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KILLING FIELDS PRESERVING MINIMAL FOLIATIONS

By

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Let $(M, g, \mathcal{F})$ be a complete Riemannian manifold with a codimension-one minimal foliation $\mathcal{F}$. In 1982 Oshikiri proved (see [3]) that if all the leaves of such a foliation are compact, any Killing field preserves $\mathcal{F}$.

Supposing M oriented and $\mathcal{F}$ transversally oriented, any Killing field X can be decomposed as $X=Z+\phi N$, where Z is tangent to $\mathcal{F}$ everywhere, N is a unit vector field orthogonal to $\mathcal{F}$ everywhere, and ϕ is a function on M.

Our proposal in this paper is to give the next generalisation of Oshikiri's result.

Theorem. *Let $(M, g, \mathcal{F})$ be a complete Riemannian manifold with a codimension-one minimal foliation $\mathcal{F}$, such that M can be oriented and $\mathcal{F}$ transversally oriented. Let $X=Z+\phi N$ be a Killing field, with ϕ verifying that in any leaf F_i there is a point p_i, such that*

$$(1/\alpha)\left[\int_{B_\alpha(p_i)}\phi^2\right]^{1/2} \longrightarrow 0 \qquad (as\ \alpha\to\infty),$$

where $B_\alpha(p_i)$ is the metric ball with center p_i and of radius α.

Then X preserves $\mathcal{F}$.

Remark. X verifies the hypothesis of the theorem if ϕN or X have finite global norm on each leaf.

In [4] Oshikiri proved that any Killing field preserves a codimension-one totally geodesic foliation if the manifold is compact. By using this theorem Oshikiri suggested the next

Corollary. *Let $(M, g, \mathcal{F})$ be a compact Riemannian manifold with a codimension-one minimal foliation $\mathcal{F}$, such that all the leaves have at most polynomial growth of 1^{st} order, then any Killing field preserves $\mathcal{F}$.*

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Remark. Codimension-one foliations with polynomial growth of finite order are very special (see [2], IX-2).

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1. Proof of the theorem

We begin introducing some technical arguments, part of them coming from [3].

We recall (see [3] or [5]) that for any normal vector field ϕN one can define:

$$J(\phi N)=\{-\Delta_{\mathcal{F}}\phi-\phi\,\mathrm{Ric}_M(N, N)-\phi\,\mathrm{Trace}\,(A^2)\}N.$$

Here $\Delta_{\mathcal{F}}$ is the Laplacian of (F, g_F) for each leaf F of $\mathcal{F}$ (i.e.

$$\Delta_{\mathcal{F}}\phi=-\sum_{i=1}^{n}\{\nabla_{\nabla_{E_i}E_i}\phi-\nabla_{E_i}\nabla_{E_i}\phi\},$$

$n=\dim F$, $\{E_1, \cdots, E_n\}$ is an orthonormal frame of $T(F)$) and A is the Weingarten map associated to N.

One says that ϕN is a Jacobi field if $J(\phi N)=0$.

In [3] it is proved:

Lemma.

$$\mathrm{Ric}_M(N, N)+\mathrm{Trace}\,(A^2)=\mathrm{div}_M(\nabla_N N).$$

So that $J(\phi N)=\{-\Delta_{\mathcal{F}}\phi-\phi\,\mathrm{div}_M(\nabla_N N)\}N.$

One sees easily that ϕN is a Jacobi field.

Let θ be the 1-form associated to $\nabla_N N$ via g.

Let F_0 be a fixed leaf of $\mathcal{F}$ and let p_0 be a fixed point of F_0, we denote $B_\alpha(p_0)=\{q\in F_0 \,|\, \rho_{F_0}(q, p_0)<\alpha\}$, where ρ_{F_0} is the geodesic distance on F_0. We can suppose that p_0 verifies that

$$\frac{1}{\alpha}\cdot\left[\int_{B_\alpha(p_0)}\phi^2\right]^{1/2}\longrightarrow 0 \qquad \text{as } \alpha\to\infty .$$

For any $\alpha\in R^+$ we take a Lipschitz function $w_\alpha : F_0\to \boldsymbol{R}$ (hence a.e. differentiable) satisfying:

$$0\le w_\alpha(q)\le 1, \qquad \forall q\in F_0$$

$$\mathrm{Supp}\, w_\alpha\subset B_{2\alpha}(p_0)$$

$$w_\alpha(q)=1, \qquad \forall q\in B_\alpha(p_0)$$

$$\lim w_\alpha=1, \qquad \text{as } \alpha\to\infty$$

$$|dw_\alpha| \leq (k/\alpha) \qquad (k=\text{constant not depending of } \alpha).$$

For more details about these functions one can see [1] or [6].

