

ISOMETRIC IMMERSIONS BETWEEN INDEFINITE RIEMANNIAN SPHERES

By

MARCOS DAJCZER and SUSANA FORNARI

(Received September 24, 1986)

Introduction.

In the present paper we consider isometric immersions of $S_s^n(c)$ into $S_t^{n+p}(c)$, where we denote by $S_s^n(c)$, $n \geq 2$, the n -dimensional indefinite Riemannian space form of constant positive curvature c and signature $(s, n-s)$, i. e., the diagonal form of the inner product has s minus signs and $(n-s)$ plus signs. For $t=s$, we extend here the main theorem of Ferus in [3] to the indefinite Riemannian case.

Theorem. *Let $f: S_s^n(c) \rightarrow S_t^{n+p}(c)$ be an isometric immersion with $1 \leq p \leq n-s-1$. Then f is totally geodesic.*

For $n-s=1$, the conclusion of the above theorem does not hold in general. Graves and Nomizu constructed examples of non totally geodesic isometric immersions of $S_{n-1}^n(c)$ into $S_{n-1}^{n+1}(c)$ [see [4], Sections 1, 3]. In case $p=n-s+1$, we obtain examples of non totally geodesic immersions of $S_s^n(c)$ into $S_{s+t}^{2n-s+1}(c)$ for $0 \leq t \leq n-s+1$. The case $p=n-s > 1$ can be related with the problem in the positive definite case (which is still open) whether one can get non totally geodesic isometric immersions of $S^n(c)$ into $S^{2n}(c)$: If there exists such an isometric immersion in the positive definite case then we know how to obtain an example in the indefinite case [see Example III in Section 3]. We observe that a much weaker version of the above theorem has been obtained by Graves and Nomizu in [4], where it is proved that there is no umbilicfree isometric embeddings of $S_1^n(c)$ into $S_1^{n+1}(c)$ for $n \geq 4$.

Finally we obtain examples of non totally geodesic isometric immersions of S_s^n into S_{s+1}^{n+2} for $0 \leq s \leq n-1$, showing that other extensions of the Theorem are not possible.

We recall that there exists a one to one correspondence between indefinite spheres of signature $(s, n-s)$ and indefinite hyperbolic spaces of signature $(n-s, s)$; consequently the above results holds also for hyperbolic spaces.

The authors sincerely thank Professor P. Dombrowski who pointed out a

serious gap in an earlier version of this paper and made several other important remarks.

1. Preliminaries.

Let R_s^n be the n -dimensional real vector space R^n with the inner product of signature $(s, n-s)$ given by

$$\langle X, Y \rangle = - \sum_{i=1}^s x_i y_i + \sum_{j=s+1}^n x_j y_j$$

for $X = (x_1, \dots, x_n)$ and $Y = (y_1, \dots, y_n)$. For a positive number c , the standard space form $S_s^n(c)$ is the hypersurface

$$\left\{ X \in R_s^{n+1} : \langle X, X \rangle = \frac{1}{c} \right\}$$

with the induced indefinite metric of signature $(s, n-s)$. For a negative number c , the standard space form $H_s^n(c)$ is the hypersurface

$$\left\{ X \in R_{s+1}^{n+1} : \langle X, X \rangle = \frac{1}{c} \right\}$$

with the induced metric of signature $(s, n-s)$.

In case $s=0$, we write $S^n(c)$ for $S_0^n(c)$, and in case $s=1$, $H^n(c)$ for $H_0^n(c) \cap (R_+ \times R^n)$.

By changing the sign of the inner product $\langle \cdot, \cdot \rangle$ in R_s^{n+1} , we obtain the inner product in R_{n-s+1}^{n+1} . In particular $S_s^n(c)$ changes to $H_{n-s}^n(-c)$. Thus, all results about indefinite spheres became results about indefinite hyperbolic spaces. See [6, p. 110].

