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CONVERGENT POWER SERIES EXPANSIONS FOR THE BIRKHOFF INVARIANTS OF MEROMORPHIC DIFFERENTIAL EQUATIONS

Part II: A closer study of the coefficients

By

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0. Introduction

In Part I of this paper (see [1] on the reference list) we were concerned with the differential equation

$$(0.1)\qquad (tI-\Lambda)y'=(\rho I-A_1)y\,,$$

where Λ is a diagonal matrix with *all distinct* diagonal entries $\lambda_1, \cdots, \lambda_n$ ($n\geqq 2$), ρ is a complex parameter, and A_1 is a constant $n\times n$ matrix which we like to decompose as

$$A_1=\Lambda'+A\,,$$

where $\Lambda'=\mathrm{diag}[\lambda_1', \cdots, \lambda_n']$ consists of the diagonal entries of A_1, and consequently

$$A=[a_{kj}]\,,\quad 1\leqq k, j\leqq n\,,\quad a_{11}=\cdots=a_{nn}=0\,,$$

contains the off—diagonal elements of A_1.

In Part I we considered a solution vector $y(t)$ of (0.1), characterized in terms of its behavior at λ_1, and gave recursion formulas for the coefficients of its power series expansion in the variables a_{kj}, $1\leqq k, j\leqq n$, $j\neq k$. Based here upon, we gave corresponding expansions for the *characteristic constants* which determine the behavior of $y(t)$ at the points $\lambda_2, \cdots, \lambda_n$. In particular we showed that $y(t)$ (for fixed t) and the characteristic constants are *entire functions* in the variables a_{kj}.

In the present paper we achieve a twofold progress compared to the results of Part I: On one hand we show that the expansion for $y(t)$ obtained in Part I is *very natural to consider*, since the coefficients occuring in the expansion satisfy differential equations very similar to (0.1) but with lower triangular coefficient matrix. Therefore a closer study of these coefficients, as functions of t as well as of the parameters which they depend upon, seems a very natural task, since in a sense they are solutions of the simplest non-trivial examples of equations (0.1), and solutions of general equations (0.1) may be obtained as power series involving these functions as coefficients. Secondly, when expanding the characteristic constants as power series in the variables a_{kj} $(1\leqq k, j\leqq n, k\neq j)$, we analyse the dependence of the coefficients upon the parameters ρ, $\lambda_1, \cdots, \lambda_n$, and $\lambda'_1, \cdots, \lambda'_n$. For certain parameter constellations we give convergent power series expansions for these coefficients.

Equation (0.1) is closely related (via Laplace's transform) to the equation

$$zx'=A(z)x\,, \quad A(z)=z\Lambda+A_1 \tag{0.2}$$

(see [2], [4]). Speaking in terms of invariants (see [3]) of equations of the form (0.2), the coefficients in the power series expansion of the characteristic constants (in the variables a_{kj}) have the following interpretation:

If we regard them as functions of the auxiliary parameter ρ, then these functions are formal *Birkhoff invariants* of (0.2), since they only depend upon the formal Birkhoff invariants $\lambda_1, \cdots, \lambda_n$ and $\lambda'_1, \cdots, \lambda'_n$. On the other hand, they are *proper meromorphic invariants* of another equation of the form (0.2), which is of a different dimension and may have equal entries in the diagonal matrix Λ, but in the corresponding coefficient A_1 only a few non-zero elements occur.

1. Improved expansion formulas

For an equation (0.1), the points $\lambda_1, \cdots, \lambda_n$ are singularities of first kind. In what follows, the points λ_1, λ_2 will play a special role, since we will be concerned with the behavior at λ_2 of a solution vector of (0.1) locally given at λ_1 by means of its convergent expansion. Since a transformation $y=P\tilde{y}$ (with a permutation matrix P) transforms (0.1) into an equation of the same kind but with the diagonal elements of Λ appearing in a different order, it is easily seen how all results derived for the pair (λ_1, λ_2) may be applied to any other pair (λ_j, λ_k) $(j\neq k,\ 1\leqq j, k\leqq n)$.

As a further normalization we will in this Section always assume

$$\lambda_1=\lambda'_1=0\,, \quad \lambda_2=1\,. \tag{1.1}$$

This is no restriction, since the changes of variables $t=\lambda_1+\tilde{t}(\lambda_2-\lambda_1)$, $\rho=\tilde{\rho}+\lambda_1$ leave the structure of (0.1) unchanged and make the new equation satisfy (1.1). Moreover, we will in this Section restrict the parameter ρ by assuming

$$\operatorname{Re}\rho>-1\,, \quad \rho\not\equiv\lambda'_2 \bmod 1\,. \tag{1.2}$$

It follows from Frobenius' method that (0.1) (with the additional assumptions (1.1), (1.2)) has a unique solution vector of the form

(1.3) $$y(t)=t^{\rho}\{\delta_1/\Gamma(1+\rho)+\mathrm{reg}_0(t)\}$$

(compare Part I for the notation). Since $\lambda_1, \cdots, \lambda_n$ generally are branching points of $y(t)$, we choose to make parallel cuts from each point λ_k to ∞. Although in principle the direction of cuts might be taken fairly arbitrary, we choose here to make them along the rays

$$\arg(t-\lambda_k)=-\delta\,, \quad 1\leqq k\leqq n\,,$$

where $\delta>0$ is taken so small that, in case $\mathrm{Im}\,\lambda_k>0$, the corresponding cut does not intersect with the closed interval from 0 $(=\lambda_1)$ to 1 $(=\lambda_2)$, and moreover we assume for every $j\neq k$, $1\leqq j, k\leqq n$, that λ_j does not *ly* on the cut λ_k to ∞. General powers of $t-\lambda_k$ are then defined consistent with the selection

(1.4) $$-\delta-2\pi<\arg(t-\lambda_k)<-\delta\,, \quad 1\leqq k\leqq n\,,$$

for t in the cut plane.

