YOKOHAMA MATHEMATICAL
JOURNAL VOL. 32, 1984

AN ANALYSIS OF NONLINEAR SYSTEMS WITH RESPECT TO JUMP

By

KIYOYUKI TCHIZAWA

(Received May 14, 1984; Revised July 20, 1984)

1. Introduction

According to Smale's formulation which describes the regularity of electric-electronic circuits with resistors, inductors and capacitors [1], we analyze local solvability [2] and jump phenomena on the following two systems. One is called Multivibrator, the other is Blocking Oscillator. In the dynamical nonlinear circuits, the property of local solvability has been already investigated [3], [4]. This property ensures that a vector field of the system is defined uniquely.

In this paper, we show that a simple geometric description of the dynamics can be obtained by choosing a suitable coordinate system which makes clear the relation between the property of local solvability and of jump phenomena.

2. Preliminaries

A state of the circuit is described by choosing a currents vector $i=(i_R, i_C, i_L)\in R^n$ and a voltages vector $v=(v_R, v_C, v_L)\in R^n$ as $(i, v)\in R^{2n}$ where n is the number of elements and R, C and L denote resistors, linear capacitors and linear inductors, respectively. Now let n_R, n_C and n_L be the numbers of resistors, capacitors and inductors, then $n_R+n_C+n_L=n$. Resistor constitutive relations are represented by

$$(1)\qquad (i_R, v_R)\in \Lambda_R\subset R^{2n_R}\ ,$$

$$(2)\qquad i_R=f(v_R)\ ,$$

where Λ_R is an n_R-dimensional smooth submanifold given by (2) (Λ_R is controlled by voltages) and $f\colon R^{n_R}\to R^{n_R}$ represents a nonlinear smooth mapping. Capacitor currents and voltages are related as follows:

$$(3)\qquad (i_C, v_C)\in R^{2n_C}\ ,$$

$$(4)\qquad i_C=C_m\dot{v}_C\ ,\quad (\dot{v}_C=dv_C/dt)\ ,$$

where C_m is an $(n_C\times n_C)$ diagonal matrix. Inductor currents and voltages are related as follows:

$$(i_L, v_L) \in R^{2n_L}\ , \tag{5}$$

$$v_L = L_m \dot{i}_L\ , \tag{6}$$

where L_m is an $(n_L \times n_L)$ diagonal matrix.

Kirchhoff's current and voltage laws restrict the possible states to an n-dimensional $(2n-n=n)$ linear subspace $K \subset R^{2n}$. The restraint of the branch characteristics denoted by Λ is $(n+n_C+n_L)$-dimensional $(2n-n_R=n+n_C+n_L)$ smooth submanifold, where

$$\Lambda = \{(i, v) \in R^{2n_R} \mid (i_R, v_R) \in \Lambda_R\}\ . \tag{7}$$

Then the configuration space Σ where the dynamics takes place is defined as follows:

$$\Sigma = \Lambda \cap K\ . \tag{8}$$

The transversality of Λ and K which the systems treated with this paper satisfy assures that Σ is an (n_C+n_L)-dimensional $(2n-n-n_R=n_C+n_L)$ submanifold.

Let $\pi_{LC}\colon \Sigma \to R^{n_C+n_L}$ be the natural projection defined by

$$\pi_{LC}(i, v) = (i_L, v_C)\ , \tag{9}$$

and let $D_p\pi_{LC}$ denote the derivatives of π_{LC} at $p=(i, v) \in \Sigma$. If the dynamics of the system can be well defined at p, then we call p local solvable point. It is known that if $\operatorname{Ker} D_p\pi_{LC}$ and $T_p(\Sigma)$, the tangent space of Σ at the above position, intersect transversally, the systems are local solvable at p.

