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MANIFOLDS OF POSITIVE RICCI CURVATURE WITH A CERTAIN FUNCTION

By

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§ 1. Introduction

Let (M, g) denote an n-dimensional $(n\geqq 2)$ connected, complete Riemannian manifold with the Riemannian metric g. The author has introduced in [4] a function $\rho_{M}: M\rightarrow R^{+}\cup\{+\infty\}$ whose boundedness gives us some informations on the geometric structure of M where R^{+} is the set of all positive real numbers. This function ρ_{M} is defined as follows. Let p be a point of M. Suppose that for an $r>0$ $\exp_{p}: \bar{B}(0_{p}, r)\rightarrow M$ is of maximal rank where $\bar{B}(0_{p}, r)=\{Y\in T_{p}M; \|Y\|\leqq r\}$. Let ξ be the outer unit normal vector field to the boundary $\partial B(0_{p}, r)=\{Y\in T_{p}M; \|Y\|=r\}$ in the Riemannian manifold $(\bar{B}(0_{p}, r), \tilde{g})$, $\tilde{g}=\exp_{p}^{*}g$, and let $\tilde{H}_{p,r}$ be the mean curvature of $\partial B(0_{p}, r)$ in $(\bar{B}(0_{p}, r), \tilde{g})$ with respect to ξ. $\tilde{H}_{p,r}(Y)$, $Y\in\partial B(0_{p}, r)$, is defined by $\tilde{H}_{p,r}(Y)=(1/(n-1))\sum_{i=1}^{n-1}g(\nabla_{e_{i}}(-\xi), e_{i})$ where ∇ denotes the Riemannian connection induced from $\tilde{g}$ and $\{e_{1}, \cdots, e_{n-1}\}$ is an orthonomal basis in $T_{Y}\partial B(0_{p}, r)$. Now let Ω_{M} be the subset of M which consists of all points $p\in M$ satisfying the following condition: There exists an $r>0$ such that $\exp_{p}: \bar{B}(0_{p}, r)\rightarrow M$ is of maximal rank and $\tilde{H}_{p,r}\geqq 0$. We define $\rho_{M}: M\rightarrow R^{+}\cup\{+\infty\}$ by $\rho_{M}(p)=\inf\{r>0; \tilde{H}_{p,r}\geqq 0\}$ if $p\in\Omega_{M}$ and $\rho_{M}(p)=+\infty$ if $p\in M\setminus\Omega_{M}$. We define $\rho(M)$ by $\rho(M)=\sup\{\rho_{M}(p); p\in M\}$.

We are interested in Riemannian manifolds M of positive Ricci curvature such that $\rho(M)$ is finite. There are many Riemannian manifolds M with $\rho(M)<+\infty$. Typical examples of such Riemannian manifolds are compact symmetric spaces of rank one. If M is an n-dimensional $(n\geqq 2)$ connected, complete Riemannian manifold whose sectional curvature K_{M} and Ricci curvature Ric_{M} satisfy $K_{M}\leqq 1$, $\mathrm{Ric}_{M}\geqq(n-1)\lambda^{2}$, $1/2<\lambda\leqq 1$, then $\pi\leqq 2\rho(M)\leqq\pi/\lambda$ ([4]).

The purpose of this paper is to show the following theorems.

Theorem A. *Let M be an n-dimensional $(n\geqq 2)$ connected, complete Riemannian manifold of positive Ricci curvature such that $\rho(M)$ is finite. Then $i(M)\leqq 2\rho(M)$ holds where $i(M)$ denotes the injectivity radius of M. Moreover, the equality holds if and only if M is isometric to a euclidean n-sphere.*

Theorem B. *Let M be an n-dimensional $(n\geqq 2)$ connected, non simply con-*

nected, compact Riemannian manifold of positive Ricci curvature such that $\rho(M)$ *is finite. Then* $i(M)\leqq\rho(M)$ *holds with the equality if and only if* M *is isometric to an n-dimensional real projective space of constant curvature.*

A Riemannian manifold (M, g) is said to be a C_L-manifold if all geodesics of (M, g) are periodic geodesics with the least period L.