Remark 1.1. When studying the singular behavior of $y(t)$ at $\lambda_2=1$, the above selection of cuts is convenient, since one can analytically continue $y(t)$ from 0 to 1 along a path which is almost a straight line. Furthermore, in view of Proposition 4 in [2], the above selection of cuts is *natural* when dealing with $\lambda_1=0$, $\lambda_2=1$, since then the characteristic constant c_{21} may directly be used to calculate the corresponding constant in the Stokes' multipliers of (0.2) which corresponds to (0.1) by means of Laplace's transform.

It was shown in Part I that the unique solution vector $y(t)$ of (0.1) is an entire function in the elements of A regarded as variables. Hence if we replace A by wA with a complex variable w, then $y(t)$, for fixed t in the cut plane, is an entire function of w and therefore may be expanded as

(1.6) $$y(t)=\sum_{p=0}^{\infty} w^p y(t\,;\,p)$$

where the coefficients

$$y(t\,;\,p)=[y^{(1)}(t\,;\,p), \cdots, y^{(n)}(t\,;\,p)]^T, \quad p\geqq 0\,,$$

are analytic functions of t, for t in the cut plane. As one may see from Lemma 1, Part I, the components of $y(t)=[y^{(1)}(t), \cdots, y^{(n)}(t)]^T$ satisfy the following system of integral equations:

$$y^{(1)}(t)=t^{\rho}\left\{1/\Gamma(1+\rho)-\sum_{j\neq 1} a_{1j}\int_0^t u^{-\rho-1}y^{(j)}(u)du\right\},$$

$$y^{(k)}(t)=-(t-\lambda_k)^{\rho-\lambda'_k}\sum_{j\neq k} a_{kj}\int_0^t (u-\lambda_k)^{\lambda'_k-\rho-1}y^{(j)}(u)du\,, \quad 2\leqq k\leqq n\,.$$

Insertion of (1.6) and termwise integration (which is justified according to Theorem 1, Part I), then gives the following recursion formulas:

$$y(t;0)=\delta_1 t^{\rho}/\Gamma(1+\rho),$$
$$y^{(k)}(t;p)=-(t-\lambda_k)^{\rho-\lambda'_k}\sum_{j\neq k} a_{kj}\int_0^t (u-\lambda_k)^{\lambda'_k-\rho-1} y^{(j)}(u;p-1)du$$

for t in the cut plane, $p\geqq 1$ and $1\leqq k\leqq n$. We inductively define functions

$$g(t)=t^{\rho}/\Gamma(1+\rho), \tag{1.7}$$

$$g(t;k)=-(t-\lambda_k)^{\rho-\lambda'_k}\int_0^t (u-\lambda_k)^{\lambda'_k-\rho-1} g(u)du \tag{1.8}$$

for $k=2,\cdots,n$ (note that the integral would not exist for $k=1$), and for $p\geqq 2$

$$g(t;k_1,\cdots,k_p)=-(t-\lambda_{k_p})^{\rho-\lambda'_{k_p}}\int_0^t (u-\lambda_{k_p})^{\lambda'_{k_p}-\rho-1} g(u;k_1,\cdots,k_{p-1})du \tag{1.9}$$

with $k_j=1,\cdots,n$ $(j=2,\cdots,p)$, $k_1=2,\cdots,n$. Then we find

$$y^{(k)}(t;1)=a_{k1}g(t;k), \quad k=2,\cdots,n,$$
$$y^{(1)}(t;1)\equiv 0,$$

and for $p\geqq 2$ (inductively) and $k=1,\cdots,n$:

$$y^{(k)}(t;p)=\sum_{k_1,\cdots,k_{p-1}} a(k,k_{p-1},\cdots,k_1,1)g(t;k_1,\cdots,k_{p-1},k), \tag{1.10}$$

where

$$a(k,k_{p-1},\cdots,k_1,j)=a_{kk_{p-1}}a_{k_{p-1}k_{p-2}}\cdots a_{k_1 j} \tag{1.11}$$

(hence $\neq 0$ only if $k\neq k_{p-1}$, $k_{p-1}\neq k_{p-2},\cdots,k_1\neq j$). So each $y(t;p)$ is a polynomial in the elements of A, and therefore $\sum y(t;p)$ coincides with the power series expansion of $y(t)$ (for $w=1$), derived in Part I.

As is seen from [2] (or compare Part I), there exist unique functions $e(t)$, $f(t)$, both analytic for $t=1$, such that

$$y(t)=(t-1)^{\rho-\lambda'_2}e(t)+f(t)$$

(observe (1.2)), and

$$e(1)=c\delta_2$$

with a scalar constant c which we like to refer to as *the characteristic constant* (corresponding to (λ_1,λ_2)). Given $p\geqq 1$ and indices k_j, $1\leqq k_j\leqq n$, $j=1,\cdots,p$, with $k_1\neq 1$ *and no two consective* k_j *equal to* 2, one can show in quite the same manner as in the proof of Proposition 4, Part I (or Proposition 1 of this paper) that

$$g(t;k_1,\cdots,k_p)=(t-1)^{\rho-\lambda'_2}e(t;k_1,\cdots,k_p)+f(t;k_1,\cdots,k_p) \tag{1.12}$$

with functions $e(t;k_1,\cdots k_p)$, $f(t;k_1,\cdots,k_p)$ which are analytic for $t=1$, and

$$e(1;k_1,\cdots,k_p)=0 \quad \text{if} \quad k_p\neq 2.$$

(If two consecutive k_j equal 2, then logarithmic terms will, in general, occur in the singular behavior of $g(t; k_1, \cdots, k_p)$ at $t=1$, however such $g(t; k_1, \cdots, k_p)$ do not occur in (1.10), since the corresponding coefficient $a(k, k_{p-1}, \cdots, k_1, 1)$ vanishes.) Defining

$$d=e(1; 2)\,, \tag{1.13}$$

$$d(k_1, \cdots, k_p)=e(1; k_1, \cdots, k_p, 2) \tag{1.14}$$

whenever $e(1; k_1, \cdots, k_p, 2)$ has been defined, and $d(k_1, \cdots, k_p)=0$ otherwise, we obtain from Theorem 2, Part I, that (for $w=1$)

$$c=da_{21}+\sum_{p=1}^{\infty}\sum_{k_1,\cdots,k_p} a(2, k_p, \cdots, k_1, 1)d(k_1, \cdots, k_p)\,,$$

and the series converges absolutely and can be rewritten to coincide with the power series expansion of c obtained in Part I.

