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SOME PROPERTIES OF SPHERICAL HARMONICS

By

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1. Introduction.

Owing to works by Takahashi [4] and by do Carmo and Wallach [1] we know that the property of spherical harmonics plays a quite important role in the theory of isometric minimal immersions of spheres into spheres. In the present paper we study some features of spherical harmonics or of homogeneous harmonic polynomials and prove a theorem which we shall use in another paper [3] where isotropic properties of isometric minimal immersions will be studied.

2. Preliminaries.

Let $\tilde{\varphi}$ be a homogeneous harmonic polynomial of degree s in R^{m+1} and let $S^m(1)$ be the unit hypersphere of R^{m+1}. Then $\tilde{\varphi}$ restricted to $S^m(1)$ is a spherical harmonic of order s. For convenience sake we use the letter φ for this function.

We fix an orthonormal frame $\{e_1, \cdots, e_{m+1}\}$ in R^{m+1} and use the following indices

$$h, i, j, \cdots=1, \cdots, m+1 ,$$
$$\kappa, \lambda, \mu, \cdots=1, \cdots, m$$

for which we adopt the summation convention if possible. This does not mean that the letters $h, i, j, \cdots$ cannot be used otherwise.

A point of $S^m(1)$ may be expressed by the position vector

$$u=u^h e_h , \quad \|u\|=1$$

and also by the local coordinates $x^1, \cdots, x^m$. The relation between them is written

$$u(x)=u^h(x)e_h .$$

The standard Riemannian metric on $S^m(1)$ is denoted by g and its local components by $g_{\mu\lambda}$ or $g(\mu\lambda)$. Then we have

$$\sum_i (u^i)^2=u_i u^i=1 ,$$
$$\nabla_\mu u^i \nabla_\lambda u^h g^{\mu\lambda}=\delta^{ih}-u^i u^h ,$$

$$\nabla_\mu u_i \nabla_\lambda u^i = g_{\mu\lambda} ,$$
$$\nabla_\mu \nabla_\lambda u^h = -g_{\mu\lambda} u^h ,$$
$$\Delta u^h = -g^{\mu\lambda}\nabla_\mu \nabla_\lambda u^h = m u^h ,$$
$$\Delta\varphi = \lambda_s \varphi , \quad \lambda_s = s(s+m-1)$$

where $u_i = u^i$, $g^{\mu\lambda}$ are given by $g^{\mu\lambda} g_{\nu\mu} = \delta_\nu^\lambda$ and ∇ denotes the Riemannian connection, hence $\nabla_\lambda u^h = \partial_\lambda u^h = \partial u^h / \partial x^\lambda$.

We can cansider φ as a symmetric tensor of degree s in R^{m+1} satisfying

(2.1) $$\sum_i \varphi(e_i, e_i, v_3, \cdots, v_s) = 0$$

for arbitrary vectors $v_3, \cdots, v_s$ of R^{m+1}.

3. Symmetric tensor fields Φ_i and $D\Phi_i$.

Let $t_1, \cdots, t_i$ be arbitrary tangent vector fields of $S^m(1)$. Then we can define a symmetric tensor field φ_i on $S^m(1)$ by

$$\varphi_i(t_1, \cdots, t_i) = \varphi(t_1, \cdots, t_i, u, \cdots, u) .$$

Here we can consider $i = 0, 1, \cdots, s$ if we put $\varphi_0 = \varphi(u, \cdots, u)$. As we can put locally

$$t_a = t_a{}^\lambda \nabla_\lambda u , \quad a = 1, \cdots, i ,$$

we have

$$\varphi_i(t_1, \cdots, t_i) = \varphi(\nabla_{\lambda_1} u, \cdots, \nabla_{\lambda_i} u, u, \cdots, u) t_1{}^{\lambda_1} \cdots t_i{}^{\lambda_i} .$$

For convenience sake we use the notation

$$\varphi(\lambda_1 \cdots \lambda_i) = \varphi(\nabla_{\lambda_1} u, \cdots, \nabla_{\lambda_i} u, u, \cdots, u)$$

so that $\varphi(\lambda_1 \cdots \lambda_i)$ are local components of the tensor field φ_i. We also adopt the abbreviation

$$\varphi_i(t, \cdots, t) = \varphi_i(t) .$$

From the tensor fields $\varphi_0, \varphi_1, \cdots, \varphi_s$ we define a symmetric tensor field Φ_i, $i = 0, 1, \cdots, s$, as follows.