3. Local solvability and jump

We mean, by "jump" at $p \in \Sigma$, an instantaneous transition Δp $(\neq 0)$ of the state p such that $p+\Delta p \in \Sigma$. It will be clear from the last statement of the previous section that the necessary condition for the system to have jump at $p \in \Sigma$ is the following:

$$T_p(\Sigma) \cap \operatorname{Ker} D_p\pi_{LC} \neq \{0\}\ . \tag{10}$$

It follows from (4), (6) that

$$v_{C_i} = C_{m_{ii}}^{-1} \int i_{C_i} dt\ , \quad i=1, \cdots, n_C\ , \tag{11}$$

$$i_{L_j} = L_{m_{jj}}^{-1} \int v_{L_j} dt\ , \quad j=1, \cdots, n_L\ . \tag{12}$$

Under the natural physical restraint, the energy of capacitors and inductors, and hence the value of (i_L, v_C) is preserved at p and $p+\Delta p$ (energy's continuity). In other words, capacitor and inductor have inertia through the jump process. On the other hand, Ker $D_p\pi_{LC}$ represents the orthogonal complement of the subspace $\pi_{LC}(R^{2n})$. On jump points, by the "inertia", the gradient vector induced from (11), (12) coincides with $D_p\pi_{LC}(\Delta p)$ which implies

$$\Delta p \in \operatorname{Ker} D_p\pi_{LC} . \tag{13}$$

Since $T_p(\Sigma)$ denotes the subspace in which the dynamics of the system at p is described, by introducing a natural convention: even if jump occurs at p, the tangent vector keeps the direction, we may conclude that

$$\Delta p \in T_p(\Sigma) . \tag{14}$$

Thus, we can examine whether Δp ($\neq 0$) exists or not by solving linear homogeneous equations induced from (13), (14). In the successive sections, we will actually show the degeneracy of the linear equation system at every jump point.

4. Multivibrator

Figure 1 describes Multivibrator which is well known as an oscillator used to generate voltages pulses. On the system, $n_R=7$ (cf. Fig. 4, (25)), $n_C=2$, $n_L=0$, therefore $n=7+2=9$.

4.1. *Phase portrait*

In this system (Fig. 1), the following condition is assumed: a gate current i_g

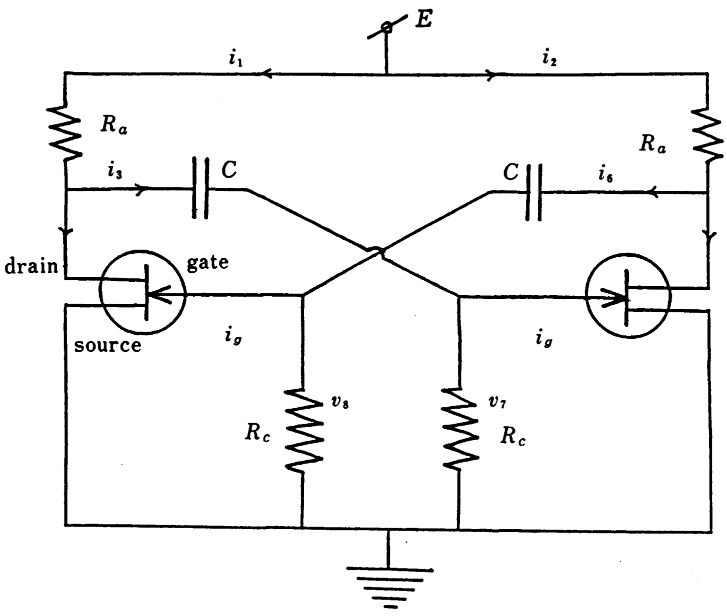


Fig. 1. Multivibrator system.

is negligible ($i_g=0$). Using gate voltages v_7, v_8 which are regarded as state variables, from voltage's relations, we obtain two first order differential equations (15) as a system representation [6], [7].

