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SOME REMARKS ON GRAPH INBEDDINGS

By

Tsuneharu Okabe

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1. Introduction

According to T. Coffin [1], he has a wire with loops at both ends, in a "figure eight" configuration. His main concern is about removing the cord which is looped arround any part of it (see fig. 1).

He said "it may be impossible to remove this cord", and also said "I am unable to offer a proof".

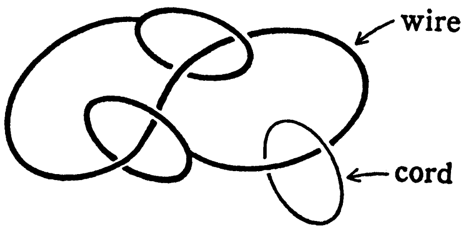


fig. 1

In this paper, we would like to show that it is impossible to remove the cord. For this purpose, we will study the graph imbedded in the 3-dimensional space R^3, since the above mentioned wire is a graph. We already have a tool so called "knot theory"—we have adopted some definitions of the knot theory from [2] and [3]. However, there is a slight and delicate difference between the knots and the graphs. So, we will take some device with which to apply this theory to the graph.

Throughout in this paper, we will assume that "the graph in R^3" is connected, and has a representation by the tame edges and vertices of finite number,

I would like to thank Prof. S. Ichiraku for his useful advice, and Mr. N. Yoshigahara for his helpful comment and his very insightful book ([4]), in the latter of which I happened to know about this problem.

2. Definitions and Notations

We will treat mainly the graph with a loop at each of both ends, in this paper. We put it "G", and we call "standard type" if it is embedded as fig. 2.

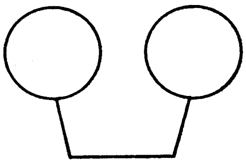


fig. 2 standard type G

We will introduce "collars" of the graph A. For this purpose, we make the following definitions.

Any imbedded graph A can be projected in a properly chosen direction onto a plane P in such a way that (1) there are no triple points and (2) no vertex of G is projected into a double point. Projection of this sort is called "regular".

Over each double point, A has an overcrossing point and an undercrossing, and there are two edges—one has the former point and the other has the latter point. These edges are called "overcrossing edge" and "undercrossing edge".

Let A be a imbedded graph and P be a plane which has a regular projection of A, then we make a new surface Q which has one handle along the overcrossing edge, at each double point. So, the graph A can be imbedded in the surface Q. Then, A have a regular neighbourhood in Q. The images of this regular neighbourhood by the homeomorphisms are called "collars", and denoted "$C_i(A)$" (fig. 3).

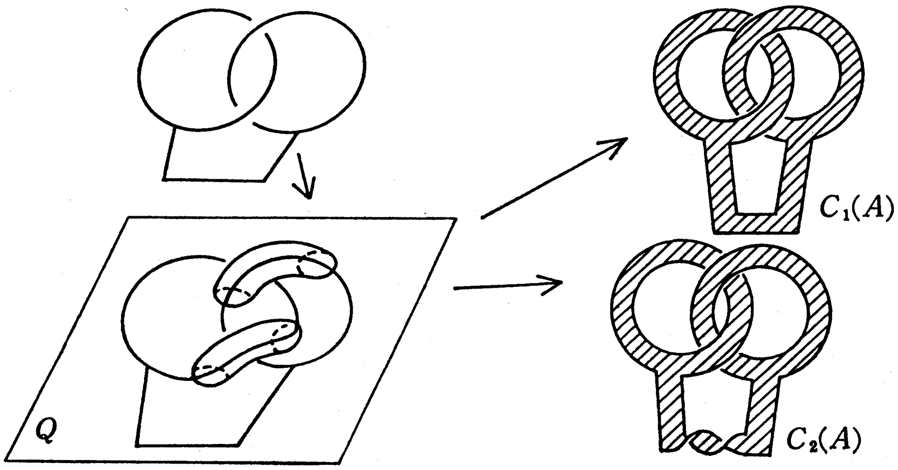
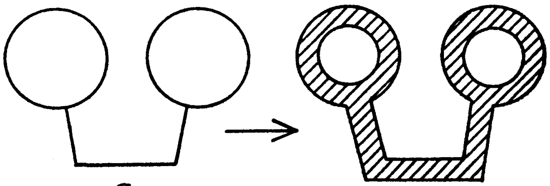


fig. 3

If A is imbedded in a plane, we have the simplest collar denoted as "$C(A)$". The collar $C(A)$ is a disc with holes. Especially, $C(G)$ is a disc with two holes (fig. 4).