Definition 1. Let t be any local unit tangent vector field and let $x_{i,q}$ ($q = 0, 1, \cdots, [i/2]$) be given by

(3.1) $$q(m+2i-2q-2) x_{i,q} = ((i-2q+2)(i-2q+1)/2) x_{i,q-1} ,$$
$$x_{i,0} = 1 .$$

Then Φ_i is defined by

(3.2) $$\Phi_i(t)=s(s-1)\cdots(s-i+1)\sum_{q=0}^{k} x_{i,q}\varphi_{i-2q}(t)$$

where $k=[i/2]$ is the largest integer satisfying $2k\leqq i$.

If t is an arbitrary tangent vector field, (3.2) should be replaced by

$$\Phi_i(t)=s(s-1)\cdots(s-i+1)\sum_{q=0}^{k} x_{i,q}\|t\|^{2q}\varphi_{i-2q}(t)\ .$$

Definition 2. $D\varphi_i$ is a symmetric tensor field of degree $i+1$ satisfying

$$(D\varphi_i)(t)=\nabla_t(\varphi_i(t))-i\varphi_i(t,\cdots,t,\nabla_t t)$$

for any tangent vector field t. Thus the local components of $D\varphi_i$ are $\mathscr{S}_{\lambda,i+1}\nabla_{\lambda_1}\varphi(\lambda_2\cdots\lambda_{i+1})$ where $\mathscr{S}_{\lambda,i+1}$ is the symmetrizer with respect to $\lambda_1,\cdots,\lambda_{i+1}$. $D\Phi_i$ is defined similarly.

4. Some lemmas and the main Theorem.

We give here some lemmas and the main Theorem which we prove in §6 and §8.

Lemma 1. *Let $y_{i,q}(q=1,\cdots,h=[(i+1)/2])$ be determined by*

(4.1) $$y_{i,q}=(s-i+2q)x_{i,q}-(i-2q+2)x_{i,q-1}-(s-i)x_{i+1,q}$$
$$\text{if}\quad q=1,\cdots,k=[i/2]\ ,\quad y_{i,h}=-x_{i,k}-(s-i)x_{i+1,h}$$

where the last one appears only when i is odd, hence $h=k+1$. Then the following identity holds at every point of $S^m(1)$ and for any unit tangent vector t,

(4.2) $$(D\Phi_i)(t)-\Phi_{i+1}(t)=s(s-1)\cdots(s-i+1)$$
$$\times\left[\sum_{q=1}^{k} y_{i,q}\varphi_{i+1-2q}(t)+y_{i,h}\varphi_0\right].$$

Lemma 2. *The local components $\Phi(\lambda_1\cdots\lambda_i)$ of the symmetric tensor field Φ_i satisfy*

(4.3) $$g^{\nu\mu}\Phi(\nu\mu\lambda_3\cdots\lambda_i)=0\ .$$

Our main result is the following theorem.

Theorem 3. *Let i be an integer, $1\leqq i<s$. Then*

(4.4) $$\Phi_{i+1}(t)=(D\Phi_i)(t)-(s-i+1)y_{i,1}\Phi_{i-1}(t)$$

holds at any point of $S^m(1)$ and for any unit tangent vector t.

In view of Lemma 1 we easily get the following equivalent.

Theorem 3′. *$y_{i,q}$ of Lemma 1 satisfy*

$$y_{i,q}=y_{i,1}x_{i-1,q-1}\ . \tag{4.5}$$

5. Some properties of symmetric tensors.

We consider the Euclidean m-space R^m where a basis $\{f_1, \cdots, f_m\}$ is fixed such that $\langle f_\mu, f_\lambda\rangle=h_{\mu\lambda}$. Let $P_i(=P_{i,0})$ be a symmetric tensor of degree i with components $P_i(\lambda_1 \cdots \lambda_i)=P_i(f_{\lambda_1}, \cdots, f_{\lambda_i})$. We define from P_i a series of symmetric tensors $P_{i,q}(q=1, \cdots, k=[i/2])$ of degree $i-2q$ inductively by

$$-h^{\nu\mu}P_{i,q}(\nu\mu\lambda_{2q+3} \cdots \lambda_i)=P_{i,q+1}(\lambda_{2q+3} \cdots \lambda_i)$$

where $h^{\nu\mu}=h^{\mu\nu}$ are given by $h^{\nu\mu}h_{\mu\lambda}=\delta_\lambda^\nu$ and $q=0, 1, \cdots, k-1$. The following lemma concerns a symmetric tensor P_{i-2p} of degree $i-2p$ and symmetric tensors $P_{i-2p,q}$.