$$\text{(15)} \qquad \begin{cases} K(\dot{v}_8+\dot{v}_7=-v_7/\tau\,, \\ \dot{v}_8+K(v_7)\dot{v}_7=-v_8/\tau\,, \end{cases} \quad (R_a \ll R_C)\,,$$

which is called "implicit form", where

$$\text{(16)} \qquad K(v_7)=R_a S(v_7)\,,$$

$$\text{(17)} \qquad \tau=CR_c$$

and $S(v_7)$ denotes a derivative of the characteristics of *FET* (Fig. 2). Rewriting the implicit form equation to the normal form one, we have

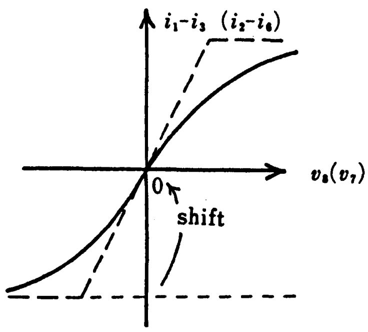


Fig. 2. Characteristic curve of Drain current.

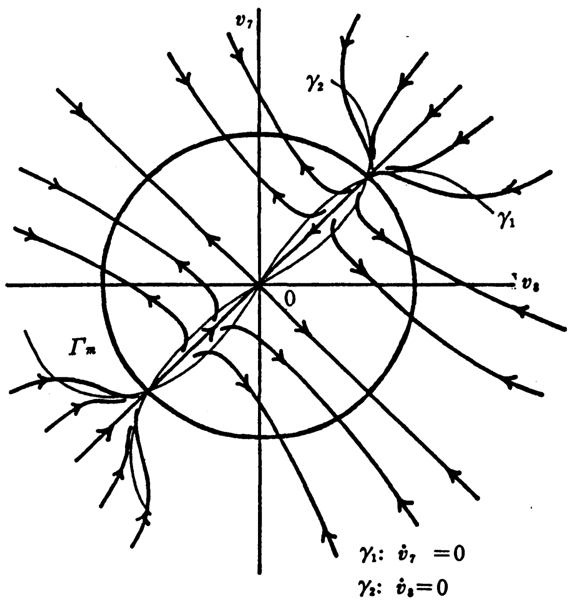


Fig. 3. Phase portait of multivibrator ($3/2>K(0)>1$).

(18)
$$\begin{cases}\dot{v}_8=(1/\tau)(K(v_7)v_7-v_8)/(1-K(v_7)K(v_8)) \\ \dot{v}_7=(1/\tau)(K(v_8)v_8-v_7)/(1-K(v_7)K(v_8))\ .\end{cases}$$

On the phase space, if $K(0)>1$, then there is a closed continuous curve Γ_m:

(19)
$$K(v_7)K(v_8)=1\ ,$$

which contains jump points of phase paths of (18). On Γ_m, only two points lying on $y=x$ are not jump points. As $1<K(0)<3/2$, we obtain phase portrait Fig. 3 of (18). Under the above assumption, a graph G_m induced from the system is represented as Fig. 4.

4.2. *Local solvability*

Kirchhoff space K, the tangent space of the branch characteristic space Λ and the configuration space Σ at $p\in\Sigma$ are represented as follows:

(20)
$$K=\{(i, v)\,|\,[Q\,|\,0](i, v)^t=0,\ [0\,|\,B](i, v)^t=0\}\ ,$$

(21)
$$T_p(\Lambda)=\{(i, v)\,|\,R(i, v)^t=0\}\ ,$$

(22)
$$T_p(\Sigma)=\{(i, v)\,|\,J(i, v)^t=0\}\ ,$$

where $(i, v)=(i_1, i_2, i_4, i_5, i_3, i_6, i_7, i_8, i_9, v_1, \cdots, v_9)$,

(23)
$$Q=\begin{bmatrix} -1 & 0 & 1 & 0 & \\ 0 & -1 & 0 & 1 & \\ -1 & 0 & 1 & 0 & I \\ 0 & -1 & 0 & 1 & \\ -1 & -1 & 0 & 0 & \end{bmatrix},\quad \text{(which is called the cut set matrix)}\ ,$$