As a result of the foregoing discussion, we state the following

Expansion Theorem. *Let an equation* (0.1) *be given, and let* (1.1), (1.2) *be satisfied. Moreover, let* $g(t)$, $g(t; k_1, \cdots, k_p)$ *and* d, $d(k_1, \cdots, k_p)$ *be as defined above. Then the unique solution vector* $y(t)=[y^{(1)}(t), \cdots, y^{(n)}(t)]^T$ *of* (0.1) *satisfying* (1.3) *can be expanded as follows:*

$$y^{(1)}(t)=g(t)+\sum_{p=1}^{\infty}\sum_{k_1,\cdots,k_p} a(1, k_p, \cdots, k_1, 1)g(t; k_1, \cdots, k_p, 1)\,, \tag{1.14}$$

and (for $k=2, \cdots, n$*)*

$$y^{(k)}(t)=a_{k1}g(t; k)+\sum_{p=1}^{\infty}\sum_{k_1,\cdots,k_p} a(k, k_p, \cdots, k_1, 1)g(t; k_1, \cdots, k_p, k)\,, \tag{1.15}$$

where the series converge absolutely, and uniformly with respect to t *in compact subsets of the cut plane which stay away from the points* $\lambda_1, \cdots, \lambda_n$. *Moreover, the characteristic constant* c *can be expanded as*

$$c=da_{21}+\sum_{p=1}^{\infty}\sum_{k_1,\cdots,k_p} a(2, k_p, \cdots, k_1, 1)d(k_1, \cdots, k_p)\,, \tag{1.16}$$

and the series converges absolutely.

In order to emphasize that expansions of the type (1.14), (1.15) are natural to consider, we show that the coefficients themselves satisfy differential equations which are *weakly coupled* in the sense that "most" of the parameters in the coefficient matrix are zero and others are one:

Lemma 1. *Under the assumptions of our Expansion Theorem, let* $p\geqq 2$ *and* $1\leqq k_j\leqq n$ $(j=1, \cdots, p)$ *with* $k_1\neq 1$ *be fixed, and define*

$$g_p(t)=[g(t), g(t; k_1), \cdots, g(t; k_1, \cdots, k_p)]^T,$$

$$\Lambda_p=\mathrm{diag}[\lambda_1, \lambda_{k_1}, \cdots, \lambda_{k_p}]\,,$$

$$A_p^{(1)}=\Lambda'_p+N_p\,,$$

$$\Lambda'_p=\mathrm{diag}\,[\lambda'_1,\lambda'_{k_1},\cdots,\lambda'_{k_p}]\,,$$

$$N_p=\begin{bmatrix} 0 & \cdot & \cdot & \cdot & 0 \\ 1 & \ddots & & & \cdot \\ 0 & \ddots & \ddots & & \cdot \\ \cdot & \ddots & \ddots & \ddots & \cdot \\ \cdot & & \ddots & \ddots & \cdot \\ \cdot & & & \ddots & \cdot \\ 0 & \cdot & 0 & 1 & 0 \end{bmatrix}.$$

Then

(1.17) $$(tI-\Lambda_p)g'_p(t)=(\rho I-A_p^{(1)})g_p(t)\,.$$

Proof. Differentiate the formulas (1.7), (1.8), (1.9).

2. Weakly coupled differential equations

Lemma 1 motivates a detailed study of equations of the form (1.17), and in view of (1.16) we are particularly interested in the singular behavior of $g(t\,;k_1,\cdots,k_p)$ at the point $t=1$ (and we may assume $k_p=2$). To simplify the notation, we will, from now on, consider equations of the form

(2.1) $$(tI-\Lambda_p)g'=(\rho I-A_p^{(1)})g\,,$$

where p is a natural number, Λ_p is a diagonal matrix having (not necessarily distinct) diagonal entries λ_j, $j=0,\cdots,p$; $A_p^{(1)}=\Lambda'_p+N_p$, with N_p as in Lemma 1 and Λ'_p a diagonal matrix with diagonal entries λ'_j, $j=0,\cdots,p$, and ρ is a complex parameter.

We will, throughout, make the following additional assumptions which do not affect the application to systems considered in Lemma 1 (as far as they are relevant in the expansion formulas (1.14)-(1.16)):

(i) *Let* $\lambda_0=\lambda'_0=0$, *and* $\lambda_p=1$.
(ii) *Let* $\lambda_1\neq 0$, *and whenever* $\lambda_j=0$, *then* $\lambda'_j=0$, *for* $j=2,\cdots,p$.
(iii) *Whenever* $\lambda_j=1$, *then* $\lambda'_j=\lambda'_p$ *and* $\lambda_{j-1}\neq 1$, *for* $j=1,\cdots,p$.

For fixed ε, $0<\varepsilon<\pi/2$, define G_ε by

(2.2) $$G_\varepsilon=\Delta_\varepsilon-\Delta_{\varepsilon/2}\,,$$

where

(2.3) $$\Delta_\varepsilon=\{t\neq 0\,;\ 0<\arg t<\varepsilon\}\cap\{t\neq 1\,;\ \pi-\varepsilon<\arg(t-1)<\pi\}\,.$$

Given any equation (2.1), we can always find ε as above such that none of the points λ_j ($j=0, \cdots, p$), except for those which are zero or one, *ly* inside or on the boundary of G_ε, and *for a given* ε, $0<\varepsilon<\pi/2$, *an equation* (2.1) *satisfying* (i)-(iii) *will be called* ε*-admissible if none of the points* λ_j *(other than those which are zero or one) lies inside or on the boundary of* G_ε. For $t\in G_\varepsilon$, general powers of t resp. $t-1$ should always be understood consistent with the restrictions of $\arg t$ resp. $\arg(t-1)$ implied by (2.3), and for points $\lambda\notin G_\varepsilon$ which are neither zero nor one, a fixed, but arbitrary selection of a branch of $\arg(t-\lambda)$ (and based here upon, of general powers of $t-\lambda$), for $t\in G_\varepsilon$, is always assumed, however in no case it will be essential which selection was made.