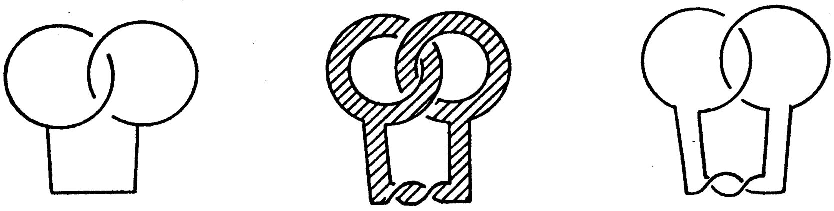


G $C(G)$

fig. 4

Let "$f: G \to R^3$" be a P. L. imbedding, "$f(G)$" is an image of f, and "$N\{f(G)\}$" is a regular neighbourhood of $f(G)$ in R^3. In generally, "$N(A)$" denotes the regular neighbourhood of A.

There exist collars $C_i\{f(G)\}$ of $f(G)$ which are homeomorphic to $C(G)$. Then, the outside boundary of $C_i\{f(G)\}$ makes a knot (fig. 5). We denote this knot as "$kC_i\{f(G)\}$".

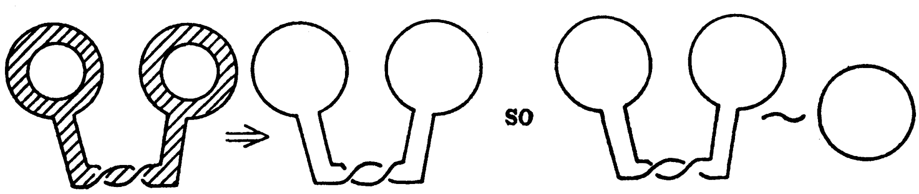


$f'(G)$ $C_1\{f'(G)\}$ $kC_1\{f'(G)\}$

fig. 5

Example 1.

In the case of the graph of the standard type G, the knots $kC_i(G)$ are all the same. It is a "trivial knot". (fig. 6) (the knot equivalent to the standard circle—circle $x^2+y^2=1$ in xy-plane—is called "trivial knot"). Because each of the knots $kC_i(G)$ is transformed into the circuit consisting of the four edges of an imbedded rectangles, and this rectangle is transformed into the imbedded disc.



$C(G)$ $kC_1(G)$ spans a disc

fig. 6

Example 2.

In the case of the Coffin's wire, one example of the knots is given by the following fig. 7.

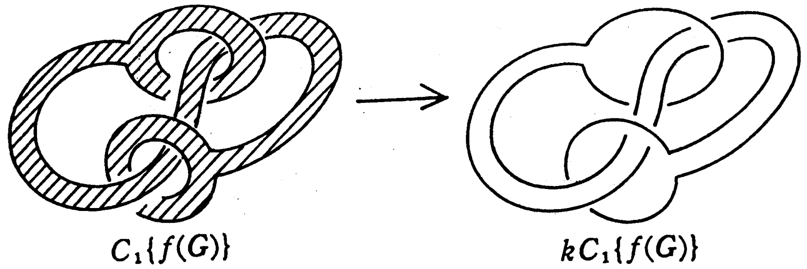


fig. 7

The problem can be essentially expressed as: can imbedded graph of Coffin's type $f(G)$ be transformed into the standard type G, in R^3?. So, we define that two figures A and B are equivalent if there exists a homeomorphism of R^3 onto itself which maps A onto B. It means that the graph $f(G)$ is transformed into the graph $g(G)$ in R^3, then $f(G)$ and $g(G)$ are equivalent.

Now, we must pay an attention to the fact that this problem can't be reduced to the following problem: are the two regular neighbourhoods $N\{f(G)\}$ and $N\{g(G)\}$ equivalent? A counter example is shown as fig. 8.

Example 3.

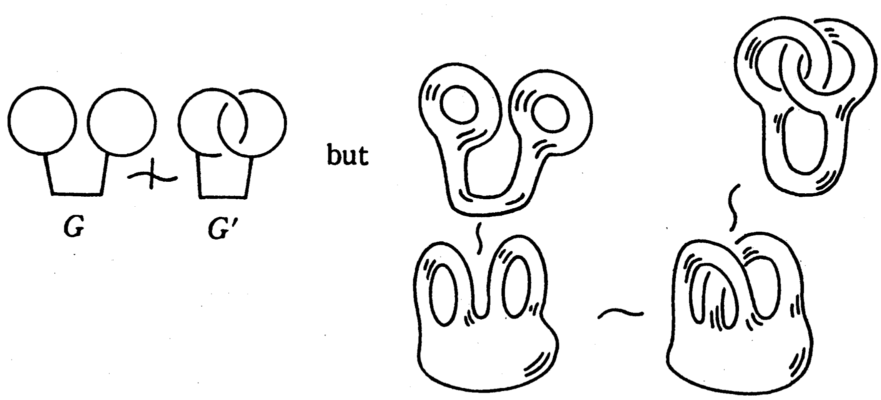


fig. 8 The graphs G and G' are not equivalent. But, their regular neighbourhoods $N(G)$ and $N(G')$ are equivalent.