For every point x of S_s^n (resp. H_s^n) one knows: Any complete, connected m -dimensional totally geodesic submanifold of S_s^n (resp. H_s^n) through x is the connected component of the intersection of S_s^n (resp. H_s^n) with an $(m+1)$ -dimensional vector subspace of R_s^{n+1} (resp. R_{s+1}^{n+1}) which contains x , and vice-versa. See [6, p. 112].

Let M_s^n denote an indefinite Riemannian manifold of dimension n and signature $(s, n-s)$ and let $f: M_s^n \rightarrow \tilde{M}_t^{n+p}$ be an isometric immersion. For each $x \in M$ set

$$T^0(x) = \{ Y \in T_x M : \alpha(Y, Z) = 0 \text{ for all } Z \in T_x M \},$$

where $T_x M$ is the tangent space of M at x and α is the second fundamental form of the immersion f . Here $T^0(x)$ denotes the relative nullity space of α at x and its dimension $\nu(x)$ is called the *index of relative nullity* of α at x . The minimum value of $\nu(x)$ for all $x \in M$ is called the *minimum relative nullity* and is denoted by ν_0 .

Lemma 1. Assume that \tilde{M}_t^{n+p} is a space form and let G be the set of points

in M where $\nu(x)=\nu_0$. Then

- i) G is an open subset of M .
- ii) The distribution $x \rightarrow T^0(x)$ in G is differentiable and involutive.
- iii) The foliation T^0 is totally geodesic in M .
- iv) Each leaf of T^0 is immersed as a totally geodesic submanifold of \tilde{M} .
- v) If M^n is complete then the relative nullity foliation is geodesically complete.

Proof. See [1] Theorems 1 and 2.

2. Proof of the Theorem.

In order to prove the theorem we need a linear algebra result. Let V, W be finite dimensional vector spaces. For a bilinear map $\beta: V \times V \rightarrow W$ we define the subspace $N(\beta)$ by

$$N(\beta) = \{X \in V : \beta(X, Y) = 0 \text{ for all } Y \in V\}.$$

In addition, we say that β is flat with respect to \langle, \rangle if

$$\langle \beta(X, Z), \beta(Y, W) \rangle - \langle \beta(X, W), \beta(Y, Z) \rangle = 0 \text{ for all } X, Y, Z, W \in V,$$

where \langle, \rangle is an inner product for W .

Lemma 2. If $\beta: V \times V \rightarrow W$ is flat with respect to either a Lorentzian or a Euclidean inner product in W and β is onto W , then

$$\dim N(\beta) \geq \dim V - \dim W.$$

Proof. See [5].

We say that \langle, \rangle is a Lorentzian (Euclidean) inner product in W if (W, \langle, \rangle) is isometric to $R_1^{\dim W}$ ($R^{\dim W}$).

Proof of the Theorem. The intersection of $S_s^n(c)$, considered as a submanifold of R_s^{n+1} , with the $(n-s+1)$ -dimensional vector subspace $P = \{0\} \times R^{n-s+1}$ of R_s^{n+1} defines the totally geodesic submanifold

$$M = \left\{ (0, \dots, 0, x_{s+1}, \dots, x_{n+1}) \in R_s^{n+1} : x_{s+1}^2 + \dots + x_{n+1}^2 = \frac{1}{c} \right\}$$

of $S_s^n(c)$, which is isometric to the standard Euclidean sphere $S^{n-s}(c)$. Therefore in the following we identify $S^{n-s}(c)$ with M and we denote the restriction of f to $S^{n-s}(c)$ by \tilde{f} . Then we have

$$\tilde{f}: S^{n-s}(c) \longrightarrow S_s^{n+p}(c).$$

Let α (resp. $\tilde{\alpha}$) denote the second fundamental form of f (resp. \tilde{f}). Then, since $S^{n-s}(c)$ is a totally geodesic submanifold of $S_s^n(c)$, we have for all points $x \in S^{n-s}(c)$

$$\bar{\alpha}(X, Y) = \alpha(X, Y) \quad \text{for all } X, Y \in T_x S^{n-s}(c). \quad (1)$$

Thus, for all $x \in S^{n-s}(c)$ we may consider $\bar{\alpha}$ as a mapping

$$\bar{\alpha}: T_x S^{n-s}(c) \times T_x S^{n-s}(c) \longrightarrow T_x S^n(c)^\perp, \quad (2)$$

where the inner product of $S^n(c)$ restricted to the normal space $T_x S^n(c)^\perp$ of f at x must be positive definite.