Theorem C. *Let* M *be an n-dimensional* $(n\geqq 2)$ *connected, compact Riemannian manifold of positive Ricci curvature. If* M *is a* C_L*-manifold with* $L=2\rho(M)$, *then* M *is isometric to an n-dimensional real projective space of constant curvature.*

The proof of theorems stated above will be given in § 3 and § 4. In § 2 we describe notations which will be used in later sections and we state some properties of ρ_M. Lemma 2.1 in § 2 is an important lemma to obtain theorems of this paper. This lemma directly follows from Theorem 3.1 and Remark 3.1 in the author's paper [4]. In § 4 we investigate compact, non simply connected Riemannian manifolds M of positive Ricci curvature such that $\rho(M)$ is finite. Theorem B is a direct consequence of Theorems 4.2 and 4.3.

§ 2. Preliminaries

In this section and throughout this paper we always assume that manifolds and apparatus on them are of class C^∞ unless otherwise stated.

In what follows let (M, g) denote an n-dimensional $(n\geqq 2)$ connected, complete Riemannian manifold with the Riemannian metric g. Let d_M denote the distance function on M which is induced from the metric g and let $d(M)$ denote the diameter of M. We denote by K_M and Ric_M the sectional curvature of M and the Ricci curvature of M respectively. For each $p\in M$ $\exp_p$ denotes the exponential map from the tangent space T_pM to M at p into M. For a $p\in M$ and an $r>0$ we put $B(p, r)=\{q\in M; d_M(p, q)<r\}$, $\partial B(p, r)=\{q\in M; d_M(p, q)=r\}$, $B(0_p, r)=\{X\in T_pM; \|X\|<r\}$ and $\partial B(0_p, r)=\{X\in T_pM; \|X\|=r\}$ where $\|X\|$ denotes the length of X. We define $i_M: M\to R\cup\{+\infty\}$ as follows: $i_M(p)=\sup\{r>0; \exp_p|B(0_p, r)$ is a diffeomorphism$\}$, $p\in M$. We define the injectivity radius $i(M)$ of M by $i(M)=\inf\{i_M(p); p\in M\}$. For each $p\in M$ and each unit tangent vector X at p we will denote by $c_{p,X}: [0, \infty)\to M$ the geodesic emanating from p with the initial velocity vector X. For a $p\in M$ we define m_p: $\partial B(0_p, 1)\to R^+\cup\{+\infty\}$ by $m_p(X)=\sup\{t>0; d_M(p, c_{p,X}(t))=t\}$, $X\in\partial B(0_p, 1)$, and we set $\tilde{C}_M(p)=\{m_p(X)X; X\in\partial B(0_p, 1), m_p(X)<+\infty\}$. The set $\tilde{C}_M(p)$ is called the tangent cut locus of p in T_pM and the set $C_M(p)=\exp_p\tilde{C}_M(p)$ is said the cut locus of p in M. It is well known that $d_M(p, C_M(p))=i_M(p)$ for any $p\in M$. For each $p\in M$ and each $\alpha\in\pi_1(M, p)$ we put $\|\alpha\|(p)=\inf\{L(c); c$ is a geodesic loop at p belonging in $\alpha\}$, where $L(c)$ denotes the length of c.

Let p be a point of Ω_M (for the definition see §1) and r a positive such that $\exp_p : \bar{B}(0_p, r) \to M$ is of maximal rank. For a unit tangent vector X at p we take unit tangent vectors $e_1, \cdots, e_{n-1}$ at p so that $\{e_1, \cdots, e_{n-1}, X\}$ is an orthonormal basis in T_pM. We extend $e_1, \cdots, e_{n-1}$ to parallel vector fields $e_i(t)$ $(0 \leqq t \leqq r)$, $1 \leqq i \leqq n-1$, along the geodesic $c_{p,X}|[0, r]$. Since $\exp_p|\bar{B}(0_p, r)$ is of maximal rank, there exist Jacobi fields $Y_i(t)$ $(0 \leqq t \leqq r)$, $1 \leqq i \leqq n-1$, along $c_{p,X}|[0, r]$ such that $Y_i(0)=0_p$ and $Y_i(r)=e_i(r)$, $1 \leqq i \leqq n-1$. Let $H_X(p, r)$ be the mean curvature of the geodesic hypersphere $S(p, r)=\exp_p \partial B(0_p, r)$ with respect to the velocity vector $\dot{c}_{p,X}(r)$. Since $\{Y_1(r), \cdots, Y_{n-1}(r)\}$ is an orthonormal basis in the tangent space to $S(p, r)$ at $c_{p,X}(r)$, using the second variation formula, $H_X(p, r)$ can be expressed by

