By Euler's identity and implicit function theorem, we obtain the following proposition.

Proposition 4.1. *For the above $V=V_m(d)$, the Gauss mapping $\Gamma_V: V=V_m(d)\to P_{m+1}$ is described as*

$$\Gamma_V([x])=\left[{}^t\left(\frac{\partial\Psi}{\partial x}\right)_x\right].$$

4.3. Pointed Gauss mappings. Let M be an m-dimensional complex submanifold of P_N. Then the pointed Gauss mapping $\hat{\Gamma}_M: M\to F_{N,m}$ is defined by

$$\hat{\Gamma}_M([x])=([x], \Gamma_M([x])) \quad \text{(see [2])}. \tag{4.5}$$

In particular, if $M=V$ is a non-singular complex hypersurface of P_{m+1}, the Gauss mapping $\tilde{\Gamma}_V: V\to H_{2m+1}$ is defined by

$$\tilde{\Gamma}_V([x])=([x], \Gamma_V([x])). \tag{4.6}$$

Then we have the following commutative diagrams:

$$\begin{tikzcd} P_N & F_{N,m} \arrow[l, "p"'] \arrow[d, "q"] \\ M \arrow[u, bend left] \arrow[ur, "\hat{\Gamma}_M"] \arrow[r, "\Gamma_M"'] & G_{N,m}, \end{tikzcd} \qquad \begin{tikzcd} P_{m+1} & H_{2m+1} \arrow[l, "\tilde{p}"'] \arrow[d, "\tilde{q}"] \\ V \arrow[u, bend left] \arrow[ur, "\tilde{\Gamma}_V"] \arrow[r, "\Gamma_V"'] & P_{m+1}. \end{tikzcd} \tag{4.7}$$

5. Results and proofs

Let M be an m-dimensional complex submanifold of P_N and V a non-singular complex hypersurface of P_{m+1}. We denote the holomorphic tangent bundles of M, V by $\tau(M)$, $\tau(V)$, respectively and the holomorphic normal bundles of M, V by $\nu(M)$, $\nu(V)$, respectively. We prove the following theorem.

Theorem 5.1. i) $\tau(M)$, $\tau(V)$ *are holomorphically isomorphic to*

$$\hat{\Gamma}_M^{-1}(\eta_F \otimes \xi_F^*) = (\hat{\Gamma}_M^{-1}\eta_F) \otimes (\gamma_{N,1}^*|_M),$$

$$\tilde{\Gamma}_V^{-1}(\eta_H \otimes \xi_H^*) = (\tilde{\Gamma}_V^{-1}\eta_H) \otimes (\gamma_{m+1,1}^*|_V),$$

respectively. ii) $\nu(M)$, $\nu(V)$ *are holomorphically isomorphic to*

$$\hat{\Gamma}_M^{-1}(\gamma_F^{\perp} \otimes \xi_F^*) = (\Gamma_M^{-1}\gamma_{N-m,m+1}^{\perp}) \otimes (\gamma_{N,1}^*|_M),$$

$$\tilde{\Gamma}_V^{-1}(\gamma_H^* \otimes \xi_H^*) = (\Gamma_V^{-1}\gamma_{m+1,1}^*) \otimes (\gamma_{m+1,1}^*|_V),$$

respectively.

Proof. We prove i), ii) only on M. On V, their proofs are analogous. Let (U_{Af}, ϕ_{Af}) be an arbitrary local graph chart and $\psi_{Af} = \phi_{Af}^{-1}: D \to U_{Af}$ the local graph parametrization associated with (U_{Af}, ϕ_{Af}), where $A \in U(N+1)$, D is an open subset of $\boldsymbol{C}^m$ and $f: D \to \boldsymbol{C}^{N-m}$ is a holomorphic mapping. Then we have

$$(\partial\psi_{Af})_z(u) = (\partial\psi_A)_{\binom{z}{f}}\left(\begin{pmatrix} E_m \\ J_f \end{pmatrix} u\right) \in T'(M)_p \subset T'(\boldsymbol{P}_N)_p$$

$$= A\begin{pmatrix} -z^* & -f^* \\ & E_N \end{pmatrix} \Lambda_{zf}^{-1} \begin{pmatrix} E_m \\ J_f \end{pmatrix} u \lambda_{zf}^{-1} (1 \;\; z^* \;\; f^*) A^{-1}$$

