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ON MANIFOLDS OF NONNEGATIVE RICCI CURVATURE AND AN EXTENSION OF MYERS' THEOREM

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1. Introduction

It is an interesting problem to investigate relations between Ricci curvature and topology of Riemannian manifolds. S. B. Myers proved in [16] that if the Ricci curvature of a connected complete Riemannian manifold M of dimension n, $n\geqq 2$, is bounded from below by a positive constant $(n-1)\lambda^2$ $(\lambda>0)$, then the diameter of M is not greater than π/λ, and hence M is compact and the fundamental group of M is finite. Until now, several extensions for Myers' theorem have been done from various points of view ([1], [2], [5], [7], [9], [21]). In the present paper we will give a certain extension of Myers' theorem. The method of our extension is different from ones in the papers quoted above. For this purpose, on a connected complete Riemannian manifold M, we introduce a function $\alpha_M: M\to R^+\cup\{+\infty\}$. The definition will be given in Section 2. Making use of this function we can give an extension of Myers' theorem.

Let M denote a connected complete Riemannian manifold of dimension n, $n\geqq 2$. We will show that if M is of nonnegative Ricci curvature and if there are distinct points p and q of M such that $\alpha_M(p)+\alpha_M(q)\leqq d_M(p,q)$, $d_M(p,q)$ is the distance between p and q, then M is homeomorphic to a standard n-sphere (see Theorem 3.1). From this we obtain that if M is of nonnegative Ricci curvature and $\alpha(M)=\sup\alpha_M$ is finite, then M is compact and its fundamental group is finite (see Theorem 3.3). If M is of positive Ricci curvature, then M is compact if and only if there exists a point p of M such that $\alpha_M(p)$ is finite. We will show that if the Ricci curvature Ric_M of M satisfies $\mathrm{Ric}_M\geqq(n-1)\lambda^2$ for a positive constant λ, then $d(M)\leqq 2\alpha(M)\leqq\pi/\lambda$ where $d(M)$ denotes the diameter of M (see Theorem 3.5). It should be noted that there is an n-dimensional $(n\geqq 4)$ compact Riemannian manifold M satisfying $\mathrm{Ric}_M\geqq(n-1)\lambda^2$, λ is a positive constant, and $2\alpha(M)<\pi/\lambda$ (see Remark 3.2). Making use of the same way as in the proof of Theorem 3.1 of this paper, we can give a proof of the so called maximal diameter theorem due to Cheng [8] which is a generalization of Toponogov's theorem [23]. Theorem 3.1 can be viewed as an extension of the maximal diameter theorem.

In the last section of this paper, for a compact, non-simply connected

Riemannian manifold M of positive Ricci curvature, we investigate the relation between the injectivity radius $i(M)$ of M and $\alpha(M)$. We will show that for such an M $i(M) \leqq \alpha(M)$ holds and equality holds if and only if M is isometric to a real projective space of constant curvature.

2. Notations and preliminaries

Throughout the present paper we always assume that manifolds and apparatus on them are of class C^∞ unless otherwise stated.

In this section let M denote a connected complete Riemannian manifold of dimension n, $n \geqq 2$, with Riemannian metric g. Let d_M be the distance function on M induced from g. We denote by $d(M)$ the diameter of M, and denote by K_M (resp. Ric_M) the sectional curvature (resp. Ricci curvature) of M, respectively. For a point p of M and a positive t, we put $B(p, t) = \{q \in M;\ d_M(p, q) < t\}$, $S(p, t) = \{q \in M;\ d_M(p, q) = t\}$, $B(0_p, t) = \{X \in T_pM;\ \|X\| < t\}$ and $S(0_p, t) = \{X \in T_pM;\ \|X\| = t\}$ where T_pM denotes the tangent vector space at p and $\|X\|$ stands for the length of X. For each point p of M, we denote by $\tilde{C}_M(p)$ the tangent cut locus of p in T_pM and by $C_M(p)$ the cut locus of p in M.

For a point p of M let $\exp_p$ denote the exponential map from the tangent space T_pM at p onto M. For each $p \in M$ and each $X \in S(0_p, 1)$ let $c_{p,X}: [0, +\infty) \to M$ denote the geodesic emanating from p with initial velocity vector X. When $c_{p,X}(t)$, $t > 0$, is the first conjugate point to p along $c_{p,X}$, we denote by $r(p, X)$ the parameter value t. If $c_{p,X}$ has no conjugate points to p, then we put $r(p, X) = +\infty$.