$$(n-1)H_X(p, r) = -\sum_{i=1}^{n-1} \int_0^r \{\|Y_i'\|^2 - g(R(Y_i, \dot{c}_{p,X})\dot{c}_{p,X}, Y_i)\} dt$$

where Y_i' is the covariant derivative of Y_i along $c_{p,X}$ and R stands for the Riemannian curvature tensor of M. We note $\tilde{H}_{p,r}(rX)=H_X(p, r)$. In the case where M is the euclidean n-sphere of curvature λ^2, $\lambda>0$, we have $H_X(p, r)=-\lambda \cot \lambda r$, $0<r<\pi/\lambda$. Thus $\rho_M(p)=\pi/2\lambda$ for any $p \in M=S^n(1/\lambda)$. We can easily show

$$(n-1)H'_X(p, t) = \mathrm{Ric}_M(\dot{c}_{p,X}(t)) + \|A_t\|^2, \qquad 0<t \leqq r.$$

where $\|A_t\|$ stands for the length of the second fundamental form A_t of $S(p, t)$ $=\exp_p \partial B(0_p, t)$. Using this formula we can show that ρ_M is continuous if M is of positive Ricci curvature and if $\rho(M)$ is finite. Let $\tilde{M}$ be a Riemannian covering manifold of M with the covering map Π. Then $\rho_{\tilde{M}}=\rho_M \circ \Pi$.

Making use of the comparison theorem with respect to the index form, we get the following.

Proposition 2.1 ([4]). *Let M be an n-dimensional $(n \geqq 2)$ connected, complete Riemannian manifold. If $K_M \leqq 1$ and $\mathrm{Ric}_M \geqq (n-1)\lambda^2$, $1<2\lambda \leqq 2$, then $\pi \leqq 2\rho_M(p)$ $\leqq \pi/\lambda$ for any $p \in M$.*

Remark 2.1. *We note that there are Riemannian manifolds M satisfying $K_M \leqq 1$, $\mathrm{Ric}_M \geqq (n-1)\lambda^2$, $0<2\lambda \leqq 1$, and $\rho(M)<+\infty$. For example, the Riemannian product manifold $S^m(1) \times S^2(1)$ $(m=3, 4)$ satisfies such conditions where $S^m(r)$ denotes the m-dimensional euclidean sphere of radius r.*

We state a lemma which plays important roles in the proof of theorems of this paper.

Lemma 2.1. *Let M be an n-dimensional $(n \geqq 2)$ connected, complete Riemannian manifold of positive Ricci curvature such that $\rho(M)$ is finite. Then $d_M(p, q)$ $\leqq \rho_M(p)+\rho_M(q)$ holds for any points p and q of M. Moreover, if $d_M(p, q)=$ $\rho_M(p)+\rho_M(q)$ for some points p and q of M, then the following properties hold.*

(1) $\partial B(p, \rho_M(p))=\partial B(q, \rho_M(q))$.

(2) $\tilde{C}_M(p)=\partial B(0_p, d_M(p, q))$ *and* $C_M(p)=\{q\}$. *The same property also holds for* q. *M is homeomorphic to a standard n-sphere.*

(3) $\partial B(p, \rho_M(p))$ *is a minimal hypersurface embedded in M.*

This lemma follows from Theorem 3.1 and Remark 3.1 in the author's paper [4]. From Lemma 2.1 we get the following.

Theorem 2.1 ([4]). *Let M be an n-dimensional* ($n\geqq 2$) *connected, complete Riemannian manifold of positive Ricci curvature such that* $\rho(M)$ *is finite. Then* $d(M)\leqq 2\rho(M)$ *and the fundamental group is a finite group. Moreover, if* $d(M)=2\rho(M)$ *holds, then M is homeomorphic to a standard n-sphere.*

Remark 2.2. *Let M be the Riemannian product manifold* $S^3(1)\times RP^2(1)$ *where* $RP^m(\lambda)$ *stands for the m-dimensional real projective space of constant curvature* λ. *Then* $\rho(M)<d(M)$. *We note that there exists a connected, non simply connected, compact Riemannian manifold of constant curvature such that* $\rho(M)=d(M)$ *and it is not homeomorphic to a real projective space* (see [6]).