(cf. (2.12) and (2.13)), where $u \in \boldsymbol{C}^m$, $z \in D$, $p = \psi_{Af}(z)$, $\lambda_{zf} = 1 + z^*z + f^*f = L_{\binom{z}{f}}$ (cf. (2.14)), $f = f(z)$, $J_f = J_f(z)$. By (1.1), from this equality, we obtain

$$\chi((\partial\psi_{Af})_z(u)) = J(\psi_{Af})_z \cdot u$$

(5.1)

$$J(\psi_{Af})_z = Y_A(z, f; J_f) \otimes x_A^*(z, f) \quad \text{(cf. (3.10))},$$

where

$$x_A^*(z, f) = \bar{A}\begin{pmatrix} 1 \\ \bar{z} \\ \bar{f} \end{pmatrix} \lambda_{zf}^{-\frac{1}{2}} \quad \text{(cf. (2.5))}.$$

$J(\psi_{Af})_z$, $Y_A(z, f; J_f)$, $x_A^*(z, f)$ give local frame fields on U_{Af} of $\tau(M)$, $\hat{\Gamma}_M^{-1}\eta_F$, $\gamma_{N,1}^*|_M$, respectively. Moreover, on $U_{Af} \cap U_{Bg} \neq \phi$, from $(\partial\psi_{Af})_z = (\partial\psi_{Bg})_w \circ (\partial(\phi_{Bg} \circ \phi_{Af}^{-1}))_z$ (3.11), (2.5) and (2.6), we obtain

$$J(\psi_{Af})_z = J(\psi_{Bg})_w \cdot J(\phi_{Bg} \circ \phi_{Af}^{-1})_z,$$

(5.2)

$$J(\phi_{Bg} \circ \phi_{Af}^{-1})_z = \left(\frac{\partial w}{\partial z}\right)_z = \hat{P}_{BA}/\hat{h}_{BA},$$

where (U_{Bg}, ϕ_{Bg}) is another local graph chart,

$$\left(\begin{pmatrix} w \\ g \end{pmatrix}, J_g\right)=(\hat{\phi}_B\circ\hat{\phi}_A^{-1})\left(\begin{pmatrix} z \\ f \end{pmatrix}, J_f\right) \quad \text{(cf. (3.2))},$$

$$w=\frac{a+Sz+D^*f(z)}{\alpha+b^*z+d^*f(z)}=(\phi_{Bg}\circ\phi_{Af}^{-1})(z) \quad \text{(cf. (3.2))},$$

$$g=g(w), \quad J_g=J_g(w)=\left(\frac{\partial g}{\partial w}\right)_w$$

and

$$\hat{P}_{BA}=S+D^*J_f(z)-w(b^*+d^*J_f(z)) \quad \text{(cf. (3.3))},$$

$$\hat{h}_{BA}=\alpha+b^*z+d^*f(z) \quad \text{(cf. (3.3))}.$$

(5.1) and (5.2) show that $\tau(M)$ is holomorphically isomorphic to $(\hat{\Gamma}_M^{-1}\eta_F)\otimes(\gamma_{N,1}^*|_M)$ by the holomorphic isomorphism

$$\chi:\tau(\boldsymbol{P}_N)|_M\longrightarrow(\gamma_{N,1}^\perp\otimes\gamma_{N,1}^*)|_M=\hat{\Gamma}_M^{-1}(\xi_F^\perp\otimes\xi_F^*)$$