For a $p \in M$ we now take an $X \in S(0_p, 1)$ and an r such that $0 < r < r(p, X)$. The geodesic sphere $S_p(r) = \exp_p(S(0_p, r))$ with center p and radius r is a regular hypersurface in a neighborhood of $c_{p,X}(r)$ because $\exp_p$ has the maximal rank at rX. Let $H_p(X, r)$ be the mean curvature of the geodesic sphere $S_p(r)$ at $c_{p,X}(r)$ with respect to the velocity vector $c'_{p,X}(r)$. Let $\{e_1, \cdots, e_{n-1}\}$ be an orthonormal basis in the tangent space to $S_p(r)$ at $c_{p,X}(r)$. Then $H_p(X, r)$ is given by $H_p(X, r) = (1/n-1)\sum_{i=1}^{n-1} g(\nabla_{e_i}(-\xi), e_i)$ where ∇ is the Riemannian connection of M and ξ is the unit normal vector field to $S_p(r)$ on a neighborhood of $c_{p,X}(r)$ in $S_p(r)$ such that $\xi(c_{p,X}(r)) = c'_{p,X}(r)$.

We can easily show the following.

Lemma 2.1. *For each $p \in M$ and each $X \in S(0_p, 1)$,*

$$(n-1)\frac{d}{dt}H_p(X, t) = \mathrm{Ric}_M(c'_{p,X}(t)) + \|A_t\|^2\,, \qquad 0 < t < r(p, X)\,,$$

where $\|A_t\|$ is the length of the second fundamental form A_t of $S_p(t)$ at $c_{p,X}(t)$.

Let p be a point of M and X a unit tangent vector at p. Let $Y(t)$ be a vector fields along $c_{p,X}\,|\,[0, r]$, $0 < r < r(p, X)$. We put $I_0^r(Y) = \int_0^r \{\|Y'\|^2 - g(R_t(Y), Y)\}dt$ where Y' is the covariant derivative of Y along $c_{p,X}$ and $R_t(Y) = R(Y(t), c'_{p,X}(t))c'_{p,X}(t)$, here R is the Riemannian curvature tensor of M. Let $e_1, \cdots, e_{n-1}$ be unit tangent vectors at p

such that $\{e_1, \cdots, e_{n-1}, X\}$ is an orthonormal basis in T_pM. We extend them to parallel vector fields $e_1(t), \cdots, e_{n-1}(t)$ along $c_{p,X}|[0, r]$. There exist Jacobi fields $Y_1(t), \cdots, Y_{n-1}(t)$ along $c_{p,X}|[0, r]$ such that $Y_i(0)=0_p$ and $Y_i(r)=e_i(r)$, $1\leqq i\leqq n-1$. Since $\{Y_1(r), \cdots, Y_{n-1}(r)\}$ is an orthonormal basis in the tangent space to $S_p(r)$ at $c_{p,X}(r)$, using the second variation formula, we get $(n-1)H_p(X, r)=-\sum_{i=1}^{n-1}I_0^r(Y_i)$. Suppose now $\mathrm{Ric}_M(c'_{p,X}(t))\geqq(n-1)\lambda^2$, $0\leqq t\leqq r$, for a positive constant λ. Let $Z_i(t)$, $1\leqq i\leqq n-1$, be vector fields along $c_{p,X}|[0, r]$ defined by $Z_i(t)=(\sin\lambda t/\sin\lambda r)e_i(t)$, $1\leqq i\leqq n-1$. As is well known, $I_0^r(Y_i)\leqq I_0^r(Z_i)$ holds and equality holds if and only if $Y_i(t)=Z_i(t)$ $(0\leqq t\leqq r)$.

Using the above fact we obtain the following.

Lemma 2.2. *Let p be a point of M and X a unit tangent vector at p. Suppose* $\mathrm{Ric}_M(c'_{p,X}(t))\geqq(n-1)\lambda^2$, $0\leqq t\leqq r(p, X)$, *for a positive constant λ. Then $H_p(X, r)\geqq-\lambda\cot\lambda r$, $0<r<\min\{r(p, X), \pi/\lambda\}$. If equality holds, then $K_M(P(t))=\lambda^2$ for all plane sections $P(t)$ containing $c'_{p,X}(t)$, $0\leqq t\leqq r$.*

It is easy to see that if M is the euclidean sphere of constant curvature λ^2, $\lambda>0$, then $H_p(X, r)=-\lambda\cot\lambda r$, $0<r<\pi/\lambda$.