Lemma 4. *Let Q_i be a symmetric tensor of degree i defined from $P_{i-2p,q}$ $(q=0, 1, \cdots, k-p)$ by*

$$Q_i(t)=\sum_{q=0}^{k-p} \alpha_{i,p+q}P_{i-2p,q}(t) \tag{5.1}$$

where $h_{\nu\mu}t^\nu t^\mu=1$. Then, putting $\alpha_{i,p-1}=0$, we get

$$\begin{aligned} -(i(i-1)/2)Q_{i,1}(t)=\sum_{q=0}^{k-p}[(p+q)(m+2i-2p-2q-2)\alpha_{i,p+q} \\ -((i-2p-2q+2)(i-2p-2q+1)/2)\alpha_{i,p+q-1}]P_{i-2p,q}(t)\ . \end{aligned} \tag{5.2}$$

Proof. (5.1) is equivalent to

$$Q_i(\lambda_1 \cdots \lambda_i)=\sum_{q=0}^{k-p} a_{i,r}\mathscr{S}_{\lambda,i}h(\lambda_1\lambda_2) \cdots h(\lambda_{2r-1}\lambda_{2r})P_{i-2p,q}(\lambda_{2r+1} \cdots \lambda_i)$$

where $h_{\mu\lambda}$ is written $h(\mu\lambda)$ and r stands for $p+q$. As we have

$$\begin{aligned} &(2r(2r-1)/2)h^{\lambda_1\lambda_2}\mathscr{S}_{\lambda,2r}h(\lambda_1\lambda_2) \cdots h(\lambda_{2r-1}\lambda_{2r}) \\ &\quad =r(m+2r-2)\mathscr{S}_{\lambda,2r-2}h(\lambda_3\lambda_4) \cdots h(\lambda_{2r-1}\lambda_{2r})\ , \end{aligned}$$

we get

$$\begin{aligned} &(i(i-1)/2)h^{\lambda_1\lambda_2}\mathscr{S}_{\lambda,i}h(\lambda_1\lambda_2) \cdots h(\lambda_{2r-1}\lambda_{2r})P_{i-2p,q}(\lambda_{2r+1} \cdots \lambda_i) \\ &\quad =(r(m+2r-2)+2r(i-2r))\mathscr{S}_{\lambda,i-2}h(\lambda_3\lambda_4) \cdots h(\lambda_{2r-1}\lambda_{2r})P_{i-2p,q}(\lambda_{2r+1} \cdots \lambda_i) \\ &\quad\quad -((i-2r)(i-2r-1)/2)\mathscr{S}_{\lambda,i-2}h(\lambda_3\lambda_4) \cdots h(\lambda_{2r+1}\lambda_{2r+2})P_{i-2p,q+1}(\lambda_{2r+3} \cdots \lambda_i)\ . \end{aligned}$$

This proves the following formula which is equivalent to (5.2),

$$\begin{aligned} &-(i(i-1)/2)Q_{i,1}(\lambda_3 \cdots \lambda_i) \\ &\quad =\sum_{q=0}^{k-p}\{r(m+2i-2r-2)\alpha_{i,r}-((i-2r+2)(i-2r+1)/2)\alpha_{i,r-1}\} \\ &\quad\quad \times\mathscr{S}_{\lambda,i-2}h(\lambda_3\lambda_4) \cdots h(\lambda_{2r-1}\lambda_{2r})P_{i-2p,q}(\lambda_{2r+1} \cdots \lambda_i)\ . \end{aligned}$$

On this occasion we prove the following lemma.

Lemma 5. *Let S_i be a symmetric tensor of degree i. We assume that there exists a symmetric tensor T_{i-2} of degree $i-2$ such that $S_i(t)=T_{i-2}(t)$ if t satisfies $h_{\mu\lambda}t^\mu t^\lambda=1$. Then we have*

$$S_i(t)+\sum_{q=1}^{k} x_{i,q}S_{i,q}(t)=0$$

where $x_{i,q}$ are given by (3.1).