Let ν_0 (resp. $\bar{\nu}_0$) denote the minimum relative nullity of the immersion f (resp. \bar{f}). Since f and \bar{f} are isometric immersions of constant curvature manifolds of the same curvature value c , it follows from the Gauss equation that α and $\bar{\alpha}$ are flat bilinear forms. From (2) and Lemma 2 it follows that

$$\nu_0 \geq n - p \geq s + 1 \quad \text{and} \quad \bar{\nu}_0 \geq n - s - p \geq 1, \quad (3)$$

where the second inequality follows from our hypothesis on the codimension p .

By Lemma 1 the leaves of the relative minimum nullity distribution of \bar{f} are complete and totally geodesic $\bar{\nu}_0$ -dimensional submanifolds. In this situation, the same argument as in [3] holds and we conclude that \bar{f} is a totally geodesic immersion of $S^{n-s}(c)$ into $S^n(c)$. For every $x \in S^{n-s}(c)$ and every $X \in T_x S^{n-s}(c)$, we have therefore $\bar{\alpha}(X, X) = 0$, which implies by (1) that $\alpha(X, X) = 0$. Consequently, since α is flat

$$\langle \alpha(X, Y), \alpha(X, Y) \rangle = \langle \alpha(X, X), \alpha(Y, Y) \rangle = 0$$

i. e. $\alpha(X, Y) = 0$ for all $Y \in T_x S^n(c)$. Thus we have shown

$$T_x S^{n-s}(c) \subset T^0(x) \quad \text{for all } x \in S^{n-s}(c), \quad (4)$$

where $T^0(x)$ is the relative nullity space of α at x . Now we are prepared to show that f is totally geodesic, for which according Lemma 1, it is sufficient to show

$$\nu_0 = n. \quad (5)$$

This is because then $\dim T^0(x) = n$ for all $x \in S^n(c)$, therefore $G = S^n(c)$ and the connected manifold $S^n(c)$ (observe $n - s \geq 2$) is the only leaf of the foliation T^0 which implies by Lemma 1 that $f: S^n(c) \rightarrow S^{n+p}(c)$ is totally geodesic.

Let L denote an arbitrary leaf of the foliation T^0 of the open subset G of $S^n(c)$ and choose $p_0 \in L$. Since, according to Lemma 1 iii) and v), L is a totally geodesic, complete ν_0 -dimensional submanifold of $S^n(c)$, there exists a $(\nu_0 + 1)$ -dimensional vector subspace Q of $R_{p_0}^{n+1}$ such that:

$$L = \text{connected component of any } x \in L \text{ in } S^n(c) \cap Q. \quad (6)$$

Let $P = \{x \in R_{p_0}^{n+1}: x_1 = \dots = x_s = 0\}$ denote the $(n + 1 - s)$ -dimensional vector subspace of $R_{p_0}^{n+1}$ with

$$S^{n-s}(c) = S^n(c) \cap P. \quad (7)$$

Then we obtain, using that $\dim(P+Q) \leq \dim R^{n+1}$, that

$$\dim(P \cap Q) = \dim P + \dim Q - \dim(P+Q) \geq \nu_0 - s + 1 \geq 2.$$

We can choose therefore $q_0 \in P \cap Q$ with $\langle q_0, q_0 \rangle = 1/c$, which means by (7), that $q_0 \in S^{n-s}(c) \cap Q$. But since $S^{n-s}(c) \cap Q$ contains $(-q_0)$ as well, we might assume without loss of generality, that $\langle p_0, q_0 \rangle \geq 0$. Thus we get

$$\begin{aligned} p_0 &\in S_s^n(c) \cap Q, & q_0 &\in S^{n-s}(c) \cap Q \\ \langle p_0, p_0 \rangle &= \langle q_0, q_0 \rangle = \frac{1}{c} > 0 & \text{and } \langle p_0, q_0 \rangle &\geq 0. \end{aligned} \quad (8)$$

