YOKOHAMA MATHEMATICAL
JOURNAL VOL. 32, 1984

SHIFT AUTOMORPHISM GROUPS OF C^*-ALGEBRAS

By

CHO-HO CHU

(Received August 12, 1983)

ABSTRACT. We study a C^*-dynamical system (A, G, α) in which G is a discrete group acting freely on A in a strong sense. We show that the enveloping von Neumann algebra $(A\underset{\alpha}{\times}G)''$ of the C^*-crossed product $A\underset{\alpha}{\times}G$ of such a system is isomorphic to the W^*-crossed product $A''\underset{\alpha''}{\times}G$ where α'' is the bitransposed action of G on A''. Consequently, $A\underset{\alpha}{\times}G$ is a type I C^*-algebra if A is a type I C^*-algebra.

1. Introduction

Let G be a discrete group with identity e. A *C^*-dynamical system* (A, G, α) consists of a C^*-algebra A and a homomorphism $\alpha: g\rightarrow\alpha_g$ from G into the group Aut(A) of $*$-automorphisms of A. In [3], the author formulated the definition of a strongly centrally free action α in a C^*-dynamical system (A, G, α). We show in section 2 that this freeness condition coincides with the notion of shift automorphisms introduced by Choda in [2]. Moreover, it is shown that the shift automorphisms are analogous to the completely dissipative transformations in ergodic theory. In fact, if we consider G acting on the state space S of A in a natural way, then G acts strongly centrally freely on A if and only if there is a *wandering* split face of S which 'generates' S. Also, similar to the *Hopf decomposition* (cf. [4; p. 48]), for any group G acting on A, its state space S is decomposed into a convex sum of two G-invariant split faces F_c and F_d where F_c contains no wandering split face (G is conservative on F_c) and F_d is either empty or is 'generated' by a wandering split face (G is completely dissipative on F_d). *I*n section 3, we show that if G acts strongly centrally freely in (A, G, α), then the enveloping von Neumann algebra $(A\underset{\alpha}{\times}G)''$ of the C^*-crossed product $A\underset{\alpha}{\times}G$ is isomorphic to the W^*-crossed product $A''\underset{\alpha''}{\times}G$ where α'' is the bitransposed action of G on the enveloping von Neumann algebra A'' of A. Under this condition, $A\underset{\alpha}{\times}G$ is a type I C^*-algebra if A is a type I C^*-algebra. The question of type I-ness of the crossed products in other circumstances has also been condidered by Rieffel [9], Takesaki [10] and Zeller-Meier [11].

2. Central shifts

In the sequel, G will denote a discrete group unless otherwise stated. Let (A, G, α) be a C^*-dynamical system and let S be the state space of A. For each $g \in G$, the transpose α_g' of the automorphism $\alpha_g: A \to A$ induces an affine homeomorphism from S onto itself which will also be denoted by α_g'. A (nonempty) subset W of S is called *G-wandering* if $\alpha_g'(W) \cap W = \emptyset$ for each $g \neq e$. Let $W^* = \overline{\text{co}} \bigcup_{g \in G} \alpha_g'(W)$ where $\overline{\text{co}}$ denotes the norm-closed convex hull. Let A'' be the universal enveloping von Neumann algebra of A which is identified with the bidual A^{**} of A. The action $\alpha: g \to \alpha_g$ also induces the bitransposed action $\alpha'': g \to \alpha_g'' \in \text{Aut}(A'')$ and (A'', G, α'') becomes a W^*-dynamical system. We say that G *acts strongly centrally freely* on A [3] if there is a family $\{p_j\}$ of mutually orthogonal central projections in A'' such that $\Sigma p_j = 1$ and $\alpha_g''(p_j)p_j = 0$ for each j and $g \neq e$.

We refer to [1] for the definition of a *split face* of a convex set and various properties of these faces in C^*-algebras.

Proposition 1. *Let (A, G, α) be a C^*-dynamical system. Then the following conditions are equivalent*:

(i) *G acts strongly centrally freely*;

(ii) *G is a central shift* [2] *in (A'', G, α''), that is, there is a central projection $p \in A''$ such that $\sum_{g \in G} \alpha_g''(p) = 1$ and $\alpha_g''(p)p = 0$ for $g \neq e$*;

(iii) *S admits a G-wandering split face F such that $S = \overline{\text{co}} \bigcup_{g \in G} \alpha_g'(F)$.*

In the above situation, we call G a *central shift* in (A, G, α) by abuse of language.