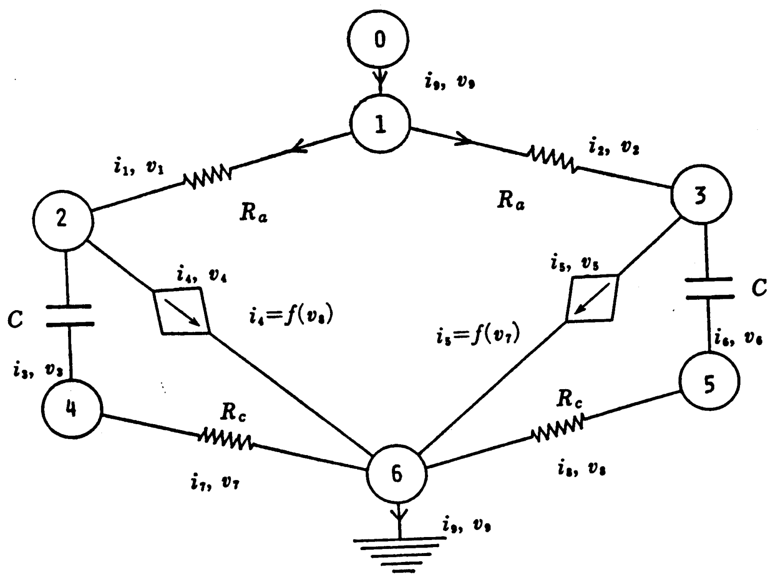


Fig. 4. Graph of multivibrator.

(24) $$B=\begin{bmatrix} & 1 & 0 & 1 & 0 & 1 \\ I & 0 & 1 & 0 & 1 & 1 \\ & -1 & 0 & -1 & 0 & 0 \\ & 0 & -1 & 0 & -1 & 0 \end{bmatrix}, \text{ which is called the loop matrix)},$$

(25) $$R=\left.\begin{bmatrix} R_a & & & & & & -1 & & & & \\ & R_a & & & & & & -1 & & & \\ & & 1 & & & & & & & -S_8 & \\ & & & 1 & & & & & -S_7 & & \\ & & & & R_c & & & & -1 & & \\ & & & & & R_c & & & & -1 & \\ & & & & & & & & & & 1 \end{bmatrix}\right\}\Lambda_R,$$

$$(S_7=S(v_7), \quad S_8=S(v_8)),$$

and

(26) $$J=\begin{bmatrix} Q & & 0 \\ 0 & & B \\ & R & \end{bmatrix}.$$

Thus, on this system,

(27) $$\dim K=n=9,$$

(28) $$\dim \Lambda=n+n_C+n_L=9+2=11,$$

and if K intersects Λ transversally,

(29) $$\dim \Sigma=n_C+n_L=2.$$

In this section, we examine the transversality between $T_p(\Sigma)$ and Ker $D_p\pi_{LC}$ which implies local solvability. If jump phenomena occur on this system, then the property of local solvability is destroyed, as precisely mentioned in Section 3. Therefore, a subset of the following set M_j:

(30) $$\left\{(i, v)\ \middle|\ \det\begin{bmatrix} J \\ D_p\pi_{LC} \end{bmatrix}=0\right\}$$

corresponds to jump points where $\pi_{LC}\colon \Sigma\to R^2$ is

(31) $$\pi_{LC}(i, v)=(v_3, v_6).$$

The derivative of π_{LC} at a jump point $p\in\Sigma$ is given by

(32) $$D_p\pi_{LC}=\begin{array}{c} \begin{bmatrix} 0 & \vdots & & & & & 1 & & & & \\ & \vdots & & & & & & 1 & & & \end{bmatrix} \\ \qquad\quad ,\ 1\ 2\ 4\ 5\ 3\ 6\ 7\ 8\ 9 \end{array}.$$