Proof. As we have

$$S_i(\lambda_1 \cdots \lambda_i)=\mathscr{S}_{\lambda,i}h(\lambda_1\lambda_2)T_{i-2}(\lambda_3 \cdots \lambda_i) ,$$

we get

$$-(i(i-1)/2)S_{i,1}(t)=(m+2i-4)T_{i-2}(t)-((i-2)(i-3)/2)T_{i-2,1}(t) .$$

Furthermore we get

$$(i(i-1)/2)S_{i,q}(t)=-q(m+2i-2q-2)T_{i-2,q-1}(t)+((i-2q)(i-2q-1)/2)T_{i-2,q}(t) ,$$

hence

$$\begin{aligned}(i(i-1)/2)&\left[S_i(t)+\sum_{q=1}^{k} x_{i,q}S_{i,q}(t)\right]\\ &=(i(i-1)/2)T_{i-2}(t)+\sum_{q=1}^{k} x_{i,q}\{-q(m+2i-2q-2)T_{i-2,q-1}(t)\\ &\quad+((i-2q)(i-2q-1)/2)T_{i-2,q}(t)\}\\ &=(i(i-1)/2)T_{i-2}(t)-\sum_{q=1}^{k} q(m+2i-2q-2)x_{i,q}T_{i-2,q-1}(t)\\ &\quad+\sum_{q=2}^{k}((i-2q+2)(i-2q+1)/2)x_{i,q-1}T_{i-2,q-1}(t) .\end{aligned}$$

As the last member vanishes because of (3.1), we have proved Lemma 5.

6. Proof of Lemma 1 and Lemma 2.

As we have

$$\begin{aligned}&\nabla_\lambda\varphi(\nabla_{\lambda_1}u, \cdots, \nabla_{\lambda_i}u, u, \cdots, u)\\ &\quad=(s-i)\varphi(\nabla_{\lambda_1}u, \cdots, \nabla_{\lambda_i}u, \nabla_\lambda u, u, \cdots, u)\\ &\qquad+\sum_{p=1}^{i}\varphi(\nabla_{\lambda_1}u, \cdots, \nabla_\lambda\nabla_{\lambda_p}u, \cdots, \nabla_{\lambda_i}u, u, \cdots, u)\end{aligned} \tag{6.1}$$

and $\nabla_\mu\nabla_\lambda u=-g_{\mu\lambda}u$, we get

$$(D\varphi_i)(t)=(s-i)\varphi_{i+1}(t)-i\varphi_{i-1}(t)$$

for any unit tangent vector t. Thus we get Lemma 1 from

$$(D\Phi_i)(t)=s(s-1)\cdots(s-i+1)\left[(D\varphi_i)(t)+\sum_{q=1}^{k}x_{i,q}(D\varphi_{i-2q})(t)\right]$$

and

$$\Phi_{i+1}(t)=s(s-1)\cdots(s-i)\left[\varphi_{i+1}(t)+\sum_{q=1}^{h}x_{i+1,q}\varphi_{i+1-2q}(t)\right]$$

after some straightforward calculation.

In order to prove Lemma 2 we take an arbitrary point x of $S^m(1)$ and consider $T_x(S^m(1))$ as the space R^m of Lemma 4. Then we can put $h_{\mu\lambda}=g_{\mu\lambda}$ and, as we have $\varphi_{i,1}(t)=\varphi_{i-2}(t)$ in view of $g^{\mu\lambda}\nabla_\mu u^i\nabla_\lambda u^h=\delta^{ih}-u^iu^h$ and (2.1), $\Phi_i(t)$ is a special case of $Q_i(t)$ in Lemma 4, such that $p=0, \alpha_{i,q}=s(s-1)\cdots(s-i+1)x_{i,q}$ and $P_{i,q}(t)=\varphi_{i-2q}(t)$. Thus we get $\Phi_{i,1}(t)=0$ from (5.2). This proves Lemma 2.

7. Some integrals.

We consider again the Euclidean m-space R^m as in §5. Let S be a symmetric tensor of any degree h with components $S(\lambda_1\cdots\lambda_h)$ and put $S(t)=S(t,\cdots,t)$. Let $d\omega_{m-1}$ be the volume element of the unit hypersphere $S^{m-1}(1)$ of R^m. Then we have the following lemma where, here and in the sequel, the domain of each integral is $S^{m-1}(1)$.

