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A NOTE ON THE BIDUAL OF A C^*-CROSSED PRODUCT

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ABSTRACT. Let G be a locally compact group. We show that, for any C^*-dynamical system (A, G, α), the bidual $(G \underset{\alpha}{\times} A)''$ of the C^*-crossed product $G \underset{\alpha}{\times} A$ is canonically isomorphic to the von Neumann algebra generated by the regular representation of $G \underset{\alpha}{\times} A$ if, and only if, G is amenable and the group C^*-algebra $C^*(G)$ is scattered.

1. Introduction

Let (A, G, α) be a C^*-dynamical system. In [4], we proved that if G is a discrete group which acts freely on A in a strong sense, then the bidual $(G \underset{\alpha}{\times} A)''$ of the C^*-crossed product $G \underset{\alpha}{\times} A$ is $*$-isomorphic to the W^*-crossed product $G \underset{\alpha''}{\times} A''$ of the W^*-dynamical system (A'', G, α'') where α'' is the bitransposed action of α on the bidual A'' of A. As $G \underset{\alpha''}{\times} A''$ is just the von Neumann algebra M generated by the regular representation of $G \underset{\alpha}{\times} A$ when G is discrete, it is natural to ask under what circumstances such an isomorphism between $(G \underset{\alpha}{\times} A)''$ and M still persists if G is nondiscrete. The purpose of this note is to show that, given any locally compact group G, the bidual $(G \underset{\alpha}{\times} A)''$ is canonically isomorphic to M for any C^*-dynamical system (A, G, α) if, and only if, G is amenable and the group C^*-algebra $C^*(G)$ is scattered which in turn, is equivalent to the condition that the Fourier algebra $A(G)$ of G coincides with the Fourier-Stieltjes algebra $B(G)$.

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As usual, we will identify the bidual A'' of a C^*-algebra A with its universal enveloping von Neumann algebra. A C^*-algebra A is called *scattered* [8] if every (nondegenerate) representation of A is a direct sum of its irreducible subrepresentations or equivalently, if its bidual A'' is a direct sum of type I factors. A separable C^*-algebra A is scattered if, and only if, its spectrum $\hat{A}$ is countable. If G is a compact group, then every continuous unitary representation of G is a direct sum of irreducible ones, so the group C^*-algebra $C^*(G)$ of G is scattered. On the other hand, the group C^*-algebra of the integers $\boldsymbol{Z}$ is not scattered. Fell (cf. [2; p. 142]) has given an example of a second countable noncompact amenable group with countable dual space and so its group C^*-algebra is scattered. It seems an interesting question to find intrinsic characterizations of G for $C^*(G)$ to be scattered. However, we will prove that the group C^*-algebra $C^*(G)$ of a discrete group G is scattered (if and) only if G is finite.

2. Fourier algebra

Let G be a locally compact group and let $\lambda_0: G \to B(L_2(G))$ be the left regular representation which extends to a representation of the group C^*-algebra $C^*(G)$ of G. In the sequel, we will use the following commutative diagram:

$$\begin{tikzcd} G^*(G) \arrow[rr, "\lambda_0"] \arrow[dr] & & B(L_2(G)) \\ & C^*(G)'' \arrow[ur, "\tau_0"'] & \end{tikzcd}$$

where τ_0 is the extension of λ_0 on the bidual $G^*(G)''$ and $\lambda_0(C^*(G))$ is the reduced group C^*-algebra of G. The weak closure $M(G)$ of $\lambda_0(C^*(G))$ in $B(L_2(G))$ is the group von Neumann algebra of G. It is well-known that the Banach dual $C^*(G)'$ of $C^*(G)$ is linearly isomorphic to the (complex) linear span $B(G)$ of all continuous positive definite functions on G (cf. [10; 7.1.8, 7.1.10]) and if $B(G)$ is equipped with the pointwise multiplication and the norm inherited from $C^*(G)'$, then it becomes a Banach algebra and is called the *Fourier-Stieltjes algebra* of G. Moreover, the closed subalgebra $A(G)$ of $B(G)$ generated by the positive definite functions with compact supports in G can be identified with the predual $M(G)_*$ of $M(G)$ and is known as the *Fourier algebra* of G.When G is abelian, $A(G)$ is the image of $L_1(\hat{G})$ under the Fourier transform, where $\hat{G}$ is the Pontryagin dual of G. We refer to [1, 6] for other properties of $A(G)$. Recently, De Cannière and Rousseau [5] proved that $A(G)$ is the smallest (nonzero) closed order and algebra ideal of $B(G)$.