§3. Proof of Theorem A

Let M be an n-dimensional ($n\geqq 2$) connected, complete Riemannian manifold. M is said to be a Wiedersehen manifold if for any $p\in M$ the cut locus $C_M(p)$ of p in M consists of a single point. It is well known that a Wiedersehen n-manifold is isometric to a euclidean n-sphere (see [1], [7]).

Proof of Theorem A. By Lemma 2.1, $i(M)\leqq i_M(p)\leqq d(M)\leqq 2\rho(M)$ for any $p\in M$. Suppose $i(M)=2\rho(M)$. Then for any $p\in M$ $i_M(p)=d(M)=2\rho(M)$. By (2) in Lemma 2.1 M is a Wiedersehen manifold. Hence M is isometric to the euclidean n-sphere of radius $2\rho(M)/\pi$.

§4. Non simply connected manifolds with $\rho(M)<+\infty$

In this section we study geometric properties of a non simply connected, compact Riemannian manifold M of positive Ricci curvature such that $\rho(M)$ is finite.

Theorem 4.1. *Let M be an n-dimensional* ($n\geqq 2$) *connected, non simply connected, compact Riemannian manifold of positive Ricci curvature such that* $\rho(M)$ *is finite. Then* $\|\alpha\|(p)\leqq 2\rho_M(p)$ *for all* $\alpha\in\pi_1(M, p)$ *at every* $p\in M$. *Moreover, if* $\|\alpha\|(p)=2\rho_M(p)$ *for an* $\alpha\in\pi_1(M, p)$ *at a* $p\in M$, *then the universal covering manifold of M is homeomorphic to a standard n-sphere and the number of elements of the fundamental group is even.*

Proof. Let $\tilde{M}$ be the universal Riemannian covering manifold of M with the covering map Π. Take a $p \in M$ and an $\alpha \in \pi_1(M, p)$, $\alpha \neq 1$. Let σ be the deck transformation of $\tilde{M}$ corresponding to α which acts on $\tilde{M}$ as an isometry of fixed point free. Take an $x \in \Pi^{-1}(p)$. By virtue of Lemma 2.1, $d_{\tilde{M}}(x, \sigma(x)) \leqq \rho_{\tilde{M}}(x) + \rho_{\tilde{M}}(\sigma(x))$. Since $\|\alpha\|(p) = d_{\tilde{M}}(x, \sigma(x))$ and $\rho_{\tilde{M}} = \rho_M \circ \Pi$, $\|\alpha\|(p) \leqq 2\rho_M(p)$. We now assume that $\|\alpha\|(p) = 2\rho_M(p)$ for some $p \in M$ and some $\alpha \in \pi_1(M, p)$. Then $d_{\tilde{M}}(x, \sigma(x)) = \rho_{\tilde{M}}(x) + \rho_{\tilde{M}}(\sigma(x))$ where $x \in \Pi^{-1}(p)$ and σ is the deck transformation of $\tilde{M}$ corresponding to α. Hence the properties (1), (2) and (3) in Lemma 2.1 hold for x and $\sigma(x)$. Therefore $\tilde{M}$ is homeomorphic to a standard n-sphere. From the property (2) in Lemma 2.1 we see that σ is an involution. Thus the number of elements of the fundamental group is even.

Theorem 4.2. *Let M be an n-dimensional $(n \geqq 2)$ connected, non simply connected, compact Riemannian manifold of positive Ricci curvature such that $\rho(M)$ is finite. Then $i_M(p) \leqq \rho_M(p)$ holds for any $p \in M$, If $i_M(p) = \rho_M(p)$ holds for some $p \in M$, then the universal covering manifold of M is homeomorphic to a standard n-sphere and moreover the following holds.*

(1) $\pi_1(M, p) \cong Z_2$.

(2) $C_M(p) = \partial B(p, \rho_M(p))$.

(3) *$C_M(p)$ is a minimal hypersurface embedded in M.*

(4) $\exp_p : \partial B(0_p, \rho_M(p)) \to C_M(p)$ *is a covering map with the covering order two.*

(5) *For distinct unit tangent vectors X and Y at p such that* $\exp_p \rho_M(p) X = \exp_p \rho_M(p) Y$, $\dot{c}_{p,X}(\rho_M(p)) = -\dot{c}_{p,Y}(\rho_M(p))$ *holds.*