From (8) it follows that the segment $\gamma: [0, 1] \rightarrow Q \subset R_s^{n+1}$ defined by

$$\gamma(t) = (1-t)p_0 + tq_0 \quad \text{for } 0 \leq t \leq 1,$$

consists only of space-like vectors and therefore $\gamma_0 = [c\langle \gamma, \gamma \rangle^{1/2}]^{-1} \gamma$ is a continuous path in $S_s^n(c) \cap Q$ with $\gamma_0(0) = p_0 \in L$. It follows from (6) that $q_0 = \gamma_0(1) \in L$, which together with (8), implies that $q_0 \in L \cap S^{n-s}(c)$. Since from (6) and (7)

$$P = T_{q_0} S^{n-s} + R_{q_0} \quad \text{and} \quad Q = T_{q_0} L + R_{q_0},$$

we obtain from (4) that $P \subset Q$ and therefore $q_0 \in S^{n-s}(c) = S_s^n(c) \cap P \subset S_s^n(c) \cap Q$ i.e., since $S^{n-s}(c)$ is connected, $S^{n-s}(c)$ is contained in the connected component of the points q_0 in $S_s^n(c) \cap Q$. But $q_0 \in L$, therefore we get $S^{n-s}(c) \subset L$. Thus we have shown: Every leaf L of T^0 contains the nonempty set $S^{n-s}(c)$, but two different leaves of any foliation are always disjoint, therefore the ν_0 -dimensional foliation T^0 of the n -dimensional open submanifold G of $S_s^n(c)$ can consist only of one single leaf, which implies (5) and completes the proof of the theorem.

3. Examples of non totally geodesic immersions between indefinite Riemannian spheres.

Let $g^{(n,s)}: S_s^n \times H^p \rightarrow S_{p+s}^{n+p}$, $0 \leq s \leq n-1$, $p \geq 1$, be the map defined by

$$g^{(n,s)}(x_1, \dots, x_{n+1}, y_0, \dots, y_p) = (y_1, \dots, y_p, y_0 x_1, \dots, y_0 x_{n+1}) \in R_{p+s}^{n+p+1}, \quad (9)$$

where $-\sum_{i=1}^s x_i^2 + \sum_{j=s+1}^{n+1} x_j^2 = 1$ and $-y_0^2 + \sum_{i=1}^p y_i^2 = -1$, since $x = (x_1, \dots, x_{n+1}) \in S_s^n \subset R_s^{n+1}$ and $y = (y_0, \dots, y_p) \in H^p \subset R_1^{p+1}$. Then, $g^{(n,s)}$ is one to one and onto S_{p+s}^{n+p} , and if $g_*^{(n,s)}$ is the differential of $g^{(n,s)}$ at (x, y) , we have

$$\begin{aligned} \langle g_*^{(n,s)}(X, 0), g_*^{(n,s)}(Z, 0) \rangle &= y_0^2 \langle X, Z \rangle \\ \langle g_*^{(n,s)}(X, 0), g_*^{(n,s)}(0, Y) \rangle &= 0 \\ \langle g_*^{(n,s)}(0, Y), g_*^{(n,s)}(0, W) \rangle &= -\langle Y, W \rangle, \end{aligned} \quad (10)$$

for all $X, Z \in T_x S_s^n$ and $Y, W \in T_y H^p$.

Lemma 3. If $\gamma: S^n_s \rightarrow S^{n+k}_{s+t}$ is a non totally geodesic isometric immersion, then $F: S^{n+p}_s \rightarrow S^{n+p+k}_{s+t}$ defined by

$$F = g^{(n+k, s+t)} \circ \gamma \times id_{H^p} \circ (g^{(n, s)})^{-1}$$

is a non totally geodesic isometric immersion.