Lemma 2. *Given* ε, $0<\varepsilon<\pi/2$, *consider any* ε*-admissible equation* (2.1). *For* $\rho\not\equiv 0 \bmod 1$, *there exists a unique solution vector*

$$g(t)=[g^{(0)}(t), \cdots, g^{(p)}(t)]^T, \quad t\in G_\varepsilon,$$

of (2.1) *which satisfies*

$$g^{(j)}(t)=t^\rho\{\delta_{0j}/\Gamma(1+\rho)+\mathrm{reg}_0(t)\}. \tag{2.4}$$

If additionally $\operatorname{Re}\rho>-1$, *the components* $g^{(j)}(t)$ *are given by*

$$g^{(0)}(t)=t^\rho/\Gamma(1+\rho), \tag{2.5}$$

$$g^{(j)}(t)=-(t-\lambda_j)^{\rho-\lambda'_j}\int_0^t (u-\lambda_j)^{\lambda'_j-\rho-1}g^{(j-1)}(u)du, \tag{2.6}$$

for $j=1, \cdots, p$.

Proof. Existence and uniqueness of $g(t)$ easily follow from Frobenius' method. If $\operatorname{Re}\rho>-1$, then (2.5), (2.6) follow by induction with respect to j.

Remark 2.1. Whenever we wish to emphasize the dependence of $g(t)$ upon ρ, we write $g(t;\rho)$ (resp. $g^{(j)}(t;\rho)$ for its components). Differentiating (2.1), we find

$$(tI-A_p)g''(t;\rho)=((\rho-1)I-A_p^{(1)})g'(t;\rho),$$

and differentiation of (2.4) implies

$$\frac{d}{dt}g^{(j)}(t)=t^{\rho-1}\{\delta_{0j}/\Gamma(\rho)+\mathrm{reg}_0(t)\},$$

hence according to the uniqueness part of Lemma 2,

$$g'(t;\rho)=g(t;\rho-1). \tag{2.7}$$

For $\operatorname{Re}\rho>-1$, formulas (2.5), (2.6) still define a solution vector of (2.1) even if ρ is an integer, and (2.7) may then be used to define $g(t;\rho)$ for negative integer ρ. Hence we note that $g(t;\rho)$ is defined *for every complex* ρ *and depends analytically upon* ρ (see also [4]).

Proposition 1. *Given* ε, $0<\varepsilon<\pi/2$, *consider any* ε*-admissible equation* (2.1).

For $\rho \not\equiv \lambda'_p \bmod 1$, *the solution vector* $g(t;\rho)$ *is, for* t $(\in G_\varepsilon)$ *close to* 1, *of the form*

$$g(t;\rho)=(t-1)^{\rho-\lambda'_p}e(t;\rho)+f(t;\rho) \tag{2.8}$$

with $e(t;\rho)$, $f(t;\rho)$ *analytic for* $t=1$. *If* $c=c(\rho)$ *denotes the value of the last component of* $e(t;\rho)$ *at* $t=1$, *then* $c(\rho)$ *is a single valued analytic function of* ρ *for* $\rho \not\equiv \lambda'_p \bmod 1$.

Proof. Let $e^{(j)}(t;\rho)$ resp. $f^{(j)}(t;\rho)$ denote the components of $e(t;\rho)$, $f(t;\rho)$ and proceed by induction with respect to j: For some fixed j, $1\leqq j\leqq p$, assume

$$g^{(j-1)}(t;\rho)=(t-1)^{\rho-\lambda'_p}e^{(j-1)}(t;\rho)+f^{(j-1)}(t;\rho),$$

with $e^{(j-1)}(t;\rho)$, $f^{(j-1)}(t;\rho)$ analytic and single valued in ρ for $\rho \not\equiv \lambda'_p \bmod 1$ (and every fixed $t\in G_\varepsilon\cup\{1\}$), and analytic in t for $t\in G_\varepsilon\cup\{1\}$ (and every fixed ρ as above). Moreover, suppose for every $\rho \not\equiv \lambda'_p \bmod 1$

$$e^{(j-1)}(1;\rho)=0 \quad \text{in case} \quad \lambda_{j-1}\neq 1.$$

Note that these assumptions hold for $j=1$, since then $e^{(0)}(t;\rho)\equiv 0$, $f^{(0)}(t;\rho)=t^\rho/\Gamma(1+\rho)$. Assume for the moment $\operatorname{Re}\rho>-1$, and write (for $t_0\in G_\varepsilon$, close to 1):

$$\begin{aligned} g^{(j)}(t;\rho)=&-(t-\lambda_j)^{\rho-\lambda'_j}\int_0^{t_0}(u-\lambda_j)^{\lambda'_j-\rho-1}g^{(j-1)}(u;\rho)du \\ &-(t-\lambda_j)^{\rho-\lambda'_j}\int_{t_0}^{t}(u-\lambda_j)^{\lambda'_j-\rho-1}(u-1)^{\rho-\lambda'_p}e^{(j-1)}(u;\rho)du \\ &-(t-\lambda_j)^{\rho-\lambda'_j}\int_{t_0}^{t}(u-\lambda_j)^{\lambda'_j-\rho-1}f^{(j-1)}(u;\rho)du. \end{aligned}$$