This example also shows that even if two figures A and B are not equivalent, by adding some figure C to figures A and B, $A\cup C$ and $B\cup C$ may be equivalent (fig. 9).

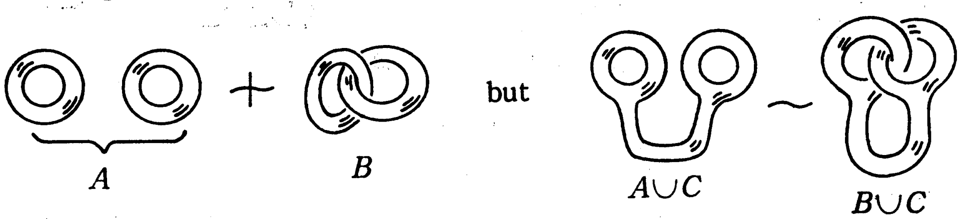


fig. 9

Therefore, we must treat carefully "the problems of transforming graph", and the following property is useful.

Proposition 1. *If two graphs $f(G)$ and $g(G)$ in R^3 are equivalent, then "the link consisting of the two loops at the ends of $f(G)$" and "the link consisting of the two loops at the ends of $g(G)$" are equivalent.*

Proof. Consider $C_i\{f(G)\}$ which is the one of the collar of $f(G)$. It have two loops with inner boundary. These are just the link of $f(G)$. The collar $C_i\{f(G)\}$ is transformed into $C_j\{g(G)\}$—is one of the collars of $g(G)$—by the homeomorphism of R^3, that transform $f(G)$ into $g(G)$.

Now, the homeomorphism maps the boundary of the surface into the boundary of the surface. Therefore, the inner boundary of $C_i\{f(G)\}$ is transformed into the inner boundary of $C_i\{g(G)\}$ in R^3. So, the link of $f(G)$ is transformed into the link of $g(G)$ in R^3. Q. E. D.

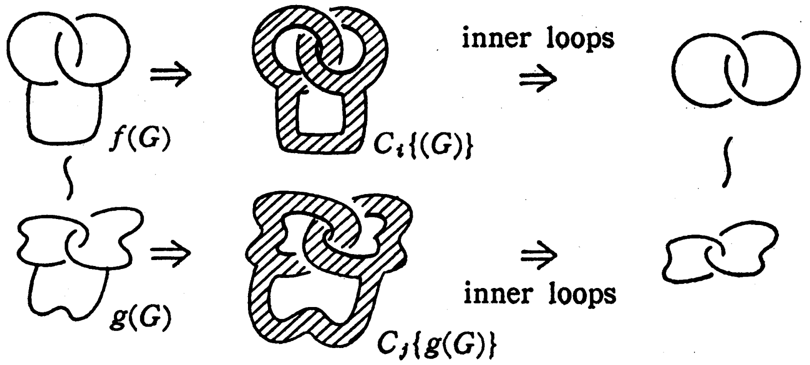


fig. 10

Returning to example 3, we see that any of the collars of G' has the two loops linked. On the other hand, any of the collars of G has the two loops unlinked. Therefore G and G' are not equivalent.

3. Property and Proofs

Proposition 2. *If two graphs $f(G)$ and $g(G)$ are equivalent, and a knot $kC_i\{f(G)\}$ is given by a collar of $f(G)$, then for a knot $kC_i\{f(G)\}$ there exists a knot $kC_j\{g(G)\}$ which is equivalent to $kC_i\{f(G)\}$.*

Proof. If, two graphs $f(G)$ and $g(G)$ are equivalent, then, there exists a homeomorphism F of R^3 onto itself, which maps $f(G)$ onto $g(G)$. This homeomorphism F maps $C_i\{f(G)\}$ onto $F[C_i\{f(G)\}]$. The image $F[C_i\{f(G)\}]$ is a collar of $g(G)$. We denote this as "$C_j\{g(G)\}$". Therefore, F maps $kC_i\{f(G)\}$ onto $kF[C_i\{f(G)\}]=kC_j\{g(G)\}$. Q. E. D.

Corollary 3. *If a graph $f(G)$ is equivalent to the standard graph G, then $kC_i\{f(G)\}$ is a trivial knot.*

Proof. This corollary is obvious from the fact that $kC_j(G)$ is the trivial knot (see Example 1). Q. E. D.