(cf. Proposition 2.2). Thus i) has been proved. The above fact shows also that the following diagram is commutative:

$$\begin{array}{ccc} \tau(M) & \subset & \tau(\boldsymbol{P}_N)|_M \\ \downarrow\chi_f & \alpha''\otimes 1 & \downarrow\chi \\ \hat{\Gamma}_M^{-1}(\eta_F\otimes\xi_F^*) & \subset & \hat{\Gamma}_M^{-1}(\xi_F^\perp\otimes\xi_F^*)\,. \end{array}$$

It follows from (3.16) that the sequence:

$$0\longrightarrow\hat{\Gamma}_M^{-1}(\eta_F\otimes\xi_F^*)\longrightarrow\hat{\Gamma}_M^{-1}(\xi_F^\perp\otimes\xi_F^*)\longrightarrow\hat{\Gamma}_M^{-1}(\gamma_F^\perp\otimes\xi_F^*)\longrightarrow 0$$

is holomorphically exact. On the other hand, we have the holomorphic short exact sequence:

$$0\longrightarrow\tau(M)\longrightarrow\tau(\boldsymbol{P}_N)|_M\longrightarrow\nu(M)\longrightarrow 0\,.$$

Therefore $\nu(M)$ is holomorphically isomorphic to $\hat{\Gamma}_M^{-1}(\gamma_F^\perp\otimes\xi_F^*)$ by the homomorphism that $\chi:\tau(\boldsymbol{P}_N)|_M\to\hat{\Gamma}_M^{-1}(\xi_F^\perp\otimes\xi_F^*)$ induces. Thus ii) has been shown. q.e.d.

Remark. The holomorphic local frame field $e_{Af}:U_{Af}\to V'_{(N+1)^2-(N-m),N-m}$ defined by

$$e_{Af}(p)=A\begin{pmatrix} -Z^* \\ E_n \end{pmatrix}M_Z^{-1}\otimes\bar{A}\begin{pmatrix} 1 \\ \bar{z} \\ \bar{f} \end{pmatrix}\lambda_{zf}^{-1} \quad \text{(cf. (2.5) and (3.12))}.$$

$$=J(\phi_A)_{\binom{z}{f}}\cdot\Lambda_{zf}\lambda_{zf}\begin{pmatrix} -J_f^* \\ E_n \end{pmatrix}(M_Z\lambda_{zf})^{-1}$$

of $(\hat{\Gamma}_M^{-1}\gamma_{N-m,m+1}^{\perp})\otimes(\gamma_{N,1}^*|_M)$ is regarded as holomorphic normal frame field on U_{Af} of M in $\boldsymbol{P}_N$ (cf. Theorem 2.1 of [1]), where $n=N-m$, $z=\phi_{Af}(p)$, $Z=(f(z)-J_f(z)\cdot z, J_f(z))$,

$$M_Z=(-J_f(z)\ E_n)\Lambda_{zf}\begin{pmatrix}-J_f(z)^* \\ E_n\end{pmatrix} \quad \text{(cf. (2.14))}$$

$$J(\phi_A)_{\binom{z}{f}}=A\begin{pmatrix}-z^* & -f(z)^* \\ & E_N\end{pmatrix}\Lambda_{zf}^{-1}\otimes\bar{A}\begin{pmatrix}1 \\ \bar{z} \\ \overline{f(z)}\end{pmatrix}\lambda_{zf}^{-1} \quad \text{(cf. (2.17))}.$$

By the short exact sequences (3.15) and

$$0\longrightarrow \xi_H \xrightarrow{\alpha'} \gamma_H^{\perp *} \xrightarrow{\beta'} \eta_H \longrightarrow 0,$$

we have the following corollary (cf. (1.11) Proposition of [2]).