Case α). Suppose $\lambda_j\neq 1$. Since $(t-\lambda_j)^{\lambda'_j-\rho-1}(t-1)^{\rho-\lambda'_p}e^{(j-1)}(t;\rho)$ is of the form $(t-1)^{\rho-\lambda'_p}\operatorname{reg}(t-1)$, it has a unique antiderivative $E(t;\rho)$ of the form $(t-1)^{\rho-\lambda'_p+1}\operatorname{reg}(t-1)$. Define

$$\begin{aligned} e^{(j)}(t;\rho)=&-(t-1)^{\lambda'_p-\rho}E(t;\rho)(t-\lambda_j)^{\rho-\lambda'_j}, \\ f^{(j)}(t;\rho)=&-(t-\lambda_j)^{\rho-\lambda'_j}\int_0^{t_0}(u-\lambda_j)^{\lambda'_j-\rho-1}g^{(j-1)}(u;\rho)du \\ &-(t-\lambda_j)^{\rho-\lambda'_j}\int_{t_0}^{t}(u-\lambda_j)^{\lambda'_j-\rho-1}f^{(j-1)}(u;\rho)du \\ &+(t-\lambda_j)^{\rho-\lambda'_j}E(t_0;\rho), \end{aligned}$$

then for fixed $\rho \not\equiv \lambda'_p \bmod 1$, $\operatorname{Re}\rho>-1$, the functions $e^{(j)}(t;\rho)$, $f^{(j)}(t;\rho)$ are analytic in t, for $t\in G_\varepsilon\cup\{1\}$, and $e^{(j)}(1;\rho)=0$. Moreover, $E(t_0;\rho)$ is analytic and single valued in ρ, for $\rho \not\equiv \lambda'_p \bmod 1$, $\operatorname{Re}\rho>-1$ (since t_0 is close to 1, we may evaluate $E(t_0;\rho)$ by termwise integration of the expansion about $t=1$ of $(t-1)^{\rho-\lambda'_p}(t-\lambda_j)^{\lambda'_j-\rho-1}e^{(j-1)}(t;\rho)$). Therefore $E(t;\rho)$, for every fixed $t\in G_\varepsilon$, is seen to be analytic and single valued in ρ (for ρ as above), and the same follows for $e^{(j)}(t;\rho)$, $f^{(j)}(t;\rho)$, even if $t=1$.

Case β). Suppose $\lambda_j=1$. According to assumption (iii) we then have $\lambda'_j=\lambda'_p$ and $\lambda_{j-1}\neq 1$. Let $F(t;\rho)$ be the unique antiderivative of $(t-1)^{\lambda'_p-\rho-1}f^{(j-1)}(t;\rho)$ which is of the form $(t-1)^{\lambda'_p-\rho}\operatorname{reg}(t-1)$, and define

$$\begin{aligned} e^{(j)}(t;\rho) &= -\int_0^{t_0}(u-1)^{\lambda'_p-\rho-1}g^{(j-1)}(u;\rho)du \\ &\quad -\int_{t_0}^{t}(u-1)^{-1}e^{(j-1)}(u;\rho)du \\ &\quad +F(t_0;\rho), \\ f^{(j)}(t;\rho) &= -F(t;\rho)(t-1)^{\rho-\lambda'_p}. \end{aligned}$$

Similar reasoning as above shows that $e^{(j)}(t;\rho)$, $f^{(j)}(t;\rho)$ have the same analyticity properties as in Case α) (in particular, note that by assumption $e^{(j-1)}(1;\rho)=0$, since $\lambda_{j-1}\neq 1$).

In both cases we obtained

$$g^{(j)}(t;\rho)=(t-1)^{\rho-\lambda'_p}e^{(j)}(t;\rho)+f^{(j)}(t;\rho)$$

for $\operatorname{Re}\rho>-1$ and $\rho\not\equiv\lambda'_p \bmod 1$. By repeated differentiation with respect to t we can, in view of (2.7), remove the restriction $\operatorname{Re}\rho>-1$ (and $e^{(j)}(t;\rho)$, $f^{(j)}(t;\rho)$ still have the same analyticity properties). Altogether, this proves (2.8) plus the fact that $e^{(p)}(1;\rho)=c(\rho)$ is analytic and single valued for $\rho\not\equiv\lambda'_p \bmod 1$. This completes the proof.

Remark 2.2. We wish to emphasize that the constant c (defined in Proposition 1) does not depend upon ε in the sense that if we take any two ε for which a given equation (2.1) is ε-admissible, then the corresponding values of c agree. Therefore, c is determined by the parameters in (2.1), which are $\lambda_1,\cdots,\lambda_{p-1}$, $\lambda'_1,\cdots,\lambda'_p$, and ρ (observe assumption (i)).

3. Calculation of the characteristic constant

In Proposition 1 we defined a constant c which we like to call the *characteristic constant* of (2.1). Obviously, c does not only depend upon ρ, but also upon $\lambda_1,\cdots,\lambda_{p-1}$ and $\lambda'_1,\cdots,\lambda'_p$ (see Remark 2.2), and we are going to investigate the nature of c as a function of some of these parameters. As a first step, we will *calculate* c (in terms of convergent series) in case the following assumption holds:

(iv) *Let* $|\lambda_j|>1$, $1\leqq j\leqq p-1$.

Given complex parameters $\beta_1,\cdots,\beta_p$, we define functions of complex variables $w_1,\cdots,w_{p-1}$ by the following recursions:

$$b=b(\beta_1)=1/\Gamma(\beta_1)\,, \tag{3.1}$$

$$b(w_1)=b(w_1;\ \beta_1,\ \beta_2)=\sum_{\nu=0}^{\infty}(\beta_2)_\nu b(\nu+\beta_1)w_1^{\nu+1}, \tag{3.2}$$

$$\begin{aligned} b(w_1,\cdots,w_j)&=b(w_1,\cdots,w_j;\ \beta_1,\cdots,\beta_{j+1}) \\ &=\sum_{\nu=0}^{\infty}(\beta_{j+1})_\nu b(w_1,\cdots,w_{j-1};\ \beta_1+\nu,\cdots,\beta_j+\nu)w_j^{\nu+1}, \end{aligned} \tag{3.3}$$

$(2\leqq j\leqq p-1)$, where $(\beta)_0=1$, $(\beta)_1=\beta$, $(\beta)_\nu=\beta(\beta+1)\cdots(\beta+\nu-1)$ for $\nu\geqq 2$.