3. C^*-crossed products

We now consider the more general set-up of a C^*-dynamical system (A, G, α). We will denote by $K(G, A)$ the linear space of continuous functions from G to A with compact support. Let $\pi_u: A \to B(H_u)$ be the universal representation of A and let $\tilde{\pi}_u: A \to B(L_2(G, H_u))$ and $\lambda: G \to B(L_2(G, H_u))$ be the representations defined by

$$(\tilde{\pi}_u(a)\xi)(t) = \pi_u(\alpha_{t^{-1}}(a))\xi(t)$$

$$(\lambda_s\xi)(t) = \xi(s^{-1}t)$$

for $a \in A$, $s, t \in G$ and $\xi \in L_2(G, H_u)$. As in [10; 7.7], we define the associated *regular representation* $\tilde{\pi}_u \times \lambda \colon G \underset{\alpha}{\times} A \to B(L_2(G, H_u))$ by

$$(((\tilde{\pi}_u \times \lambda)f)\xi)(t) = \int_G \pi_u(\alpha_{t^{-1}}(f(s)))\xi(s^{-1}t)ds$$

where ds is the left Haar measure on G, $f \in K(G, A)$ and $\xi \in L_2(G, H_u)$. We have the following commutative diagram:

$$\begin{tikzcd} G \underset{\alpha}{\times} A \arrow[rr, "\tilde{\pi}_u \times \lambda"] \arrow[dr, "\pi"'] & & B(L_2(G, H_u)) \\ & (G \underset{\alpha}{\times} A)'' \arrow[ur, "\tau"'] & \end{tikzcd}$$

where π is the universal representation of $G \underset{\alpha}{\times} A$, τ is the extension of $\tilde{\pi}_u \times \lambda$ on the bidual $(G \underset{\alpha}{\times} A)''$ and $\tau((G \underset{\alpha}{\times} A)'')$ is the weak closure of $(\tilde{\pi}_u \times \lambda)(G \underset{\alpha}{\times} A)$ in $B(L_2(G, H_u))$. We will denote this weak closure by $M(A, G, \alpha)$. We note that the previous diagram is a special case of the above one in which $A = \mathbf{C}$, α reduces to the trivial action, $G \underset{\alpha}{\times} \mathbf{C}$ is the group C^*-algebra $C^*(G)$ and $M(\mathbf{C}, G, \alpha)$ the group von Neumann algebra $M(G)$. We also note that $(\tilde{\pi}_u \times \lambda)(G \underset{\alpha}{\times} A)$ is the reduced C^*-crossed product $G \underset{\alpha r}{\times} A$.

If G is discrete, then the bitransposed action $\alpha'' \colon g \in G \to \alpha''_g \in \mathrm{Aut}(A'')$ on the bidual A'' induces a W^*-dynamical system (A'', G, α'') and in this case, $M(A, G, \alpha)$ is just the W^*-crossed product $G \underset{\alpha''}{\times} A''$. We have shown in [4] that if α is a strongly centrally free action, then the map τ in the above diagram is faithful and so $(G \underset{\alpha}{\times} A)''$ can be identified with $G \underset{\alpha''}{\times} A''$. We now investigate the faithfulness of τ in the nondiscrete situation. We say that $(G \underset{\alpha}{\times} A)''$ is *canonically isomorphic* to $M(A, G, \alpha)$ if τ is faithful.

We first observe that if B is any C^*-algebra and if $\phi \colon B \to M$ is a $*$-homomorphism into a von Neumann algebra M with predual M_*, then the extension $\bar{\phi}$ of ϕ on the universal envelope B'' in the diagram

$$\begin{tikzcd} B \arrow[rr, "\phi"] \arrow[dr] & & M \\ & B'' \arrow[ur, "\bar{\phi}"'] & \end{tikzcd}$$

is faithful if, and only if, $\phi^t(M_*) = B'$ where $\phi^t \colon M' \to B'$ is the transpose of ϕ. Note that $\bar{\phi}$ is the transpose of the restriction of ϕ^t to $M_* \subset M'$ and if $\phi(B)$ is weakly dense in M, then ϕ^t is faithful on M_*.