By applying elementary operations to the matrix in (30), we have the following:

$$\det\begin{bmatrix} J \\ D_p\pi_{LC} \end{bmatrix} = \det\begin{bmatrix} R_a+R_C & R_CR_aS_8 \\ R_CR_aS_7 & R_a+R_C \end{bmatrix}. \tag{33}$$

Consequently, we obtain the set M_j as follows:

$$M_j = \{(i, v) \mid K(v_7)K(v_8) = (1+R_a/R_C)^2\}\ . \tag{34}$$

Then we assume that a drain resistor R_a is small enough than a gate resistor R_C ($R_a \ll R_C$). So, we reduce the same results as the phase plane analysis, i.e.,

$$M_j = \Gamma_m\ , \tag{35}$$

which is defined in (19). On the other hand, rank $[J]$ is the full rank as follows:

$$\det\begin{bmatrix} 1+R_C/R_a & R_CS_8 \\ R_CS_7 & 1+R_C/R_a \end{bmatrix} = 1 - R_C{}^2S_7S_8(1-R_C/(R_a+R_C))^2 = 1\ . \tag{36}$$

It follows from a similar argument to (33) that the transversality of K and Λ holds.

5. Blocking Oscillator

Although Blocking Oscillator which is shown by Fig. 5 does not satisfy (5), (6), under the following assumption (i), we can reduce the system which satisfies (1)-(7). Considering mutual inductance, coupled inductors are transformed into another inductor (50). At the same time, there is a new current's relation (51) and there are new two voltage's relations. One is a Kirchhoff's voltage law (53) and the other is a relation between v_2 and v_5 (52).

5.1. *Phase portrait*

In this system (Fig. 5), it is natural to assume that

(i) the magnetic leakage flux is zero ($M^2 = LL_a$),

(ii) the anode current i_2 is a function of v_3, v_4 ($i_2 = \phi(v_4, v_3)$),

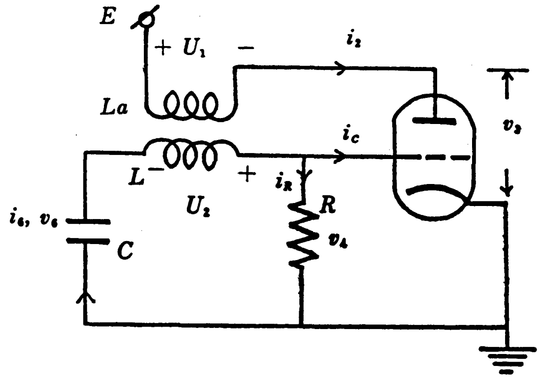


Fig. 5. Blocking oscillator system.

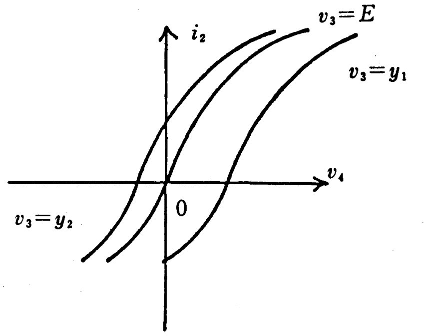


Fig. 6_a. Characteristic curves of $i_2=S_a(0, E)Z-Z^3$ $Z=v_4+\frac{v_3-E}{u}$, $u=$constant.

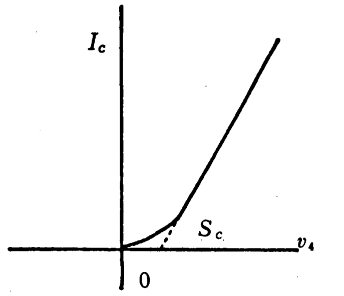


Fig. 6_b. Characteristic curve of I_c.