Proposition 2. *For arbitrarily given $\beta_1,\cdots,\beta_p$ and every j, $1\leqq j\leqq p$, the series in* (3.3), *when regarded as a power series in several variables, converges absolutely for $|w_k|<1$, $1\leqq k\leqq j$, hence $b(w_1,\cdots,w_j)$ is analytic in each variable for these values of $w_1,\cdots,w_j$. If*

$$\beta_1=\cdots=\beta_p=\beta\,, \tag{3.4}$$

then

$$b(w_1,\cdots,w_j)=w_1(1-w_1)^{-1}\cdots w_j(1-w_j)^{-1}/\Gamma(\beta)\,. \tag{3.5}$$

Proof. We first assume (3.4). Then

$$(\beta)_\nu b(\nu+\beta)=\beta(\beta+1)\cdots(\beta+\nu-1)/\Gamma(\beta+\nu)=1/\Gamma(\beta)\,,$$

hence (3.5) holds for $j=1$ and every complex β. If we now assume (3.5) for some fixed $j\geqq 1$ and arbitrary β, then

$$\begin{aligned} b(w_1,\cdots,w_{j+1})&=\sum_{\nu=0}^{\infty}(\beta)_\nu b(w_1,\cdots,w_j;\ \beta+\nu,\cdots,\beta+\nu)w_{j+1}^{\nu+1} \\ &=\Big\{\sum_{\nu=0}^{\infty}w_{j+1}^{\nu+1}\Big\}w_1(1-w_1)^{-1}\cdots w_j(1-w_j)^{-1}/\Gamma(\beta)\,, \end{aligned}$$

i.e. (3.5) holds with $j+1$ in place of j.

In order to show convergence of (3.3), let $c\geqq 0$ be the maximum of $|\beta_j|$, $1\leqq j\leqq p$. Since for every complex β we have $|(\beta)_\nu|\leqq(|\beta|)_\nu$, we obtain by induction with respect to j (with $|w_k|\leqq\rho<1$, $1\leqq k\leqq j$, and sufficiently large $K=K(j,\ \rho)>0$):

$$\begin{aligned} |b(w_1,\cdots,w_j;\ \beta_1,\cdots,\beta_{j+1})|&\leqq Kb(|w_1|,\cdots,|w_j|\,;\ |\beta_1|,\cdots,|\beta_{j+1}|) \\ &\leqq Kb(|w_1|,\cdots,|w_j|\,;\ c,\cdots,c)\,, \end{aligned}$$

which completes the proof.

Proposition 3. *Given ε, $0<\varepsilon<\pi/2$, and an ε-admissible equation* (2.1), *the unique solution vector $g(t;\rho)$ has, for $t\in G_\varepsilon$, $|t|$ sufficiently small, a convergent expansion*

$$g(t;\rho)=\sum_{\mu=0}^{\infty}f(\mu)t^{\mu+\rho}/\Gamma(1+\mu+\rho)\,,$$

with coefficient vectors $f(\mu)=[f^{(0)}(\mu),\cdots,f^{(p)}(\mu)]^T$ which are independent of ρ. If *we additionally assume* (iv), *then*

$$\lim_{\mu\to\infty} f^{(p)}(\mu)/\Gamma(\mu+\lambda'_p)=b(\lambda_{p-1}^{-1},\cdots,\lambda_1^{-1};\lambda'_p+p,\cdots,\lambda'_1+1)\,. \tag{3.6}$$

Proof. For arbitrarily fixed $\rho\not\equiv 0 \bmod 1$, if we expand $g(t;\rho)$ as above and insert into (2.1), we obtain the following identities for the components of the coefficients (observe $\lambda_0=\lambda'_0=0$):

$$0=\mu f^{(0)}(\mu)\,,\quad \mu\geqq 0\,, \tag{3.7}$$

$$\lambda_j f^{(j)}(\mu+1)=(\mu+\lambda'_j)f^{(j)}(\mu)+f^{(j-1)}(\mu)\,,\quad \mu\geqq 0,\ 1\leqq j\leqq p\,. \tag{3.8}$$

According to (2.4) we have $f^{(j)}(o)=\delta_{jo}$, $0\leqq j\leqq p$, and (3.7), (3.8) determine the coefficients $f(\mu)$ completely (for $\mu\geqq 1$; observe that $\lambda_j=0$ implies $\lambda'_j=0$, due to assumption (ii)). Therefore, the coefficients do not depend upon ρ.

If we now assume (iv), then we find (by induction with respect to j)

$$f^{(j)}(0)=\cdots=f^{(j)}(j-1)=0\,,\quad 1\leqq j\leqq p\,.$$

Moreover, $f^{(0)}(\mu)=\delta_{\mu o}$ ($\mu\geqq 0$), and since (3.8) is an inhomogeneous difference equation, by the usual "variation of constants" technique we obtain

$$f^{(j)}(\mu)=\lambda_j^{-\mu}\sum_{\nu=j-1}^{\mu-1}\lambda_j^{\nu}f^{(j-1)}(\nu)(\nu+\lambda'_j+1)_{\mu-\nu-1}\,,\quad 1\leqq j\leqq p,\ \mu\geqq j \tag{3.9}$$

(observe that $f^{(j)}(\mu)$ depends analytically upon the parameters $\lambda'_1,\cdots,\lambda'_p$, hence to obtain (3.9) one can assume that no λ'_j is an integer, $j=1,\cdots,p$). Using (iv), we may estimate (3.9) to show that for sufficiently small $\delta>0$ and large enough $K>0$

$$|f^{(j)}(\mu)|\leqq K(1-\delta)^{\mu}\Gamma(\mu)\,,\quad \mu\geqq j,\ 1\leqq j\leqq p-1\,,$$

hence $\lim_{\mu\to\infty} f^{(p)}(\mu)/\Gamma(\mu+\lambda'_p)$ exists, and is equal to

$$\sum_{\nu=p-1}^{\infty} f^{(p-1)}(\nu)b(\lambda'_p+1+\nu)$$

(use (3.9) with $j=p$ and observe $(\nu+\lambda'_p+1)_{\mu-\nu-1}=\Gamma(\mu+\lambda'_p)/\Gamma(\nu+\lambda'_p+1)$, for sufficiently large μ). By induction (with respect to $k=p-j$) one obtains, using (3.9), (3.2), and (3.3), for $j=0,\cdots,p-2$:

$$\begin{aligned}&\lim_{\mu\to\infty} f^{(p)}(\mu)/\Gamma(\mu+\lambda'_p)\\ &\quad=\sum_{\nu=j}^{\infty} f^{(j)}(\nu)b(\lambda_{p-1}^{-1},\cdots,\lambda_{j+1}^{-1};\lambda'_p+p-j+\nu,\cdots,\lambda'_{j+1}+1+\nu)\,.\end{aligned}$$

For $j=0$ (observe $f^{(0)}(\nu)=\delta_{\nu 0}$) this implies (3.6). Hence the proof is completed.