Proposition 1. *Let (A, G, α) be a C^*-dynamical system. Then the following* *conditions are equivalent*:

(i) $(G \underset{\alpha}{\times} A)''$ *is canonically isomorphic to* $M(A, G, \alpha)$;

(ii) *For any* $\psi \in (G \underset{\alpha}{\times} A)'$, *there exist sequences* (ξ_n) *and* (η_n) *in* $L_2(G, H_u)$ *with* $\sum_n \|\xi_n\|^2 < \infty$ *and* $\sum_n \|\lambda_n\|^2 < \infty$ *such that*

$$\psi(y) = \sum_n \langle (\tilde{\pi}_u \times \lambda) y \xi_n, \eta_n \rangle \qquad (y \in G \underset{\alpha}{\times} A).$$

Proof. Let M_* be the predual of $M(A, G, \alpha)$. By the above remark, τ is faithful if and only if $(\tilde{\pi}_u \times \lambda)^t(M_*) = (G \underset{\alpha}{\times} A)'$. But if $\psi \in (G \underset{\alpha}{\times} A)'$ is equal to $(\tilde{\pi}_u \times \lambda)^t(\omega)$ for some $\omega \in M_*$, then there exist sequences (ξ_n) and (η_n) in $L_2(G, H_u)$ with $\sum \|\xi_n\|^2 < \infty$, $\sum \|\lambda_n\|^2 < \infty$ and $\omega(\cdot) = \sum \langle \cdot\, \xi_n, \eta_n \rangle$ and so we have $\psi(y) = (\tilde{\pi}_u \times \lambda)^t(\omega)(y) = \omega((\tilde{\pi}_u \times \lambda)(y)) = \sum \langle (\tilde{\pi}_u \times \lambda) y \xi_n, \eta_n \rangle$.

Remark. The faithfulness of τ implies that of $\tilde{\pi}_u \times \lambda$ in which case $G \underset{\alpha}{\times} A$ is isomorphic to the reduced C^*-crossed product $G \underset{\alpha r}{\times} A$. Therefore τ need not be faithful in general.

We write $A(G \underset{\alpha}{\times} A)$ for $(\tilde{\pi}_u \times \lambda)^t(M_*)$ which is the norm-closed subspace of $(G \underset{\alpha}{\times} A)'$ consisting of functions of the form $\psi(y) = \sum \langle (\tilde{\pi}_u \times \lambda) y \xi_n, \eta_n \rangle$. Thus, $(G \times A)''$ is canonically isomorphic to $M(A, G, \alpha)$ if, and only if, $A(G \underset{\alpha}{\times} A) = (G \underset{\alpha}{\times} A)'$. Note that if $A = \boldsymbol{C}$, then $A(G \underset{\alpha}{\times} \boldsymbol{C}) = \lambda_0^t(M(G)_*)$ which is just the Fourier algebra $A(G)$ and so the last condition is $A(G) = B(G)$.

As before, we can identify the Banach dual $(G \underset{\alpha}{\times} A)'$ with the linear span $B(G \underset{\alpha}{\times} A)$ of A'-valued functions $\Phi\colon G \to A'$ which are positive definite with respect to α (cf. [10; 7.6.10]). Moreover, if $\Phi \in B(G \times A)$ is positive definite, then $\|\Phi\| = \|\Phi(e)\|$ where e is the identity of G. Also, if $\psi \in A(G)$ is positive definite, then $\psi \cdot \Phi \in B(G \underset{\alpha}{\times} A)$ is positive definite with respect to α and also $\|\psi \cdot \Phi\| = \|\psi(e)\Phi(e)\| = \|\psi\| \cdot \|\Phi\|$ (cf. [10; 7.6.9]). Hence for $\phi \in A(G)$ with positive decomposition $\phi = \phi_1 - \phi_2$ where $\|\phi\| = \|\phi_1\| + \|\phi_2\|$, we have

$$\phi \cdot \Phi = \phi_1 \cdot \Phi - \phi_2 \cdot \Phi \in B(G \underset{\alpha}{\times} A)$$

and

$$\begin{aligned} \|\phi \cdot \Phi\| &= \|\phi_1 \cdot \Phi - \phi_2 \cdot \Phi\| \leqslant \|\phi_1 \cdot \Phi\| + \|\phi_2 \cdot \Phi\| = \|\phi_1\| \cdot \|\Phi\| + \|\phi_2\| \cdot \|\Phi\| \\ &= (\|\phi_1\| + \|\phi_2\|) \cdot \|\Phi\| = \|\phi\| \cdot \|\Phi\| . \end{aligned}$$

Lemma 2. *Let* A *be a* C^**-algebra and suppose that there is a minimal central projection* p *in* A''. *If* z *is a central projection in* A'' *with* $pz = 0$, *then* $A' \cdot z$ *is not* w^**-dense in* A', *where we define as usual* $(f \cdot z)(a) = f(az)$ *for* $f \in A'$ *and* $a \in A$.

Proof. We note that the split (invariant) faces of the state space of a unital C^*-algebra B are in natural one-one correspondence with the central projections in B'' (cf. [11; III.6]).