We shall give the definition of the function $\alpha_M\colon M\to R^+\cup\{+\infty\}$ mentioned in the introduction. For each $p\in M$ and each $X\in S(0_p, 1)$ we define $\alpha_M(p, X)\in R^+\cup\{+\infty\}$, R^+ is the set of all positive real numbers, as follows: $\alpha_M(p, X)=\inf\{t>0;\ H_p(X, t)\geqq 0\}$ if there exists an r such that $0<r<r(p, X)$ and $H_p(X, r)\geqq 0$, and $\alpha_M(p, X)=r(p, X)$ if $H_p(X, t)<0$ for any t such that $0<t<r(p, X)$. Then $\alpha_M\colon M\to R^+\cup\{+\infty\}$ is defined by $\alpha_M(p)=\sup\{\alpha_M(p, X);\ X\in S(0_p, 1)\}$, $p\in M$. We put $\alpha(M)=\sup\alpha_M$. By the definition, if $\tilde{M}$ is a Riemannian covering manifold of M with covering map π, then $\alpha_{\tilde{M}}=\alpha_M\circ\pi$.

Proposition 2.1. *Let M be a connected complete Riemannian manifold of dimension n, $n\geqq 2$. Suppose $\mathrm{Ric}_M\geqq(n-1)\lambda^2$ for a positive constant λ. Then we have $\alpha(M)\leqq\pi/2\lambda$.*

Proof. Let p be a point of M. Let X be an arbitrary unit tangent vector at p. If $r(p, X)>\pi/2\lambda$, then we have $H_p(X, \pi/2\lambda)\geqq 0$ by Lemma 2.2. This implies $\alpha_M(p, X)\leqq\pi/2\lambda$. If $r(p, X)\leqq\pi/2\lambda$, then $\alpha_M(p, X)\leqq\pi/2\lambda$ by the definition of $\alpha_M(p, X)$. Thus we obtain $\alpha_M(p, X)\leqq\pi/2\lambda$ for any $X\in S(0_p, 1)$, which implies $\alpha_M(p)\leqq\pi/2\lambda$. Hence we have $\alpha(M)\leqq\pi/2\lambda$.

Proposition 2.2. *Let M be an n-dimensional $(n\geqq 2)$ connected compact Riemannian manifold of constant curvature λ^2, $\lambda>0$. Then $\alpha_M(p)=\pi/2\lambda$ for any point p of M.*

3. Extension of Myers'theorem

In this section we give a certain extension of Myers'theorem from our point of view.

The following lemma will be useful for the proof of Theorem 3.1.

Lemma 3.1 ([10]). *Let M be an n-dimensional $(n\geqq 2)$ connected Riemannian manifold of nonnegative Ricci curvature. Let W_1 and W_2 be hypersurfaces embedded in M with unit normal vector fields ξ_1 and ξ_2, respectively. Let H_1 (resp. H_2) be the mean curvature of W_1 (resp. W_2) with respect to ξ_1 (resp. ξ_2). Suppose that W_1 and W_2 have a common point $p\in M$ satisfying the following conditions: (1) $\xi_1(p)=\xi_2(p)$; (2) A neighborhood of p in W_2 lies in the same side as the direction of ξ_1 with respect ot W_1. Moreover assume that $H_1\geqq 0$ on W_1 and $H_2\leqq 0$ on W_2. Then there exists a minimal hypersurface W embedded in M such that $p\in W\subset W_1\cap W_2$.*

For the proof of this lemma, see Lemma 1.3 in the author's paper [10].