Proof. From (10) it follows that F is an isometric immersion. F is non totally geodesic if and only if $F(S^{n+p}_s)$ can not be contained in an $(n+p+1)$ -dimensional vector subspace of $R^{n+p+k+1}_{s+t}$, thus there must exist $n+p+2$ points p_1, \dots, p_{n+p+2} in S^{n+p}_s such that $F(p_1), \dots, F(p_{n+p+2})$ are linearly independent in $R^{n+p+k+1}_{s+t}$.

Since γ is non totally geodesic, there exists $\theta_1, \dots, \theta_{n+2}$ in S^n_s such that $\gamma(\theta_1), \dots, \gamma(\theta_{n+2})$ are linearly independent in R^{n+k+1}_{s+t} . Consider the $p+1$ points in $H^p: e_0 = (1, 0, \dots, 0), e_1 = (\sqrt{2}, 1, 0, \dots, 0), \dots, e_p = (\sqrt{2}, 0, \dots, 1)$ and let θ_0 be in S^n_s . We define the following $n+2+p$ points in S^{n+p}_s

$$p_i = g^{(n, s)}(\theta_i, e_0) \quad \text{for } 1 \leq i \leq n+2$$

and

$$p_{n+2+j} = g^{(n, s)}(\theta_0, e_j) \quad \text{for } 1 \leq j \leq p.$$

Therefore

$$F(p_i) = (0, \gamma(\theta_i)) \quad \text{for } 1 \leq i \leq n+2$$

$$F(p_{n+2+j}) = (a_j, \sqrt{2} \gamma(\theta_0)) \quad 1 \leq j \leq p,$$

where $a_j = (0, \dots, \overset{j}{1}, \dots, 0) \in R^p_s$. Clearly, the set $\{F(p_i), F(p_{n+2+j}); 1 \leq i \leq n+2, 1 \leq j \leq p\}$ is linearly independent in $R^{n+p+k+1}_{s+t}$.

I. Examples of non totally geodesic immersions of S^n_t into S^{n+s+1}_{t+s+1} for $0 \leq t \leq n-s+1$.

First we will construct an isometric immersion $G: S^n_1(1) \rightarrow S^{n+s+1}_{t+s+1}(1)$ for $0 \leq t \leq n$, where the relative nullity is a 1-dimensional distribution, thus G is non totally geodesic.

Let us consider the mapping g of $(r, x) \in R \times S^{n-1}$ into S^n_1 given by

$$g(r, x) = \cosh r(0, x) + \sinh r(1, 0). \quad (11)$$

Then g is one to one and onto S^n_1 and if g_* is the differential of g at (r, x) , we have

$$g_*\left(\frac{\partial}{\partial r}\right) = \sinh r(0, x_1, \dots, x_{n-1}, D) + \cosh r(1, 0, \dots, 0)$$

$$g_*\left(\frac{\partial}{\partial x_k}\right) = \cosh r(0, \dots, 1, \dots, \frac{-x_k}{D}) \quad \text{for } 1 \leq k \leq n-1$$

where $D = \left(1 - \sum_{j=1}^{n-1} x_j^2\right)^{1/2}$. Thus

$$\left\langle g_*\left(\frac{\partial}{\partial r}\right), g_*\left(\frac{\partial}{\partial r}\right) \right\rangle = -1$$

$$\left\langle g_*\left(\frac{\partial}{\partial r}\right), g_*\left(\frac{\partial}{\partial x_k}\right) \right\rangle = 0$$

$$\left\langle g_*\left(\frac{\partial}{\partial x_i}\right), g_*\left(\frac{\partial}{\partial x_j}\right) \right\rangle = \left(\delta_{ij} + \frac{x_i x_j}{D}\right) \cosh^2 r \quad \text{for } 1 \leq i, j \leq n-1.$$