Let A_1 be the C^*-algebra obtained by adjunction of an identity to A. Let $Q=\{f\in A'_+ : \|f\|\leqslant 1\}$ be the quasi-state space of A which is affine w^*-homeomorphic to the state space of A_1 [11; p. 166]. Now $A_1''=A''\oplus \mathbf{C}$ and p is a minimal central projection in A_1''. So $F=\{f\in Q: f(p)=0\}$ is a (proper) maximal split face of Q. But the w^*-closure $\bar{F}$ of F is also a split face of Q and $\bar{F}\neq Q$. Hence $F=\bar{F}$ is w^*-closed. Now $V_+=\bigcup_{\lambda\geqslant 0}\lambda F$ is a hereditary subcone of A'_+ which is w^*-closed since F is the intersection of V_+ with the closed unit ball of A' [11; p. 146]. Therefore, by [11; Proposition III.4.13], $V_+=V\cap A'_+$ for some proper w^*-closed (invariant) subspace V of A'. It follows from $pz=0$ that $A'\cdot z\subset V$ and so the w^*-closure of $A'\cdot z$ is properly contained in A'.

Now we prove the main result.

Theorem 3. *Let G be a locally compact group. Then the following conditions are equivalent*:

(i) *G is amenable and $C^*(G)$ is a scattered C^*-algebra*;

(ii) $A(G)=B(G)$;

(iii) *For any C^*-dynamical system (A, G, α), the bidual $(G\underset{\alpha}{\times} A)''$ is canonically isomorphic to $M(A, G, \alpha)$.*

Proof. (i)$\Rightarrow$(ii). We prove that $\lambda_0^t(M(G)_*)=C^*(G)'$ since there is a linear isomorphism which identifies $A(G)$ with $\lambda_0^t(M(G)_*)$ and $B(G)$ with $C^*(G)'$. As G is amenable, $\lambda_0^t(M(G)_*)$ is w^*-dense in $C^*(G)'$ [10; 7.3.9]. To show that they are actually equal, we prove that τ_0 is faithful. As $C^*(G)$ is scattered, there is a family $\{p_j\}$ of minimal central projections in $C^*(G)''$ with $\sum_j p_j=1$. Let z be a central projection in $C^*(G)''$ such that $\ker\tau_0=C^*(G)''(1-z)$. By minimality, we have either $p_jz=0$ or $p_jz=p_j$ for each j. Suppose $z\neq 1$, then $p_jz=0$ for some j. By Lemma 2, $C^*(G)'\cdot z$ is not w^*-dense in $C^*(G)'$. If $\omega\in M(G)_*$, then $(1-z)(\lambda_0^t(\omega))=\tau_0(1-z)(\omega)=0$. Therefore $\lambda_0^t(M(G)_*)\subset C^*(G)'\cdot z$ which implies that $\lambda_0^t(M(G)_*)$ is not w^*-dense in $C^*(G)'$. This is impossible. Hence $z=1$ and τ_0 is faithful. So we have $A(G)=B(G)$.

(ii)$\Rightarrow$(iii). Let (A, G, α) be a C^*-dynamical system. By Proposition 1, the faithfulness of τ is equivalent to $A(G\underset{\alpha}{\times} A)=(G\underset{\alpha}{\times} A)'$. We prove the latter. It suffices to show that the positive definite A'-valued functions Φ in $B(G\underset{\alpha}{\times} A)$ are contained in $A(G\underset{\alpha}{\times} A)$. As $A(G)=B(G)$, the constant 1-function on G is the norm-limit of a sequence (ϕ_n) of positive definite functions with compact supports in G [10; 7.2.5]. By [10; 7.7.6], we have $\phi_n\cdot\Phi\in A(G\underset{\alpha}{\times} A)$. Now $\|\Phi-\phi_n\cdot\Phi\|=\|(1-\phi_n)\cdot\Phi\|\leqslant \|1-\phi_n\|\cdot\|\Phi\|\to 0$ as $n\to\infty$. Hence $\Phi\in A(G\underset{\alpha}{\times} A)$. This proves that $A(G\underset{\alpha}{\times} A)=$

$B(G \times_\alpha A)$ and so τ is an isomorphism from $(G \times_\alpha A)''$ onto $M(A, G, \alpha)$.