As ϕN is a Jacobi field and Stokes' Theorem holds for Lipschitz differential forms we have:

$$0=\int_{F_0}\langle -\Delta_{\mathcal{F}}\phi-\phi\,\mathrm{div}_M(\nabla_N N),\ w_\alpha^2\phi\rangle=\int_{F_0}\langle \delta_{\mathcal{F}}d_{\mathcal{F}}\phi,\ w_\alpha^2\phi\rangle-\int_{F_0}(w_\alpha\phi)^2\,\mathrm{div}_M(\nabla_N N)$$

$$=\int_{F_0}\langle d_{\mathcal{F}}\phi,\ d_{\mathcal{F}}(w_\alpha^2\phi)\rangle-\int_{F_0}\mathrm{div}_M((w_\alpha\phi)^2\nabla_N N)+\int_{F_0}(\nabla_N N)((w_\alpha\phi)^2)$$

$$=\int_{F_0}|w_\alpha d_{\mathcal{F}}\phi|^2+2\int_{F_0}\langle w_\alpha d_{\mathcal{F}}\phi,\ \phi d_{\mathcal{F}}w_\alpha\rangle-\int_{F_0}\mathrm{div}_{\mathcal{F}}((w_\alpha\phi)^2\nabla_N N)$$

$$+\int_{F_0}|w_\alpha\phi\theta|^2+2\int_{F_0}\langle\theta,\ w_\alpha^2\phi d_{\mathcal{F}}\phi\rangle+2\int_{F_0}\langle\theta,\ \phi^2 w_\alpha d_{\mathcal{F}}w_\alpha\rangle$$

$$=\int_{F_0}|w_\alpha(d_{\mathcal{F}}\phi+\phi\theta)|^2+2\int_{F_0}\langle w_\alpha(d_{\mathcal{F}}\phi+\phi\theta),\ \phi d_{\mathcal{F}}w_\alpha\rangle.$$

As Supp $w_\alpha \subset B_{2\alpha}(p_0)$ we have

$$(1/2)\|w_\alpha(d_{\mathcal{F}}\phi+\phi\theta)\|_{B_{2\alpha}}^2=|\langle\!\langle w_\alpha(d_{\mathcal{F}}\phi+\phi\theta),\ \phi d_{\mathcal{F}}w_\alpha\rangle\!\rangle_{B_{2\alpha}}|$$

$$\leq\|w_\alpha(d_{\mathcal{F}}\phi+\phi\theta)\|_{B_{2\alpha}}\cdot\|\phi d_{\mathcal{F}}w_\alpha\|_{B_{3\alpha}}.$$

So that

$$(1/2)\|w_\alpha(d_{\mathcal{F}}\phi+\phi\theta)\|_{B_{2\alpha}}\leq\|\phi d_{\mathcal{F}}w_\alpha\|_{B_{2\alpha}}\leq\|\phi\|_{B_{2\alpha}}\cdot(k/\alpha).$$

But $\|\phi\|_{B_{2\alpha}}\cdot(k/\alpha)\to 0$, as $\alpha\to\infty$, so

$$d_{\mathcal{F}}\phi+\phi\theta=0 \tag{A}$$

But (A) is a necessary and sufficient condition in order to X preserve $\mathcal{F}$ (see [3] or [4]). q. e. d.

2. Examples

Finally we shall give two examples of Killing fields verifying the hypothesis of the Theorem.

Example 1. Let us consider on $R^3=R^2\times R$ the Riemannian metric $g=dx^2+dy^2+\psi(x, y)^2dt^2$, i.e., a warped product $R^2\times_\psi R$. Then $R^2\times(\cdot)$ is a totally geodesic foliation of (R^3, g). The vector field ∂_t is a Killing field and $\partial_t=\psi N$. Taking a function $\psi(x, y)$ such that $\int_{R^2}\psi^2dxdy<\infty$, the hypothesis is satisfied.

Example 2. We begin by looking for a minimal and not totally geodesic foliation. On $R^3=R^2\times R$, let us consider the Riemannian metric $g=dx^2+dy^2+2\alpha dxdt+f^2dt^2$, where α and f are "a priori" functions on R^3.

To ensure the Riemannian character, we need $f^2>\alpha^2$ everywhere.

The foliation $R^2\times(\text{-})$ is minimal and not totally geodesic if $\alpha_y\neq 0$.

One can see easily that ∂_t is Killing if and only if $\alpha_t=0$ and $f_t=0$ everywhere, and that $\partial_t=A\partial_x+BN$, with $A=\alpha$ and $B=\sqrt{f^2-\alpha^2}$.

If $\int_{R^2\times(-)}(f^2-\alpha^2)dxdy<\infty$, the hypothesis of the Theorem is satisfied. For instance we can take $f=f(x, y)$ such that $\int_{R^2}f^2dxdy<\infty$, and as $f^2-\alpha^2>0$, we have $\int_{R^2\times(-)}(f^2-\alpha^2)dxdy<\infty$.

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