For the later applications, we recall the useful theorem that is called "Dehn's lemma".

Theorem 4 (T. Homma [5], C. D. Papakyriakopoulos [6]). *Let M be a 3-manifold such that the induced map*

$$\pi_1(\partial M) \longrightarrow \pi_1(M)$$

has non-trivial kernel. Then there exists an essential simple closed curve in ∂M which spans an embedded disc in M.

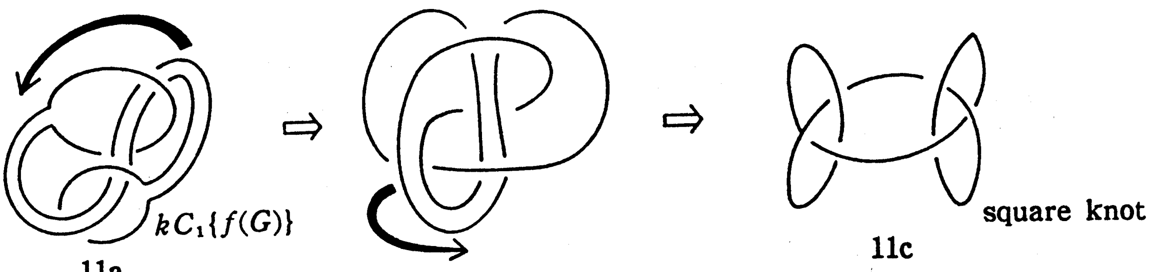
Consider the case of $M=R^3-\mathrm{Int}\,\{N(A)\}$. Since we have $\partial M=\partial N(A)$, we may regard the loop arround graph A as imbedded in ∂M. Therefore, we can guarantee the following: if this loop shrinks in $R^3-\mathrm{Int}\,\{N(A)\}$, then it spans a disc in $R^3-\mathrm{Int}\,\{N(A)\}$.

4. Applications to Coffin's Problem

Now, we prove that it is impossible to remove the cord of Coffin's wire. For this purpose, we look at this graph.

Proposition 5. *Let $f(G)$ be a graph which comes from Coffin's wire. Then, there exists a knot $kC_i\{f(G)\}$ which is not a trivial knot* (*see Example* 2).

Proof. We take the knot $kC_1\{f(G)\}$ as is shown in fig. 11a. Then it can be transformed as fig. 11. Finally it would be transformed into the so called "square knot".



11a 11b 11c

fig. 11

We have a following result in the knot theory: the equivalent knots have the same "fundamental groups", and therefore, the same "Alexander polynomials".

The fundamental group of the trivial knot is infinite cyclic group and Alexander polynomial is 1. On the other hand, the fundamental group of the square knot is $|x, y, a: a^{-1}xa=xax^{-1}, a^{-1}ya=yay^{-1}|$, and Alexander polynomial is $(1-t+t^2)^2$ (see R. H. Crowell and R. H. Fox [2]). Therefore, the knot $kC_1\{f(G)\}$ is never equivalent to the trivial knot. Q. E. D.

Finally, we get the following,

Proposition 6. *The cord of Coffin's graph cannot be removed.*

Proof. If this loop of the cord is removed, the loop can shrink in the space $R^3-f(G)$. Therefore, the loop spans a disc in $R^3-f(G)$. This means that this disc does't intersect the wire. Then, we can remove the loop of the wire along the disc (fig. 12).

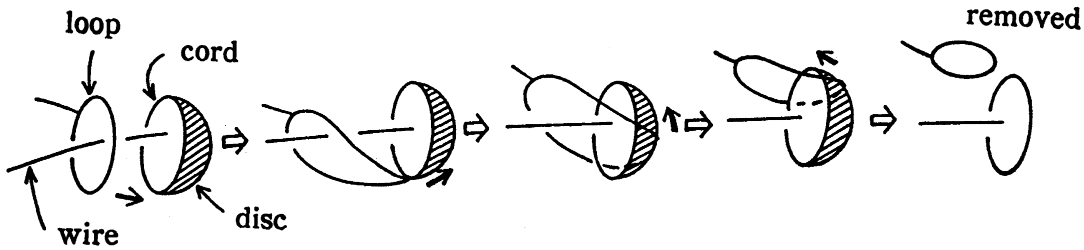


fig. 12

fig. 13

This means that the graph $f(G)$ and standard graph are equivalent (see, (fig. 13). And therefore, the $kC_1\{f(G)\}$ is trivial knot. But, this contradicts to Proposition 5. Q. E. D.

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TSUNEHARU OKABE
College of Liberal Arts
Saitama University
Urawa, Saitama, Japan