Theorem 1. *Let a differential equation* (2.1) *be given, and assume that* (i)-(iv) *are satisfied. Then the characteristics constant c is given by*

$$c = c(\lambda_1, \cdots, \lambda_{p-1}; \lambda'_1, \cdots, \lambda'_p; \rho) \tag{3.10}$$
$$= e^{i\pi(\lambda'_p - \rho)} \Gamma(\lambda'_p - \rho) b(\lambda_{p-1}^{-1}, \cdots, \lambda_1^{-1}; \lambda'_p + p, \cdots, \lambda'_1 + 1)$$

(*for* $\rho \not\equiv \lambda'_p$ mod 1).

Proof. Since both sides of (3.10) are analytic functions of ρ, provided $\rho \not\equiv \lambda'_p$ mod 1 (compare Proposition 1), it is sufficient to show (3.10) in case

$$\mathrm{Re}(\lambda'_p - \rho) > 0 .$$

In this case, we obtain from (2.8) (with suitably small $\varepsilon > 0$)

$$c = c(\rho) = e^{(p)}(1; \rho) = \lim_{t \to 1} (t-1)^{\lambda'_p - \rho} g^{(p)}(t; \rho), \quad t \in G_\varepsilon . \tag{3.11}$$

From Proposition 3 resp. its proof, we conclude

$$g^{(p)}(t; \rho) = \sum_{\mu=0}^{\infty} f^{(p)}(\mu) t^{\mu+\rho} / \Gamma(1+\mu+\rho),$$
$$f^{(p)}(\mu) = \Gamma(\mu + \lambda'_p)\{b + O(1-\delta)^\mu)\} \qquad \text{for} \quad \mu > -\mathrm{Re}\,\lambda'_p ,$$

with sufficiently small $\delta > 0$ and $b = b(\lambda_{p-1}^{-1}, \cdots, \lambda_1^{-1}; \lambda'_p + p, \cdots, \lambda'_1 + 1)$. Obviously, the limit in (3.11) stays fixed, if we subtract terms from $g^{(p)}(t; \rho)$ which stay bounded as $t \to 1$, hence for arbitrary $\mu_0 > -\mathrm{Re}\,\lambda'_p$:

$$c = d(\rho) b(\lambda_{p-1}^{-1}, \cdots, \lambda_1^{-1}; \lambda'_p + p, \cdots, \lambda'_1 + 1),$$

where

$$d(\rho) = \lim_{t \to 1} (t-1)^{\lambda'_p - \rho} f(t),$$
$$f(t) = \sum_{\mu=\mu_0}^{\infty} \Gamma(\mu + \lambda'_p) t^{\mu+\rho} / \Gamma(1+\mu+\rho) .$$

Since (for $\mu_0 > -\mathrm{Re}\,\rho$)

$$\Gamma(\rho+\mu_0) f(t) = -(t-1)^{\rho - \lambda'_p} \int_0^t (u-1)^{\lambda'_p - \rho - 1} u^{\rho+\mu_0-1} du\, \Gamma(\lambda'_p + \mu_0),$$

(observe that both sides behave the same for $t \to 0$ and satisfy the same inhomogeneous differential equation), we find

$$d(\rho) = -\int_0^1 (u-1)^{\lambda'_p - \rho - 1} u^{\rho+\mu_0-1} du\, \Gamma(\lambda'_p + \mu_0) / \Gamma(\rho + \mu_0),$$

and observing $\arg(u-1) = \pi$, this implies

$$d(\rho) = e^{i\pi(\lambda'_p - \rho)} \Gamma(\lambda'_p - \rho) .$$

4. Analyticity properties of the characteristic constant

As a consequence of Theorem 1, we obtain that the characteristic constant c of a system (2.1) which satisfies (i)-(iii) is an analytic function of $\lambda_1, \cdots, \lambda_{p-1}$ provided $|\lambda_j| > 1$ $(1 \leqq j \leqq p-1)$. In this Section we are going to show the analyticity of c (in these variables) in a bigger region.

Lemma 3. *Given* ε, $0<\varepsilon<\pi/2$, *consider an arbitrarily fixed* ε*-admissible equation* (2.1), *and assume*

$$0<\mathrm{Re}(\lambda'_p-\rho)<1\,. \tag{4.1}$$

Then for every $k=0, \cdots, p$, *the vector* $g_k(t)=[g_k^{(0)}(t), \cdots, g_k^{(p)}(t)]^T$ *with*

$$g_k^{(j)}(t)\equiv 0 \qquad if \quad 0\leqq j\leqq k-1\,, \tag{4.2}$$

$$g_k^{(k)}(t)=(t-\lambda_k)^{\rho-\lambda'_k}\,, \tag{4.3}$$

$$g_k^{(j)}(t)=-(t-\lambda_j)^{\rho-\lambda'_j}\int_1^t (u-\lambda_j)^{\lambda'_j-\rho-1}g_k^{(j-1)}(u)du \qquad if \quad k+1\leqq j\leqq p\,, \tag{4.4}$$

is a solution vector of (2.1) (*for* $t\in G_\varepsilon$). *Obviously* $g_0(t), \cdots, g_p(t)$ *are linearly independent, and if we expand*

$$g(t)=\sum_{k=0}^{p} c_k g_k(t)\,, \tag{4.5}$$

then the characteristic constant c *of* (2.1) *equals* c_p.