Corollary 5.2. (Euler sequence)

$$0\longrightarrow \gamma_{N,1}|_M \longrightarrow \Gamma_M^{-1}\gamma_{N-m,m+1} \longrightarrow \tau(M)\otimes(\gamma_{N,1}|_M)\longrightarrow 0,$$

$$0\longrightarrow \gamma_{m+1,1}|_V \longrightarrow \Gamma_V^{-1}\gamma_{m+1,1}^{\perp *} \longrightarrow \tau(V)\otimes(\gamma_{m+1,1}|_V)\longrightarrow 0$$

are holomorphic short exact sequences of vector bundles over M, V, respectively.

If the non-singular complex hypersurface V is of degree d, then it follows from Proposition 4.1 that $\Gamma_V^{-1}\gamma_{m+1,1}^*$ is holomorphically isomorphic to $\gamma_{m+1,1}^{-(d-1)}|_V$, where $\gamma_{m+1,1}^{-p}$ is a p-fold tensor product of $\gamma_{m+1,1}^*$ for any non-negative integer p. Hence we have the following well-known fact.

Corollary 5.3. *If V is of degree d, then $\nu(V)$ is holomorphically isomorphic to $\gamma_{m+1,1}^{-d}|_V$.*

Example 1. (Tangent bundles of m-dimensional complex quadric hypersurfaces) Let

$$\begin{aligned}\boldsymbol{Q}_m&=\{[x]\in\boldsymbol{P}_{m+1}\,|\,x\in\boldsymbol{C}^{m+2}-\{0\}\,;\,{}^t xx=0\}\\ &=\{Q\in\boldsymbol{M}_{m+1,1}\,|\,{}^tQQ=O\}.\end{aligned}$$

By Proposition 4.1, the Gauss mapping $\tilde{\Gamma}_{Q_m}:\boldsymbol{Q}_m\to\boldsymbol{P}_{m+1}$ is given by $\tilde{\Gamma}_{Q_m}(Q)=(Q, Q)$. Hence we have

$$\eta_m:=\tilde{\Gamma}_{Q_m}^{-1}\eta_H=\{(Q, b)\in\boldsymbol{Q}_m\times\boldsymbol{C}^{m+2}\,|\,Qb=0={}^tQb\}.$$

By i) of Theorem 5.1, the holomorphic tangent bundle $\tau(\boldsymbol{Q}_m)=\{(Q, X)\in\boldsymbol{Q}_m\times gl(m+2, \boldsymbol{C})\,|\,XQ=X,\ QX=0={}^tQX\}$ is holomorphically isomorphic to $\eta_m\otimes\gamma_{Q_m}^{-1}$ by the homomorphism χ, where $\gamma_{Q_m}^{-1}=\gamma_{m+1,1}^*|_{Q_m}$.

Example 2. (Normal bundles of the Segre embeddings) The holomorphic normal bundle $\nu(M)$ of the Segre embedding $\sigma: \boldsymbol{P}_m \times \boldsymbol{P}_n \to \boldsymbol{P}_{m+n+mn}$ defined by $\sigma([x], [y]) = [x \otimes y]$ is holomorphically isomorphic to $\tau(\boldsymbol{P}_m) \otimes \tau(\boldsymbol{P}_n)$, where $M = \sigma(\boldsymbol{P}_m \times \boldsymbol{P}_n)$. In fact, since $\tau(\boldsymbol{P}_m)$, $\tau(\boldsymbol{P}_n)$ are holomorphically isomorphic to $\gamma_{m,1}^{\perp} \otimes \gamma_{m,1}^{*}$, $\gamma_{n,1}^{\perp} \otimes \gamma_{n,1}^{*}$, respectively, by ii) of Theorem 5.1, we have only to show that $\Gamma_M^{-1} \gamma_{mn,m+n+1}^{\perp}$, $\gamma_{m+n+mn,1}^{*}|_M$ are holomorphically isomorphic to $(\gamma_{m,1}^{\perp} \otimes \gamma_{n,1}^{\perp})|_M$, $(\gamma_{m,1}^{*} \otimes \gamma_{n,1}^{*})|_M$, respectively. But since the proofs of these facts are elementary, we omit them.

References

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