Theorem 3.1. *Let M be an n-dimensional $(n\geqq 2)$ connected complete Riemannian manifold of nonnegative Ricci curvature. Suppose that there are distinct points p and q of M such that $\alpha_M(p)+\alpha_M(q)\leqq d_M(p, q)$. Then M is homeomorphic to a standard n-sphere.*

Proof. Let p and q be distinct points of M such that $\alpha_M(p)+\alpha_M(q)\leqq d_M(p, q)$. Let us consider the subset $A=\{X\in S(0_p, 1);\ \exp_p dX=q\}$, $d=d_M(p, q)$, in the unit (n−1)-sphere $S(0_p, 1)$. By completeness of M and by continuity of $\exp_p$, A is a nonempty closed subset in $S(0_p, 1)$. We shall show that A is open in $S(0_p, 1)$. Take an $X\in A$. For simplicity, we put $2r=d-\alpha_M(p)-\alpha_M(q)$, $r_1=\alpha_M(p)+r$ and $r_2=\alpha_M(q)+r$. Since $c_{p,X}|[0, d]$ is a minimal geodesic segment from p to q, $r(p, X)\geqq d$. We can choose a connected neighborhood U_X of X in $S(0_p, 1)$ such that $\exp_p|CU_X$, $CU_X=\{tZ\in T_pM;\ 0\leqq t\leqq r_1,\ Z\in U_X\}$, is a diffeomorphism. Since $\alpha_M(p, Z)\leqq r_1$ and $H_p(Z, \alpha_M(p))\geqq 0$ for any $Z\in U_X$, by Lemma 2.1 $H_p(Z, r_1)\geqq 0$ for any $Z\in U_X$. Hence the mean curvature of the hypersurface $W_1=\exp_p(r_1 U_X)$, $r_1 U_X=\{r_1 Z\in T_pM;\ Z\in U_X\}$, with respect to the unit nomal vector field ξ_1 to W_1, which is defined by $\xi_1(c_{p,Z}(r_1))=c'_{p,Z}(r_1)(Z\in U_X)$, is nonnegative. By the same argument as in the above, we can take a connected neighborhood U_Y of Y, $Y=-c'_{p,X}(d)$, in $S(0_q, 1)$ such that $W_2=\exp_q(r_2 U_Y)$ is a hypersurface embedded in M and such that the mean curvature of W_2 with respect to the unit nomal vector field ξ_2 to W_2, which is defined by $\xi_2(c_{q,Z}(r_2))=-c'_{q,Z}(r_2)$ $(Z\in U_Y)$, is nonpositive. Note that both open metric balls $B(p, r_1)$ and $B(q, r_2)$ have no common points and that $c_{p,X}(r_1)=c_{q,Y}(r_2)\in W_1\cap W_2$, $W_1\subset\bar{B}(p, r_1)$ and $W_2\subset\bar{B}(q, r_2)$. Thus we can apply Lemma 3.1 to the present situation. Therefore there exists a minimal hypersurface W embedded in M such that $c_{p,X}(r_1)\in W\subset W_1\cap W_2$. From this we see that there are neighborhoods V_X of X in $S(0_p, 1)$ and V_Y of Y in $S(0_q, 1)$ such that $\exp_p(r_1 V_X)=\exp_q(r_2 V_Y)\subset W$. Hence $V_x\subset A$. We have proved that A is open in $S(0_p, 1)$, which implies $A=S(0_p, 1)$. Since for each $X\in S(0_p, 1)$ $c_{p,X}|[0, d]$ is a minimal geodesic segment from p to q, $\exp_p\colon B(0_p, d)\to B(p, d)$ is a diffeomorphism and $\exp_p(S(0_p, d))=\{q\}$. Hence M is homeomorphic to a standard n-sphere. We complete the proof.

From the proof of the above theorem we have the following.

Corollary 3.1. *Let M be as in Theorem 3.1. Suppose that there are distinct points p and q of M such that $\alpha_M(p)+\alpha_M(q)\leqq d_M(p, q)$. Then $\tilde{C}_M(p)=S(0_p, d_M(p, q))$ and $C_M(p)=\{q\}$. The same property also holds for q.*

Remark 3.1. Let M be as in Theorem 3.1. Suppose that there are distinct points p and q of M such that $\alpha_M(p)+\alpha_M(q)<d_M(p, q)$. Let r be the positive number defined by $2r=d_M(p, q)-\alpha_M(p)-\alpha_M(q)$. From the proof of Theorem 3. 1 we see that $\exp_p: B(0_p, d)\to B(p, d)$ is a diffeomorphism and $\exp_p(S(0_p,d))=\{q\}$, $d=d_M(p, q)$, and that $S(p, t)=S(q,d-t)$ for any t, $0<t<d$. We note that both mean curvatures of $S(p, \alpha_M(p))$ and $S(q, \alpha_M(q))$ with respect to the outer nomal direction are nonnegative. Making use of Lemma 2.1 we can show that $S(p, t)$ is totally geodesic for each t, $\alpha_M(p)\leqq t\leqq\alpha_M(p)+2r$. From this $M\setminus(B(p, \alpha_M(p))\cup B(q, \alpha_M(q)))$ is isometric to the Riemannian product manifold $S(p, \alpha_M(p))\times[0, 2r]$.