Proof. For every fixed k, by induction with respect to j, one can show that the integrals in (4.4) all exist (observe that if $k<p$ and $\lambda_k=1$, then $\lambda'_k=\lambda'_p$ and $\lambda_{k+1}\neq 1$, according to assumption (iii)). By differentiation of (4.3), (4.4) it is easily seen that $g_k(t)$ satisfies (2.1) (for every $k=0, \cdots, p$). If $\lambda_k\neq 1$, then $g_k(t)$ is analytic for $t=1$, and if $\lambda_k=1$ and $k<p$, (using again $\lambda_{k+1}\neq 1$ and $\lambda'_k=\lambda'_p$) we find $g_k^{(j)}(t)=(t-1)^{\rho-\lambda'_p+1}\mathrm{reg}(t-1)$ for $j=k+1, \cdots, p$. Hence comparing (4.5) to (2.8), we see

$$e(t\,;\rho)=(t-1)^{\lambda'_p-\rho}\sum_{\substack{k=0\\ \lambda_k=1}}^{p} c_k g_k(t)\,,$$

and in particular

$$\begin{aligned} c=e^{(p)}(1\,;\rho)&=\sum_{\substack{k=0\\ \lambda_k=1}}^{p} c_k\,\{(t-1)^{\lambda'_p-\rho}g_k(t)\}_{t=1}\\ &=c_p\,. \end{aligned}$$

By E_ε we denote the complement of the closure of G_ε.

Theorem 2. *For arbitrarily given* ε, $0<\varepsilon<\pi/2$, *consider a fixed, but arbitrary* ε*-admissible equation* (2.1) *and its characteristic constant* c. *For some fixed* j ($1\leqq j\leqq p-1$), *replace* λ_j *by any number* $\lambda\in E_\varepsilon$. *Then the resulting equation is again* ε*-admissible, and its characteristic constant* c_λ (*for* $\rho\not\equiv\lambda'_p \bmod 1$) *is an analytic function of* λ (*for* $\lambda\in E_\varepsilon$). *In case* $\lambda_j=0$ *resp.* $\lambda_j=1$, *we have*

$$c=\lim_{\lambda\to 0} c_\lambda \quad (\lambda\in E_\varepsilon)\,, \tag{4.6}$$

resp.

$$c=\lim_{\lambda\to 1} c_\lambda \quad (\lambda\in E_\varepsilon). \tag{4.7}$$

Proof. Let us refer to equation (2.1) (with the original value λ_j) as *the original equation,* and to the one obtained when replacing λ_j by λ as the λ-equation. Then according to the definition every λ-equation clearly is admissible. Let $g(t)$ resp. $\tilde{g}(t)$ be the unique solutions of the original resp. the λ-equation which were defined in Lemma 2 (also observe Remark 2.1). For $\operatorname{Re}\rho>-1$, it may be seen from (2.5), (2.6) that for fixed $t\in G_\varepsilon$, the components of $\tilde{g}(t)$, regarded as functions of $\lambda\in E_\varepsilon$, are locally analytic (although not necessarily single valued), and in case $\lambda_j=0$ resp. $\lambda_j=1$, we have

$$\lim_{\lambda\to 0} \tilde{g}(t)=g(t),$$

resp.

$$\lim_{\lambda\to 1} \tilde{g}(t)=g(t).$$

Using (2.7) and Cauchy's integral formula for derivatives, it is easily seen that these statements generalize to arbitrary values of ρ. If, in particular, we assume (4.1) and define $g_k(t)$ resp. $\tilde{g}_k(t)$ $(0\leqq k\leqq p)$ (for the original resp. the λ-equation) as in Lemma 3, then again we obtain from (4.2)-(4.4) that $\tilde{g}_k(t)$ (for fixed $t\in G_\varepsilon$) is locally analytic in λ, for $\lambda\in E_\varepsilon$, and tends to $g_k(t)$ as $\lambda\to 0$ (if $\lambda_j=0$) resp. $\lambda\to 1$ (if $\lambda_j=1$), for $k=0, \cdots, p$. Hence if we expand

$$\tilde{g}(t)=\sum_{k=0}^{p} \tilde{c}_k \tilde{g}_k(t),$$

then obviously $\tilde{c}_k$ $(0\leqq k\leqq p)$ are locally analytic functions of λ and tend to c_k (as in (4.5)) if $\lambda\to\lambda_j$, even if $\lambda_j=0$ or $\lambda_j=1$. However, $\tilde{c}_p=c_\lambda$ and $c_p=c$ (according to Lemma 3). This proves Theorem 2 for ρ restricted by (4.1), since the single-valuedness of c_λ is clear according to Theorem 1. In view of (3.10), we see that $b(\lambda_{p-1}^{-1}, \cdots, \lambda_1^{-1}; \lambda_p'+p, \cdots, \lambda_1'+1)$ (if we replace λ_j by λ) becomes an analytic function for $\lambda\in E_\varepsilon$, and (3.10) generalizes, first to arbitrary $\lambda\in E_\varepsilon$ and ρ as in (4.1), and then to arbitrary $\rho\not\equiv\lambda_p'$ mod 1, in view of the analyticity of c with respect to ρ (compare Proposition 1).

Remark 4.1. For arbitrarily given values $\beta_1, \cdots, \beta_p$ $(p\geqq 2)$, it follows from the proof of Theorem 2 that $b(w_1, \cdots, w_{p-1}; \beta_1, \cdots, \beta_p)$ is a single valued analytic function in the variables w_j in the region

$$\arg(w_j-1)\not\equiv 0 \bmod 2\pi, \quad 1\leqq j\leqq p-1.$$

For an explicit calculation of the characteristic constants it is important to have explicit formulas for $b(w_1, \cdots, w_{p-1})$ in the above region. For example, if $p=2$, then $b(w_1)$ is a hypergeometric function, and formulas for its analytic continuation are known. In a separate paper, we will try to find similar formulas for higher values of p.

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