Proof. Since the central projections in A'' are in one-one correspondence with the split faces of S (cf. [1]), the equivalence of (ii) and (iii) is evident. We only need to prove (i)$\Rightarrow$(ii). Let $\mathscr{F} = \{p \in A'': p$ is a central projection and $\alpha_g''(p)p = 0 \forall g \neq e\}$ with the usual ordering of projections. Then $\mathscr{F} \neq \emptyset$. Suppose $\mathscr{C}$ is a totally ordered subset of $\mathscr{F}$. Let $q = \sup\{p: p \in \mathscr{C}\}$. Then for $g \neq e$, we have $\alpha_g''(q) = \sup\{\alpha_g''(p): p \in \mathscr{C}\}$ and if $p_1, p_2 \in \mathscr{C}$ with $p_1 \leqq p_2$ say, then $\alpha_g''(p_2)p_1 \leqq \alpha_g''(p_2)p_2 = 0$. It follows that $\alpha_g''(q) = 0$ and $q \in \mathscr{F}$. By Zorn's lemma, there is a maximal element p in $\mathscr{F}$. A routine argument using (i) and the maximality of p shows that $\sum_{g \in G} \alpha_g''(p) = 1$ which completes the proof.

We note that if G is a central shift in (A, G, α) and if A has a G-invariant state, then G must be finite. Also we remark that in a (discrete) W^*-dynamical system (M, G, β), G is a shift if and only if β is a 'dual' action in the sense of

Nakagami (cf. [7; Theorem II. 2.4]). Hence the investigation of shift automorphisms may be of some interest.

A maximality argument similar to the above proof yields the following result.

Proposition 2. *Let (A, G, α) be a C^*-dynamical system. Then there is a G-invariant central projection p in A'' such that G is a central shift in $(Ap, G, \alpha|_{Ap})$ and G is not a central shift in $(Aq, G, \alpha|_{Aq})$ for any G-invariant central projection $q \leqq 1-p$.*

The above result can be expressed in terms of split faces, analogous to the Hopf decomposition in ergodic theory, as follows.

Corollary 3. *Let (A, G, α) be a C^*-dynamical system and let S be the state space of A. Then there are G-invariant split faces F_c and F_d of S such that*

(i) $S = \mathrm{co}(F_c \cup F_d)$ *where F_d is the complementary face of F_c;*

(ii) *F_c contains no G-wandering split face of S;*

(iii) *$F_d = \emptyset$ or $F_d = W^*$ for some G-wandering split face W of S.*

Example 1. Any discrete group G acts as a central shift in some C^*-dynamical system. In fact, let $C_0(G)$ be the C^*-algebra of continuous functions on G vanishing at infinity and let $\tau\colon \mathrm{G} \to \mathrm{Aut}(\mathrm{C_0(G)})$ be the translation

$$(\tau_s f)(t) = f(s^{-1}t)$$

where $f \in C_0(G)$ and $s, t \in G$. Then G is a central shift in $(C_0(G), G, \tau)$.

We note that if G is a central shift in (A, G, α), then G *acts centrally freely* in the sense of Zeller-Meier [11], that is, for $g \neq \mathrm{e}$, α_g'' acts freely on the centre A'' as in [6]. The converse is true if G is finite (cf. [3]) and false in general if G is infinite (cf. Example 2).

A C^*-algebra A is *scattered* [3] if if A'' is the direct sum of type I factors. If A is a separable commutative scattered C^*-algebra, then $A'' = l_\infty$ and obviously there is a natural central shift acting on A''. On the other hand, there is no nontrivial central shift acting on the compact operators $K(H)$.

Remark. Note that if G acts centrally freely in (A, G, α) where A is a scattered C^*-algebra, then G is a central shift. Ideed, if A'' is the full operator algebra $B(H)$, then trivially $G=\{e\}$. Otherwise there is a family $\{p_j\}$ of nontrivial minimal central projections in A'' such that $\sum p_j = 1$. Then for each j and $g \neq e$, we have either $\alpha_g''(p_j) = p_j$ or $\alpha_g''(p_j)p_j = 0$ by minimality. Since α_g'' is free, we must have $\alpha_g''(p_j)p_j = 0$. So G is a central shift. Further, if G acts on a nonscattered C^*-algebra B with action β, then the C^*-tensor product $A \otimes B$ is not scattered by [3; Proposition 1] and G is a central shift in $(A\otimes B, G, \alpha\otimes\beta)$ where $(A\otimes B)''$ is isomorphic to the W^*-tensor product $A''\bar{\otimes}B''$ since A is scattered (cf. [5]). So there exist central shifts on nonscattered C^*-algebras.