(iii) the grid current i_c depends only on the grid voltage v_4 ($i_c=\psi(v_4)$), where ϕ and ψ are given in Fig. 6. We choose grid voltages v_3, v_4 as the state variables. Then, rewriting the implicit form differential equation (first order), we can obtain the following normal form one as a representation of the system [7]:

$$\left\{\begin{array}{l}\dot{v}_4=-\dfrac{v_3-E}{n\theta}+\dfrac{n^2L}{\tau R_i\theta}(v_4+R\psi(v_4)) \\ \dot{v}_3=\dfrac{v_3-E}{\theta}-\dfrac{n}{\tau}\left(1+\dfrac{n^2L}{R_i\theta}\right)(v_4+R\psi(v_4))\end{array}\right. \tag{37}$$

where θ, S_a and R_i denote abbreviation of $\theta(v_4, v_3)$, $S_a(v_4, v_3)$ and $R_i(v_4, v_3)$, respectively. In (37), E denotes a supply voltage, L denotes the grid self-inductance and other notations are defined as follows:

$$\theta(v_4, v_3)=\tau_c[(1-n/u(v_4, v_3))nRS_a(v_4, v_3)-1-RS_c(v_4)]\,, \tag{38}$$

$$S_a(v_4, v_3)=\partial\phi/\partial v_4\,, \tag{39}$$

$$1/R_i(v_4, v_3)=\partial\phi/\partial v_3\,, \tag{40}$$

$$S_c(v_4)=d\psi/dv_4\,, \tag{41}$$

$$u(v_4, v_3)=R_i(v_4, v_3)S_a(v_4, v_3)\,, \tag{42}$$

$$\tau=CR\,, \tag{43}$$

$$\tau_c=L/R\,. \tag{44}$$

As an approximation, we suppose that the value of (42) holds constant on a neighborhood of the equilibrium point. If $\theta(0, E)>0$, then the set Γ_b:

$$\{(v_4, v_3)\,|\,\theta(v_4, v_3)=0\} \tag{45}$$

which consists of two lines and includes jump points, is constructed on the phase space as Fig. 7. On the phase portrait Fig. 8 of (37), the set Γ_b deleted the region of two dotted lines shows jump points.

On the other hand, under the assumptions (i), (ii) and (iii), we can obtain a graph G_b induced from the system in Fig. 9.

5.2. *Local solvability*

Let $(i, v)=(i_r, i_1, i_3, i_4, i_6, i_2, i_5, v_r, \cdots, v_5)$, where i_r and v_r are defined by (51),

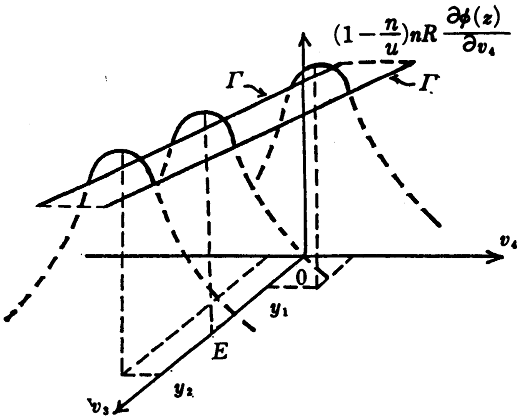
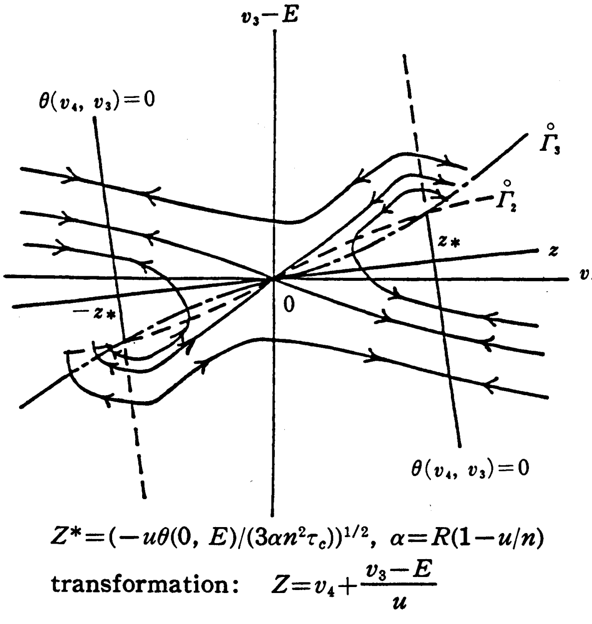