We consider the mapping Z of S^{n-1} into $S_t^{2n-1} \subset R_t^{2n}$ given by

$$\begin{aligned} Z(x) = & \frac{1}{\sqrt{t}} \left(\cosh(\sqrt{t} x_1), \dots, \cosh(\sqrt{t} x_t), \sinh(\sqrt{t} x_1), \dots, \right. \\ & \left. \sinh(\sqrt{t} x_t), 0, \dots, 0 \right) + \frac{\sqrt{2}}{\sqrt{n-t}} \left(0, \dots, 0, \sin\left(\sqrt{\frac{n-t}{2}} x_{t+1}\right), \right. \\ & \left. \cos\left(\sqrt{\frac{n-t}{2}} x_{t+1}\right), \dots, \sin\left(\sqrt{\frac{n-t}{2}} D\right), \cos\left(\sqrt{\frac{n-t}{2}} D\right) \right). \end{aligned}$$

In case $t=0$, Z is the canonical isometric immersion of the n -dimensional Euclidean space as a flat torus in the sphere S^{2n-1} .

Now we define the mapping f of $(r, x) \in R \times S^{n-1}$ into $S_{t+1}^{2n} \subset R_{t+1}^{2n+1}$ by

$$f(r, x) = \cosh r(0, Z(x)) + \sinh r(1, 0). \quad (12)$$

For the differential f_* of f at (r, x) we have

$$f_*\left(\frac{\partial}{\partial r}\right) = \sinh r(0, Z(x)) + \cosh r(1, 0)$$

$$f_*\left(\frac{\partial}{\partial x_k}\right) = \cosh r\left(0, \frac{\partial Z}{\partial x_k}\right) \quad \text{for } 1 \leq k \leq n-1.$$

Thus

$$\left\langle f_*\left(\frac{\partial}{\partial r}\right), f_*\left(\frac{\partial}{\partial r}\right) \right\rangle = -1$$

$$\left\langle f_*\left(\frac{\partial}{\partial r}\right), f_*\left(\frac{\partial}{\partial x_k}\right) \right\rangle = 0 \quad (13)$$

$$\left\langle f_*\left(\frac{\partial}{\partial x_i}\right), f_*\left(\frac{\partial}{\partial x_j}\right) \right\rangle = \left(\delta_{ij} + \frac{x_i x_j}{D}\right) \cosh^2 r \quad \text{for } 1 \leq i, j \leq n-1.$$

We define $G = f \circ g^{-1}: S_1^n(1) \rightarrow S_{t+1}^{2n}(1)$. The relations (12) and (13) show that G is an isometric immersion. The relative nullity distribution of G is 1-dimensional. Finally we define $F: S_{p+1}^{n+p} \rightarrow S_{p+1+t}^{2n+p}$ by the composition

$$F = g^{(2n, t+1)} \circ G \times id_{H^{p \circ} (g^{(n, 1)})^{-1}} \quad \text{for } 0 \leq t \leq n.$$

It follows from Lemma 3 that F is a non totally geodesic isometric immersion.

By putting $k=n+p$ and $s=p+1$, we obtain $F: S_s^k \rightarrow S_{s+t}^{2k-s+1}$ for $t \leq k-s+1$, as we wanted.

II. Examples of non totally geodesic isometric immersions of S_s^n into S_{s+1}^{n+2} for $0 \leq s \leq n-1$.

Let $\phi: S^n \rightarrow R$ be a C^∞ function and let $\phi: S^n \rightarrow R^{n+1}$ be the standard isometric immersion. Let $\gamma: S^n \rightarrow S_1^{n+2}$ be defined by

$$\gamma(x) = (\phi(x), \phi(x), \phi(x)) \quad \text{for all } x \in S^n.$$

Since $\langle \gamma(x), \gamma(x) \rangle = -\phi^2(x) + \phi^2(x) + \langle \phi(x), \phi(x) \rangle = 1$, we have that $\gamma(S^n) \subset S_1^{n+2}$. Then γ is an isometric immersion because

$$\begin{aligned} \langle \gamma_*(X), \gamma_*(Y) \rangle &= \langle (X(\phi), X(\phi), \phi_*(X)), (Y(\phi), Y(\phi), \phi_*(Y)) \rangle \\ &= \langle \phi_*(X), \phi_*(Y) \rangle \\ &= \langle X, Y \rangle, \quad \text{for all } X, Y \in T_x S^n. \end{aligned}$$