The proof of Theorem 3.1 and Remark 3.1 imply the following.

Theorem 3.2. *Let M be an n-dimensional ($n\geqq 2$) connected complete Riemannian manifold of positive Ricci curvature. Then $d_M(p, q)\leqq\alpha_M(p)+\alpha_M(q)$ for any points p and q of M. If equality holds for some points p and q of M, then the following properties hold:*

(1) *$\tilde{C}_M(p)=S(0_p, d_M(p, q))$ and $C_M(p)=\{q\}$. The same property also holds for q.*
(2) *$S(p, \alpha_M(p))=S(q, \alpha_M(q))$.*
(3) *$S(p, \alpha_M(p))$ is a minimal hypersurface embedded in M.*
(4) *M is homeomorphic to a standard n-shpere.*

Theorem 3.3. *Let M be an n-dimensional ($n\geqq 2$) connected complete Riemannian manifold of nonnegative Ricci curvature. If $\alpha(M)$ is finite, then M is compact and its fundamental group is finite. Moreover, if $\alpha(M)$ is finite and $d(M)\geqq 2\alpha(M)$, then M is homeomorphic to a standard n-sphere.*

Proof. Let $\tilde{M}$ be the universal Riemannian covering manifold of M with covering map π. Since $\alpha_{\tilde{M}}=\alpha_M\circ\pi$, $\alpha(\tilde{M})$ is finite by the hypothesis. We shall show that $\tilde{M}$ is compact. Suppose $\tilde{M}$ is noncompact. Then we can choose distinct points x and y of $\tilde{M}$ such that $\alpha_{\tilde{M}}(x)+\alpha_{\tilde{M}}(y)<d_{\tilde{M}}(x, y)$. It follows from Theorem 3.1 that $\tilde{M}$ is homeomorphic to a standard n-sphere. This is a contradiction. Hence $\tilde{M}$ is compact. This implies that M is compact and its fundamental group is finite. The later part follows from Theorem 3.1.

By the same way as in the proof of Theorem 3.3 we can easily show the following.

Theorem 3.4. *Let M be an n-dimensional ($n\geqq 2$) connected complete Riemannian manifold of nonnegative Ricci curvature. If there exists a point p of M such that $\alpha_M(p)$ is finite, then the fundamental group of M is finite.*

Theorem 3.5. *Let M be an n-dimensional ($n\geqq 2$) connected complete Riemannian manifold of positive Ricci curvature. Then M is compact if and only if there exists a point p of M such that $\alpha_M(p)$ is finite.*

Proof. We first assume that M is compact. Then $\mathrm{Ric}_M \geqq (n-1)\lambda^2$ holds for some positive constant λ. By Proposition 2.1 $\alpha_M(p) \leqq \pi/2$ holds for any $p \in M$. Conversely, assume that there exists a point p of M such that $\alpha_M(p)$ is finite. Suppose M is noncompact. Then we can choose a unit tangent vector X at p so that $c_{p,X}: [0, +\infty) \to M$ is minimal, that is, $d_M(p, c_{p,X}(t)) = t$ holds for any $t > 0$. For simplicity, we put $c(t) = c_{p,X}(t)$, $t \geqq 0$. Since $\alpha_M(p)$ is finite and there are no conjugate points to p along c, using Lemma 2.1 and the hypothesis $\mathrm{Ric}_M > 0$, we have $H_p(X, t) > 0$ for any $t > \alpha_M(p)$. We now take a positive $r > \alpha_M(p)$. Then $H_p(X, r) \leqq -H_{c(t)}(-c'(t), t-r) < 1/(t-r)$ for any $t > r$. From this we get $H_p(X, r) = 0$ as $t \to +\infty$, which is a contradiction. Hence M is compact. We complete the proof.