Fig. 7. Curves of $\left(1-\frac{n}{u}\right)nR\frac{\partial\phi(Z)}{\partial v_4}$, and Γ, $z=v_4+\frac{v_3-E}{u}$, u=constant.

$$\mathring{\Gamma}_2=\frac{n^3L_c}{\tau R_i(v_4, v_3)}v_4+E \quad : \quad \dot{v}_4=0$$

$$\mathring{\Gamma}_3=\left(\frac{n^3L_c}{\tau R_i(v_4, v_3)}+\frac{n\theta(v_4, v_3)}{(1-u/n)\tau}\right)v_4+E \quad : \quad \dot{z}=0$$



$$Z^*=(-u\theta(0, E)/(3\alpha n^2\tau_c))^{1/2}, \quad \alpha=R(1-u/n)$$

$$\text{transformation:} \quad Z=v_4+\frac{v_3-E}{u}$$

Fig. 8. Phase portrait of blocking oscillator ($\theta(0, E)>0$).

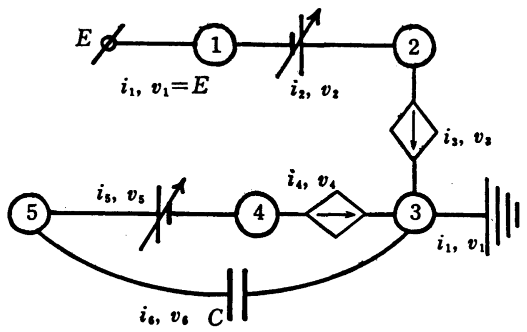


Fig. 9. Graph of blocking oscillator.

(53), then each matrix A, R and J which determines K, $T_p(\Lambda)$ and $T_p(\Sigma)$ at $p \in \Sigma$ are represented as follows respectively:

$$\text{(46)} \qquad A=\begin{bmatrix} 0 & 1 & & & & -1 & & & & & & & & \\ & & 1 & & & -1 & & & & & & & & \\ & & & 1 & & & -1 & & & & & & & \\ & & & & 1 & & -1 & & & & & & & \\ & & & & & & & & 1 & 1 & & & 1 & \\ & & & & & & & & & & 1 & 1 & & 1 \\ & & & & & & & 1 & & & & & & -1 \end{bmatrix}\begin{matrix} \\ \\ \\ \\ \\ \\ \}\Lambda_{L_r} \end{matrix},$$

$$\text{(47)} \qquad R=\begin{bmatrix} & 1 & & & & & & & & *1 & *2 & & & \\ & & 1 & & & & & & & & *3 & & & \\ & & & & & & & & 1 & & & & & \\ & & & & & & & & & & & & 1 & n \\ 1 & & & -n & 1 & & & & & & & & & \end{bmatrix}\begin{matrix} \left.\begin{matrix} \\ \\ \\ \end{matrix}\right\}\Lambda_R \\ \left.\begin{matrix} \\ \\ \end{matrix}\right\}\Lambda_{L_r} \end{matrix},$$

$$\left(*1=-1/R_i,\ *2=-S_a,\ *3=-\frac{1+RS_c}{R}\right),$$

$$\text{(48)} \qquad J=\begin{bmatrix} A \\ R \end{bmatrix}.$$

By the way Λ_{L_r}:

$$\text{(49)} \qquad \left\{(i_2, i_5, v_2, v_5)\,\middle|\,\begin{pmatrix} v_2 \\ v_5 \end{pmatrix}=\begin{bmatrix} L_a & -M \\ M & -L \end{bmatrix}\begin{pmatrix} \dot{i}_2 \\ \dot{i}_5 \end{pmatrix}\right\}$$