Now, we calculate the second fundamental form α of γ . Let ∇ be the Levi-Civita connection associated to the standard inner product of S^n and let $\tilde{\nabla}$ be the Levi-Civita connection associated to the Lorentzian inner product of R_1^{n+3} . For $X, Y \in T_x S^n$ we have $\tilde{\nabla}_{\gamma_* X} \gamma_* Y = \gamma_*(\nabla_X Y) + \alpha(X, Y) - \langle X, Y \rangle \gamma$. On the other hand

$$\begin{aligned} \tilde{\nabla}_{\gamma_* X} \gamma_* Y &= ((X(Y\phi), X(Y\phi), \nabla_X Y - \langle X, Y \rangle \phi)) \\ &= ((\nabla_X Y)\phi, (\nabla_X Y)\phi, \nabla_X Y) - \langle X, Y \rangle (0, 0, \phi) + (X(Y\phi) - (\nabla_X Y)\phi)(1, 1, 0) \\ &= \gamma_*(\nabla_X Y) - \langle X, Y \rangle \gamma + (X(Y\phi) - (\nabla_X Y)\phi + \langle X, Y \rangle \phi)(1, 1, 0). \end{aligned}$$

Let H_ϕ denote the Hessian of the function ϕ in the riemannian sphere S^n . Then

$$\alpha(X, Y) = (H_\phi(X, Y) + \langle X, Y \rangle \phi)(1, 1, 0), \quad \text{for all } X, Y \in T_x S_1^n.$$

Observe that the image of α is a degenerate subspace of $T_{\gamma(x)} S^{n+2}$. Obviously, there exists functions $\phi \in C^\infty(S^n)$ such that $\alpha \neq 0$, then there exists a non totally geodesic immersion of S^n into S_1^{n+2} .

Now, using the functions $g^{(n,s)}$ defined in (9) we may obtain non totally geodesic immersions of S_s^n into S_{s+1}^{n+2} for $0 \leq s \leq n-1$. For other examples of non totally geodesic immersion between indefinite spheres see [2].

III. Existence of non totally geodesic isometric immersion of $S_s^n \rightarrow S_s^{2n-s}$.

If there exists an non totally geodesic isometric immersion $Z: S^{n-1}(1) \rightarrow S^{2(n-1)}(1)$, then we can define a mapping f of $(t, x) \in R \times S^{n-1}$ into S_1^{2n-1} by

$$f(t, x) = \cosh t(0, Z(x)) + \sinh t(1, 0), \quad x \in S^{n-1}(1).$$

Thus, if g is the diffeomorphism defined in (11) the mapping $G = f \circ g^{-1}$ of $S_1^n(1)$ into $S_1^{2n-1}(1)$, is a non totally geodesic isometric immersion.

References

- [1] Abe, K. and Magid, M.: Relative nullity foliations and indefinite isometric immersions. *Pacific J. of Math.* 124 (1986), 1-20.
- [2] Dajczer, M. and Dombrowski, P.: Examples of 1-codimensional non totally geodesic isometric immersions of pseudo-Riemannian space forms with the same positive constant curvature and the same spacelike rank. *Global Differential Geometry and Global Analysis* 1984. Springer Verlag 1156 (1984), 59-73.
- [3] Ferus, D.: Isometric immersions of constant curvature manifolds. *Math. Ann.* 217 (1975), 155-156.
- [4] Graves, L. and Nomizu, K.: Isometric immersions of Lorentzian space forms. *Math. Ann.* 233 (1978), 123-136.
- [5] Moore, J.D.: Submanifolds of constant curvature I. *Duke Math. J.* 44 (1977), 450-484.
- [6] O'Neill, B.: *Semi-Riemannian Geometry with applications to Relativity*. Academic Press, 1983.

IMPA
Estrada Dona Castorina, 110
22460, Rio de Janeiro, RJ
Brasil

ICEX—Depto. Mat. U.F.M.G.
Caixa Postal 702
30000, Belo Horizonte, MG
Brasil