From the proof of the above theorem we obtain the following.

Corollary 3.2. *Let M be as in Theorem 3.5. If there is a point p of M such that $\alpha_M(p)$ is finite, then $\alpha(M) \leqq \pi/2\lambda$ where λ is the positive constant defined by* $\lambda^2 = \inf\{\mathrm{Ric}_M(X)/(n-1);\ \|X\| = 1\}$.

Summing up the results obtained above, we have the following.

Theorem 3.6. *Let M be an n-dimensional ($n \geqq 2$) connected complete Riemannian manifold of positive Ricci curvature. If there exists a point p of M such that $\alpha_M(p)$ is finite, then $\alpha(M)$ is finite and $d(M) \leqq 2\alpha(M) \leqq \pi/\lambda$ where is a positive constant such that $\lambda^2 = \inf\{\mathrm{Ric}_M(X)/(n-1);\ \|X\| = 1\}$, and the fundamental group is finite. If $\mathrm{Ric}_M \geqq (n-1)\lambda^2$ for a positive constant λ, then $d(M) \leqq 2\alpha(M) \leqq \pi/\lambda$. The equality $d(M) = 2\alpha(M)$ implies that M is homeomorphic to a standard n-sphere.*

Remark 3.2. There are n-dimensional ($n \geqq 4$) compact connected Riemannian manifolds satisfying $\mathrm{Ric}_M \geqq (n-1)\lambda^2$, λ is a positive constant, and $2\alpha(M) < \pi/\lambda$. For example, the complex projective space CP^m with $1/4 \leqq K_M \leqq 1$ and the Riemannian product manifold $S^k(1) \times S^m(1)$ ($k > 4(m-1)$, $m \geqq 2$) satisfy such conditions where $S^k(1)$ is the euclidean unit sphere of dimension k.

Making use of the same way as in the proof of Theorem 3.1 we can give a proof of the maximal diameter theorem due to Cheng.

Theorem ([8]). *Let M be an n-dimensional ($n \geqq 2$) complete connected Riemannian manifold with $\mathrm{Ric}_M \geqq (n-1)\lambda^2$, λ is a positive constant. If $d(M) = \pi/\lambda$, then M is isometric to the standard n-sphere of curvature λ^2.*

Proof. Since M is compact, we can choose points p and q such that $d(M) = d_M(p, q)$. Theorem 3.2 and 3.6 imply $\alpha_M(p) = \alpha_M(q) = \pi/2\lambda$. Using the same argument as in the proof of Theorem 3.1, we can show that $M = B(p, \pi/2\lambda) \cup B(q, \pi/2\lambda) \cup S(p, \pi/2\lambda)$ and $S(p, \pi/2\lambda) = S(q, \pi/2\lambda)$ and that $S(p, \pi/2\lambda)$ is a minimal hypersurface embedded in M. By Lemma 2.1, for each $X \in S(0_p, 1)$ $\alpha_M(p, X) = \alpha_M(q, Y) = \pi/2\lambda$, $Y = -c'_{p,X}(\pi/\lambda)$. It follows from Lemma 2.2 that for any $X \in S(0_p, 1)$ (resp. $S(0_q, 1)$) $K_M(P(t)) = \lambda^2$ where $P(t)$ is any plane section containing $c'_{p,X}(t)$ (resp. $c'_{q,X}(t)$), $0 \leqq t \leqq \pi/2\lambda$. By virtue of E. Cartan's theorem ([6], p. 38), both closed balls $\bar{B}(p, \pi/2\lambda)$ and $\bar{B}(q, \pi/2\lambda)$ are isometric to a closed ball with radius $\pi/2\lambda$ in the standard n-sphere of curvature λ^2. Therefore M is isometric to the standard n-sphere of curvature λ^2.

Remark 3.3. We note that there are different proofs for the maximal diameter theorem ([12], [13], [19], [22]).