is a set which shows the relation of a coupled inductors given initially in the network. Considering mutual inductance M, from assumption (i), which implies that the matrix in (49) is singular, we reduce the relation in (49) to the followings:

$$\text{(50)} \qquad v_5=L\dot{i}_r\,,$$

$$\text{(51)} \qquad i_r=ni_2-i_5\,,\quad \text{(which is called the magnetization current)}\,,$$

$$\text{(52)} \qquad v_2=nv_5\,,$$

(53) $v_r = v_5$, (which is regarded as a Kirchhoff's relation) .

On this system, let n_r be the number of equations which are transformed, then $n_R' = n_R + n_r = 3+2=5$, $n_C=1$, $n_L=1$, therefore $n = n_R' + n_C + n_L = 7$.

(54) $$\dim K = n = 7\,,$$

(55) $$\dim \Lambda = n + n_C + n_L = 9\,,$$

in case K intersects Λ transversally,

(56) $$\dim \Sigma = n_C + n_L = 2\,.$$

The derivative of π_{LC} at $p \in \Sigma$ is given by

(57) $$D_p\pi_{LC} = \begin{array}{c}\begin{bmatrix}1 & & \\ & & 1\end{bmatrix}\\ {\scriptstyle r1\cdots 5,\quad r1\ 3\ 4\ 6\ 2\ 5}\end{array},$$

where $\pi_{LC}\colon \Sigma \to R^2$

(58) $$\pi_{LC}(i, v) = (i_r, v_6)\,.$$

Since, applying elementary operations,

(59) $$\det\begin{bmatrix} J \\ D_p\pi_{LC}\end{bmatrix} = \det\begin{bmatrix}1 & *1 & *2 \\ n & 0 & *3 \\ 0 & -1 & -n\end{bmatrix} = 0$$

which gives a necessary condition for the property of local solvability to be destroyed, the following set B_j:

(60) $$\{(i, v) \mid \theta(v_4, v_3) = 0\}$$

includes jump points. This set coincides with (45).

Since,

(61) $$\det\begin{bmatrix}-1/R_i & -S_a \\ 0 & -(1+RS_c)/R\end{bmatrix} = (1+RS_c)/RR_i > 0\,,$$

the matrix J has the full rank, and hence the transversality of K and Λ holds.

Acknowledgment

The author would like to record a special debt of gratitude to Professor T. Sekine and Professor H. Nishino of Keio University, who furnished a number of stimulating discussions and useful suggestions. And he would particularly like to thank Professor S. Ichiraku of Yokohama City University for many valuable suggestions and corrections.

References

[1] S. Smale, *On the mathematical foundations of electrical circuit theory*, J. Diff. Geometry, **7** (1972), 193–210.

[2] T. Matsumoto, L. Chua, H. Kawakami and S. Ichiraku, *Geometric properties of dynamic nonlinear networks, transversality, local solvability and eventual passivity*, IEEE Trans. Circuits & Syst., Vol. CAS-28, May, (1981), 406–428.

[3] T. Matsumoto, *On the dynamics of electrical networks*, J. Diff. Equations, **21** (1976), 179–196.

[4] S. Ichiraku, *Connecting electrical circuits, transversality and well-posedness*, Yokohama Math. J., **27** (1979), 111–126.

[5] L. Chua, T. Matsumoto and S. Ichiraku, *Geometric properties of resistive nonlinear n-ports, transversality, structural stability, reciprocity, anti-reciprocity*, IEEE Trans. Circuits & Syst. Vol. CAS-27, July, (1980), 577–603.

[6] K. Tchizawa, *The topological analysis of jump phenomena in nonlinear systems*, SICE, J., **18** (1982), 363–370.

[7] A. Andronov, A. Vitt and S. Khaikin, *Theory of oscillator*, Pergmamon Press (1965).

Department of Administration Engineering
Faculty of Science and Technology
Keio University