Proof. By Theorem 4.1 $i_M(p) \leqq \rho_M(p)$ holds for any $p \in M$. Suppose $i_M(p) = \rho_M(p)$ for some $p \in M$. Let $\tilde{M}$ be the universal Riemannian covering manifold of M with the covering map Π and Γ the deck transformation group of $\tilde{M}$ corresponding to $\pi_1(M, p)$. Each element of $\Gamma_1 = \Gamma \setminus \{\text{identity}\}$ acts on $\tilde{M}$ as an isometry of fixed point free. Take an $x \in \Pi^{-1}(p)$. There exists a $\sigma \in \Gamma_1$ such that $d_{\tilde{M}}(x, \sigma(x)) \leqq d_{\tilde{M}}(x, \gamma(x))$ for any $\gamma \in \Gamma_1$. The assumption $i_M(p) = \rho_M(p)$ implies $d_{\tilde{M}}(x, y) = \rho_{\tilde{M}}(x) + \rho_{\tilde{M}}(y) = 2\rho_M(p)$, $y = \sigma(x)$. Then the properties (1), (2) and (3) in Lemma 2.1 hold for x and y. By (2) in Lemma 2.1 we see that $\Gamma = \{\text{identity}, \sigma\}$ and that $\partial B(x, \rho_M(p))$ is invariant by σ. Hence $\tilde{M} = B(x, \rho_M(p)) \cup \sigma(B(x, \rho_M(p))) \cup \partial B(x, \rho_M(p))$. The properties (1) to (5) then follow from the facts stated above. We complete the proof.

In the proof of Theorem 4.2 we have shown the following.

Corollary 4.1. *Let M be as in Theorem 4.2 and $\tilde{M}$ the universal Riemannian covering manifold of M with the covering map Π. Suppose that $i_M(p) = \rho_M(p)$ holds at a $p \in M$. Then $\Pi^{-1}(p) = \{p_1, p_2\}$, $i_{\tilde{M}}(p_j) = 2\rho_M(p)$, $j = 1, 2$, and $C_{\tilde{M}}(p_j) = \{p_k\}$, $j \neq k$.*

Theorem 4.3. *Let M be an n-dimensional ($n \geqq 2$) connected, non simply connected, compact Riemannian manifold of positive Ricci curvature. If $i_M(p)=\rho_M(p)$ holds for any $p \in M$, then M is isometric to an n-dimensional real projective space of constant curvature.*

Proof. Let $\tilde{M}$ be the universal Riemannian covering manifold of M. By Theorem 4.2 and Corollary 4.1, $\tilde{M}$ is a Wiedersehen manifold. Therefore $\tilde{M}$ is isometric to a euclidean n-sphere. Since $\pi_1(M) \cong Z_2$, M is isometric to a real projective space.

Theorem B follows from Theorems 4.2 and 4.3.

Proof of Theorem C. Since ρ_M is continuous, there exists a $p \in M$ such that $\rho_M(p)=\rho(M)$. By the definition of $\rho_M(p)$ and by the assumption, $\exp_p$: $\bar{B}(0_p, \rho(M)) \to M$ is a surjective map of maximal rank. From this we see that M is not simply connected. Moreover, we see $i(M)=L/2$. Then, by Theorem B, M is isometric to the real projective space of constant curvature $(\pi/L)^2$.

Theorem 4.4. *Let M be an n-dimensional ($n \geqq 2$) connected, compact Riemannian manifold of positive Ricci curvature. If $i_M(p)=\rho_M(p)=d(M)$ holds for some $p \in M$, then M is diffeomorphic to an n-dimensional real projective space.*

Proof. By the definition of $\rho_M(p)$ and by the assumption $\rho_M(p)=d(M)$, $\exp_p$: $\bar{B}(0_p, d(M)) \to M$ is a surjective map of maximal rank. From this we see that M is not simply connected. Take a unit tangent vector X at p. By (4) and (5) in Theorem 4.2, the geodesic $c_{p,X}(t)$ $(0 \leqq t \leqq 2d(M))$ is a simple loop at p. Since $i_M(p)=d(M)$, using Berger's lemma (Lemma 6.2 in [3], p. 106) and (4) in Theorem 4.2, $\dot{c}_{p,X}(0)=\dot{c}_{p,X}(2d(M))$. Thus we have shown that for each unit tangent vector X at p $c_{p,X}: [0, 2d(M)] \to M$ is a simple closed geodesic. Then, using (4) and (5) in Theorem 4.2, we can construct a diffeomorphism from M to the real projective space of constant curvature $(\pi/2d(M))^2$.

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