4. Non-simply connected manifolds

In this section we investigate topological and geometrical properties of compact, non-simply connected Riemannian manifolds of nonnegative Ricci curvature. Let M be a connected, non-simply connected, complete Riemannian manifold and $\tilde{M}$ the universal Riemannian covering manifold of M with covering map π. $\tilde{M}$ is complete. For a $p \in M$ and a $\gamma \in \pi_1(M, p)$, we put $\|\gamma\|(p) = \inf\{L(c);\ c$ is a geodesic loop at p belonging in $\gamma\}$ where $L(c)$ is the length of c. Let Γ be the deck transformation group of $\tilde{M}$ corresponding to the fundamental group of M. Each element of Γ is an isometry on $\tilde{M}$ and Γ acts freely on $\tilde{M}$. For each $\gamma \in \pi_1(M, p)$, the element of Γ corresponding to γ will be denoted by T_γ. Then for each $\gamma \in \pi_1(M, p) \|\gamma\|(p) = d_{\tilde{M}}(x, T_\gamma(x))$, $x \in \pi^{-1}(p)$.

Proposition 4.1. *Let M be an n-dimensional ($n \geqq 2$) connected, non-simply connected, complete Riemannian manifold of nonnegative Ricci curvature. Suppose $\|\gamma\|(p) \geqq 2\alpha_M(p)$ for a $p \in M$ and a $\gamma \in \pi_1(M, p)$. Then the universal covering manifold of M is homeomorphic to a standard n-sphere, and hence M is compact and its fundamental group is finite. Moreover the number of elements of the fundamental group is even.*

Proof. Suppose $\|\gamma\|(p) \geqq 2\alpha_M(p)$ for a $p \in M$ and a $\gamma \in \pi_1(M, p)$. Let $\tilde{M}$ be the universal Riemannian covering manifold of M with covering map π and Γ the deck transformation group of $\tilde{M}$ corresponding to the fundamental group of M. Take an $x \in \pi^{-1}(p)$. Since $\alpha_{\tilde{M}} = \alpha_M \circ \pi$ and $\|\gamma\|(p) \geqq 2\alpha_M(p)$, we have $\alpha_{\tilde{M}}(x) + \alpha_{\tilde{M}}(y) \leqq d_{\tilde{M}}(x, y)$, $y = T_\gamma(x)$. It follows from Theorem 3.1 that $\tilde{M}$ is homeomorphic to a standard n-sphere. Hence the fundamental group of M is finite. By Corollary 3.1, $C_{\tilde{M}}(x) = \{y\}$ and $C_{\tilde{M}}(y) = \{x\}$. From this we see $T_\gamma^2(x) = x$, which implies that T_γ is an involution because Γ acts freely on $\tilde{M}$. Hence the number of elements of the fundamental group is even.

Corollary 4.1. *Let M be as in Proposition* 4.1. *Let γ and δ be elements of $\pi_1(M, p)$. If $\|\gamma\|(p)$ and $\|\delta\|(p)$ are not less than $2\alpha_M(p)$, then $\gamma = \delta$.*

Proof. Let $\tilde{M}$ be the universal Riemannian covering manifold of M with covering map π. Take an $x \in \pi^{-1}(p)$. From the proof of Proposition 4.1 we see $C_{\tilde{M}}(x) = \{T_\gamma(x)\} = \{T_\delta(x)\}$. This implies $\gamma = \delta$.

Theorem 4.1. *Let M be an n-dimensional ($n \geqq 2$) compact, connected, non-simply connected Riemannian manifold of positive Ricci curvature. Then $\|\gamma\|(p) \leqq 2\alpha_M(p)$ holds for any $p \in M$ and for any $\gamma \in \pi_1(M, p)$. If equality holds for a $p \in M$ and a $\gamma \in \pi_1(M, p)$, then the universal covering manifold of M is homeomorphic to a standard n-sphere and* *the number of elements of* $\pi_1(M, p)$ *is even.*

Proof. Suppose $\|\gamma\|(p) > 2\alpha_M(p)$ for a $p \in M$ and a $\gamma \in \pi_1(M, p)$. Let $\tilde{M}$ be the universal Riemannian covering manifold with covering map π. Since $\alpha_{\tilde{M}} = \alpha_M \circ \pi$, $\alpha_{\tilde{M}}(x) + \alpha_{\tilde{M}}(y) > d_{\tilde{M}}(x, y)$ where $x \in \pi^{-1}(p)$ and $y = T_\gamma(x)$. This is a contradiction (see Theorem 3.2). The other assertions follow from Proposition 4.1.