Lemma 6. *Let x be a moving point of $S^{m-1}(1)$ and t be the position vector x in R^m. Then*

$$\int S(t)d\omega_{m-1}=0 \quad \textit{if} \quad h \quad \textit{is odd}\ ,$$

$$\int S(t)d\omega_{m-1}=k_pS(\lambda_1\cdots\lambda_{2p})h^{\lambda_1\lambda_2}\cdots h^{\lambda_{2p-1}\lambda_{2p}} \quad \textit{if} \quad h=2p\ .$$

Proof. We only need to consider the case $\{f_1,\cdots,f_m\}$ is an orthonormal basis. It is clear that, if h is odd, the integral vanishes. Putting $h=2p$ we get

$$\int S(t)d\omega_{m-1}=S(\lambda_1\cdots\lambda_{2p})\int t^{\lambda_1}\cdots t^{\lambda_{2p}}d\omega_{m-1}\ ,$$

where

$$\int t^{\lambda_1}\cdots t^{\lambda_{2p}}d\omega_{m-1}=k_p\mathscr{S}_{\lambda,2p}\delta^{\lambda_1\lambda_2}\cdots\delta^{\lambda_{2p-1}\lambda_{2p}}\ ,$$

$$k_p=((m-2)!!(2p-1)!!/(2p+m-2)!!)\int d\omega_{m-1}$$

(see [2], pages 321, 322). This proves Lemma 6.

Let x be a point of $S^m(1)$ and t be the moving unit tangent vector at x. Considering $T_x(S^m(1))$ as R^m we can prove the following lemma where the integral is taken over the unit hypersphere $S^{m-1}(1)$ of $T_x(S^m(1))$.

Lemma 7. *The symmetric tensor fields Φ_i, Φ_j satisfy*

$$\int \Phi_j(t)\Phi_i(t)d\omega_{m-1}=0 \quad \textit{if} \quad j\neq i\ .$$

Proof. In terms of local components we can write

$$\Phi_j(t)\Phi(t)=\Phi(\mu_1\cdots\mu_j)\Phi(\lambda_1\cdots\lambda_i)t^{\mu_1}\cdots t^{\mu_j}t^{\lambda_1}\cdots t^{\lambda_i}\ .$$

Hence, if $j+i$ is odd, the integral vanishes. Thus we need to consider only the cases $i\geqq j+2$ such that $j+i$ is even. Hence, in view of Lemma 2 and Lemma 6 we can deduce Lemma 7 immediately.

Lemma 8. *The following integral vanishes if $j\leqq i-2$,*

$$\int \Phi_j(t)(D\Phi_i)(t)d\omega_{m-1}\ .$$

Proof. If $j=i-2$, then $(D\Phi_i)(t)\Phi_j(t)$ is a homogeneous polynomial of degree $2i-1$ in t and the integral vanishes. We consider the case $j=i-3$. From Lemma 2, which is valid at every point of $S^m(1)$, we get

$$g^{\nu\mu}\nabla_\kappa\Phi(\nu\mu\lambda_3\cdots\lambda_i)=0\ .$$

Thus, applying Lemma 6 to the integral

$$\int \Phi(\mu_1\cdots\mu_{i-3})\nabla_\kappa\Phi(\lambda_1\cdots\lambda_i)t^{\mu_1}\cdots t^{\mu_{i-3}}t^\kappa t^{\lambda_1}\cdots t^{\lambda_i}d\omega_{m-1}\ ,$$

we find that this integral vanishes. Proof for the cases $j\leqq i-3$ follows similarly.

8. Proof of the main Theorem.

It is clear that we get from (3.2) the following formulas where $\zeta_{i,q}$ are certain constants and $k=[i/2]$,

$$s(s-1)\cdots(s-i+1)\varphi_i(t)=\Phi_i(t)+\sum_{q=1}^{k}\zeta_{i,q}\Phi_{i-2q}(t)\ .$$

From this and Lemma 1 we get

$$(D\Phi_i)(t)-\Phi_{i+1}(t)-(s-i+1)y_{i,1}\Phi_{i-1}(t)+\sum_{q=2}^{h}z_{i,q}\Phi_{i+1-2q}(t)=0$$

weher $z_{i,q}$ are some constants and $h=[(i+1)/2]$. Then, as an application of Lemma 7 and Lemma 8, we get

$$z_{i,q}\int(\Phi_{i-2q+1}(t))^2 d\omega_{m-1}=0\ ,\quad q\geqq 2\ .$$

This shows that, if $z_{i,q}\neq 0$, then $\Phi_{i-2q+1}(t)$ vanishes identically, hence $z_{i,q}\Phi_{i+1-2q}(t)$ always vanishes. This proves the main theorem. In fact $z_{i,q}$ vanishes and $\Phi_{i+1-2q}(t)$ does not vanish except the trivial case, as we can prove the following lemma.