(iii) $\Rightarrow$ (i). First, by considering the trivial C^*-dynamical system $(\mathbf{C}, G, \iota)$, $\tau_0: C^*(G)'' \to B(L_2(G))$ is faithful and so is the regular representation $\lambda_0: C^*(G) \to B(L_2(G))$. Therefore G is amenable. Let A be any C^*-algebra and consider the C^*-dynamical system (A, G, ι) in which ι is the trivral action. The bitranspose ι'' of ι induces the W^*-dynamical system (A'', G, ι'') and it is not difficult to verify that $M(A, G, \iota)$ is naturally isomorphic to the W^*-tensor product $A'' \bar{\otimes} M(G)$. Also the C^*-crossed product $G \times_\iota A$ is naturally isomorphic to the projective C^*-tensor product $A \overset{\vee}{\otimes} C^*(G)$ where $C^*(G)''$ is isomorphic to $M(G)$ via τ_0. Therefore we have the following canonical isomorphisms

$$(A \overset{\vee}{\otimes} C^*(G))'' \approx (G \times_\iota A)'' \approx M(A, G, \iota) \approx A'' \bar{\otimes} M(G) \approx A'' \bar{\otimes} C^*(G)'' .$$

As A was arbitrary, by a result of Huruya [7; p. 23], $C^*(G)$ is a scattered C^*-algebra. The proof is complete.

We conclude with two relevant results.

Proposition 4. *Let G be a discrete group. Then the group C^*-algebra $C^*(G)$ is scattered if, and only if, G is finite.*

Proof. We need only prove the sufficiency. As $C^*(G)$ is a type I C^*-algebra, by Thoma's characterization of type I groups [12], there is an abelian normal subgroup Δ of G with finite index. It suffices to show that Δ is finite. Since C^*-subalgebras of a scattered C^*-algebra are also scattered [3], the group C^*-algebra $C^*(\Delta)$ is also scattered. But $C^*(\Delta)$ is the C^*-algebra $C(\hat{\Delta})$ of continuous functions on the dual group $\hat{\Delta}$ which is compact and so has finite Haar measure μ. By scatteredness of $C(\hat{\Delta})$, μ must be atomic [8] and hence $\hat{\Delta}$ must be finite since μ is finite and invariant. It follows that Δ is finite by Pontryagin duality.

If G is discrete, then A'' can be embedded into $(G \times_\alpha A)''$ as in [3, 4] where it has been shown that if the relative commutant of the centre of A'' in $(G \times_\alpha A)''$ is contained in A'', then $(G \times_\alpha A)''$ is canonically isomorphic to the W^*-crossed product $G \times_{\alpha''} A''$. Conversely we have the following result.

Proposition 5. *Let (A, G, α) be a C^*-dynamical system in which G is a discrete group. Then the following two conditions are equivalent:*

(i) *If $m \in (G \times_\alpha A)''$ commutes with the centre of A'', then $m \in A''$;*

(ii) *The bitranspose α'' acts freely on the centre of A'' and $(G \times_\alpha A)''$ is canonically isomorphic to the W^*-crossed product $G \times_{\alpha''} A''$.*

Proof. This follows readily from a result of Nakagami and Takesaki [9; p. 102] which states that α'' acts freely on the centre of A'' if and only if the relative commutant of the centre of A'' in $G \times_{\alpha''} A''$ is contained in A''.

References

[1] W. Arendt and J. De Cannière: *Order isomorphisms of Fourier algebras*, J. Funct. Anal., **50** (1983), 1–7.

[2] L. Baggett: *A separable group having a discrete dual space is compact*, J. Funct. Anal., **10** (1972), 131–148.

[3] C.-H. Chu: *Crossed products of scattered C^*-algebras*, J. London Math. Soc., **26** (1982), 317–324.

[4] C.-H. Chu: *Shift automorphism groups of C^*-algebras*, Yokohama Math. J., **32** (1984), 31–37.

[5] J. De Cannière and R. Rouseau: *The Fourier algebra as an order ideal of the Fourier-Stieltjes algebra*, Math. Z., **186** (1984), 501–507.

[6] P. Eymard: *L'algèbre de Fourier d'un groupe localement compact*, Bull. Soc. Math. France, **92** (1964), 181–236.

[7] T. Huruya: *A spectral characterization of a class of C^*-algebras*, Sci. Reports, Niigata University, **15** (1978), 21–24.

[8] H. E. Jensen: *Scattered C^*-algebras*, Math. Scand., **41** (1977), 308–314.

[9] Y. Nakagami and M. Takesaki: *Duality for crossed products of von Neumann algebras*, Lecture Notes in Math., Springer-Verlag, 1979.

[10] G. K. Pedersen, *C^*-algebras and their automorphism groups*, Academic Press, 1979.

[11] M. Takesaki, *The theory of operator algebras* I, Springer-Verlag, 1979.

[12] E. Thoma, *Eine Charakterisierung disketer Gruppen vom* Typ I, Invent. Math., **6** (1968), 190–196.

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