Theorem 4.2. *Let M be an n-dimensional* ($n \geqq 2$) *compact, connected, non-simply connected Riemannian manifold of positive Ricci curvature. Then* $i_M(p) \leqq \alpha_M(p)$ *holds for any* $p \in M$ *where* $i_M(p)$ *denotes the injectivity radius of* $\exp_p$ *at p. If equality holds at some* $p \in M$*, then the universal covering manifold of M is homeomorphic to a standard n-sphere and the following properties hold*:

(1) $\pi_1(M,p) \cong Z_2$.

(2) $\tilde{C}_M(p) = S(0_p, \alpha_M(p))$, $C_M(p) = S(p, \alpha_M(p))$.

(3) $C_M(p)$ *is a minimal hypersurface embedded in M.*

(4) $\exp_p$: $\tilde{C}_M(p) \to C_M(p)$ *is a twofold covering map.*

(5) *For each* $X \in S(0_p, 1)$, $c_{p,X} \mid [0, 2\alpha_M(p)]$ *is a simple geodesic loop at p.*

Proof. The inequality follows from Theorem 4.1. Suppose there exists a $p \in M$ such that $i_M(p) = \alpha_M(p)$. Then $\|\gamma\|(p) = 2\alpha_M(p)$ holds for any $\gamma \in \pi_1(M, p)$, $\gamma \neq 1$. By Proposition 4.1 the universal covering manifold of M is homeomorphic to a standard n-sphere. Corollary 4.1 implies $\pi_1(M, p) \cong Z_2$. Let $\tilde{M}$ be the universal Riemannian covering manifold of M with covering map π and $\Gamma = \{1, T\}$ the deck transformation group of $\tilde{M}$ corresponding to the fundamental group of M. Take an $x \in \pi^{-1}(p)$. Since $\alpha_{\tilde{M}} = \alpha_M \circ \pi$ and $\alpha_{\tilde{M}}(x) + \alpha_{\tilde{M}}(T(x)) = d_{\tilde{M}}(x, T(x))$, all properties in Theorem 3.2 hold for x and $T(x)$. Hence $\tilde{M} = B(x, \alpha_M(p)) \cup T(B(x, \alpha_M(p))) \cup S(x, \alpha_M(p))$ and $S(x, \alpha_M(p))$ is invariant by T. From this we see that other assertions of the theorem hold.

Remark 4.1. Let M be an n-dimensional ($n \geqq 2$) compact, connected, non-simply connected Riemannian manifold of positive Ricci curvature having a point p at which $i_M(p) = \alpha_M(p)$ holds. By the above theorem M is homeomorphic to a quotient manifold $S^n(1)/\Gamma$ where $\Gamma = \{1, T\}$, T is a homeomorphic involution on $S^n(1)$ without fixed points. Then M has the homotopy type of the real projective space of dimension n (see [15], p. 43).

Theorem 4.3. *Let M be an n-dimensional* ($n \geqq 2$) *compact, connected, non-simply connected Riemannian manifold of positive Ricci curvature. Then the injectivity radius* $i(M)$ *of M satisfies* $i(M) \leqq \alpha(M)$. *Moreover, equality holds if and only if M is isometric to an n-dimensional real projective space of constant curvature.*

Proof. Since, by definition, $i(M) = \inf\{i_M(p);\ p \in M\}$ and $\alpha(M) = \sup \alpha_M$, the inequality $i(M) \leqq \alpha(M)$ follows from Theorem 4.2. Suppose $i(M) = \alpha(M)$. Then $i_M(p) = \alpha_M(p)$ holds for any $p \in M$. By Theorem 4.2, $\pi_1(M) \cong Z_2$. Let $\tilde{M}$ be the universal Riemannian covering manifold of M with covering map π and $\Gamma = \{1, T\}$ the deck transformation group of $\tilde{M}$ corresponding to $\pi_1(M)$. From the proof of Theorem 4.2 we see that for each $p \in M C_{\tilde{M}}(x) = \{T(x)\}$, $x \in \pi^{-1}(p)$. This implies that

$\tilde{M}$ is a Wiedersehen manifold. Thus $\tilde{M}$ is isometric to a standard n-sphere of constant curvature (see Appendix D in [3]). Since $\pi_1(M) \cong Z_2$, M is isometric to an n-dimensional real projective space of constant curvature. The converse follows from Proposition 2.2.

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