Lemma 9. *Let φ be a non-trivial spherical harmonic of order s and j be an integer, $2\leqq j\leqq s$. Then there exists no set of numbers $\alpha_{j,q}(q=1,\cdots,h(=[j/2]))$ such that*

$$\varphi_j(t)+\sum_{q=1}^{h}\alpha_{j,q}\varphi_{j-2q}(t)=0$$

holds identically.

Proof. First, suppose we have the identity

$$\varphi(\mu\lambda)+\alpha g_{\mu\lambda}\varphi_0=0\ .$$

Then we get $\alpha=1/m$ from $g^{\mu\lambda}(\varphi(\mu\lambda)+\alpha g_{\mu\lambda}\varphi_0)=0$, hence $\varphi(\mu\lambda)+(1/m)g_{\mu\lambda}\varphi_0=0$. Differentiating partially we get

$$(s-2)\varphi(\nu\mu\lambda)+((s/m-2)/3)(g_{\nu\mu}\varphi(\lambda)+g_{\nu\lambda}\varphi(\mu)+g_{\mu\lambda}\varphi(\nu))=0$$

and further

$$\{-(s-2)+((m+2)(s/m-2)/3)\}\varphi(\lambda)=0\ ,$$

namely, $(m-1)(m+s)\varphi(\lambda)=0$. But, if $\varphi(\lambda)=0$, φ vanishes identically which we have excluded. Thus Lemma 9 is proved for $j=2$.

Suppose there exists a certain integer i such that Lemma 9 is valid for $j=2,\cdots,i-1$ but fails for $j=i$, namely, there exists a set of numbers $\alpha_{i,q}$ such that the identity

$$\varphi_i(t)+\sum_{q=1}^{k}\alpha_{i,q}\varphi_{i-2q}(t)=0\ ,\quad k=[i/2]$$

holds. This identity is just the identity (5.1) of Lemma 4 if we put $Q_i(t)=0$, $p=0$, $\alpha_{i,0}=1$ and $P_{i,q}(t)=\varphi_{i,q}(t)$ where $\varphi_{i,q}(t)=\varphi_{i-2q}(t)$. Hence we get

$$0=\sum_{q=1}^{k}\{q(m+2i-2q-2)\alpha_{i,q}-((i-2q+2)(i-2q+1)/2)\alpha_{i,q-1}\}\varphi_{i-2q}(t)$$

and consequently $\alpha_{i,q}=x_{i,q}$.

Now suppose

$$\varphi_i(t)+\sum_{q=1}^{k} x_{i,q}\varphi_{i-2q}(t)=0 ,$$

namely, $\Phi_i(t)=0$ identically. Then we get $(D\Phi_i)(t)=0$ and (4.2) becomes

$$-\Phi_{i+1}(t)=s(s-1)\cdots(s-i+1)\left[\sum_{q=1}^{k} y_{i,q}\varphi_{i+1-2q}(t)+y_{i,h}\varphi_0\right].$$

This identity is the special case of (5.1) such that $p=0$, i is replaced by $i+1$ and

$$Q_{i+1}(t)=-\Phi_{i+1}(t) ,$$
$$\alpha_{i+1,0}=0 , \quad \alpha_{i+1,q}=s(s-1)\cdots(s-i+1)y_{i,q} ,$$
$$P_{i+1,q}(t)=\varphi_{i+1-2q}(t) .$$

As, in view of Lemma 2, we have $\Phi_{i+1,1}(t)=0$, we get

$$q(m+2i-2q)\alpha_{i+1,q}-((i-2q+3)(i-2q+2)/2)\alpha_{i+1,q-1}=0$$

and, especially for $q=1$, $(m+2i-2)y_{i,1}=0$. But we get from (4.1)

$$y_{i,1}=-i(a-i+1)(a+s-i+2)/(a(a+2))$$

where $a=m+2i-4$, hence $y_{i,1}<0$. Thus $\Phi_i(t)=0$ identically cannot occur and we have proved Lemma 9.

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