3. Crossed products

Recall that the crossed product $A\underset{\alpha}{\times}G$ of (A, G, α), for discrete G, is the enveloping C^*-algebra of the $*$-algebra $k(G, A)$ of A-valued functions on G with finite support, where the $*$-algebraic structure of $k(G, A)$ is defined as follows:

$$(fh)(s)=\sum_{t\in G}f(t)\alpha_t(h(t^{-1}s))$$
$$f^*(s)=\alpha_s(f(s^{-1}))^*$$

where $f, g\in k(G, A)$ and $s\in G$ (cf. [8]). We shall identify A as a subalgebra of $A\underset{\alpha}{\times}G$ via the embedding $a\mapsto f_a\in k(G, A)$ where $f_a(e)=a$ and $f_a(t)=0$ for $t\neq e$.

Proposition 4. *If G acts centrally freely in (A, G, α), then $A\underset{\alpha}{\times}G$ is scattered if (and only if) A is scattered.*

This follows immediately from [3; Theorem 5] since we have remarked before that if G acts centrally freely on a scattered C^*-algebra, then G is in fact a central shift.

Given a C^*-dynamical system (A, G, α) and the universal representation $\pi_u\colon A\to B(H_u)$ of A, we define the covariant representations $\tilde{\pi}_u\colon A\to B(l_2(G, H_u))$ and $\lambda\colon G\to B(l_2(G, H_u))$ by

$$(\tilde{\pi}_u(a)\xi)(t)=\pi_u(\alpha_{t^{-1}}(a))\xi(t)$$
$$(\lambda_s\xi)(t)=\xi(s^{-1}t)$$

for $a\in A$, s, $t\in G$ and $\xi\in l_2(G, H_u)$ (cf. [8]). Further we define the induced regular representation $\tilde{\pi}_u\underset{\alpha}{\times}\lambda\colon A\underset{\alpha}{\times}G\to B(l_2(G, \mathrm{H}_u))$ by

$$(3.1)\qquad (((\tilde{\pi}_u\times\lambda)f)\xi)=\sum_{s\in G}\pi_u(\alpha_{t^{-1}}(f(s)))\xi(s^{-1}t)\quad (f\in k(G, A))\ .$$

If we identify A as a subalgebra of $A\underset{\alpha}{\times}G$ via $a\mapsto f_a$ as aforementioned, then we have, for $a\in A$,

$$(3.2)\qquad (((\tilde{\pi}_u\times\lambda)\mathrm{a})\xi)(t)=\sum_{s\in G}\pi_u(\alpha_{t^{-1}}(f_a(s)))\xi(s^{-1}t)=\pi_u(\alpha_{t^{-1}}(a))\xi(t)\ .$$

We now defie the W^*-crossed product $A''\underset{\alpha''}{\times}G$ of the W^*-dynamical system (A'', G, α'') where A'' is the σ-weak closure of $\pi_u(A)\subset B(H_u)$ and $\alpha''\colon G\to\mathrm{Aut}(A'')$ is the bitransposed action such that $\alpha_g''(\pi_u(a))=\pi_u(\alpha_g(a))$ for $a\in A$ and $g\in G$. As usual, we define the faithful normal *-representation $\iota\colon A''\to B(l_2(G, H_u))$ by

$$(\iota(m)\xi)(t)=\alpha''_{t^{-1}}(m)\xi(t)$$

for $m\in A''$, $\xi\in l_2(G, H_u)$ and $t\in G$. The *W*-crossed product* $A''\times_{\alpha''} G$ is the von Neumann algebra in $B(l_2(G, H_u))$ generated by $\iota(A'')\cup\lambda(G)$ and it is not difficult to see from (3.1) and (3.2) above that $A''\times_{\alpha''} G$ is just the σ-weak closure of $(\tilde{\pi}_u\times\lambda)(A\times_\alpha G)$ with $(\tilde{\pi}_u\times\lambda)(a)=\iota(\pi_u(a))$ for $a\in A$.

Now let $\pi\colon A\times_\alpha G\to B(H)$ be the universal representation of $A\times_\alpha G$ so that $(A\times_\alpha G)''$ is the σ-weak closure of $\pi(A\times_\alpha G)$ in $B(H)$. The by [8; 3.7.9] and as in [3], there is a $*$-isomorphism $\tilde{\rho}$ of A'' into $(A\times_\alpha G)''$ such that $\tilde{\rho}(\pi_u(a))=\pi(a)$ for each $a\in A$. Let $\tau\colon (A\times_\alpha G)''\to B(l_2(G, H_u))$ be the σ-weakly continuous extension of $\tilde{\pi}_u\times\lambda\colon A\times_\alpha G\to B(l_2(G, H_u))$ so that $\tau((A\times_\alpha G)'')$ is the σ-weak closure of $(\tilde{\pi}_u\times\lambda)(A\times_\alpha G)$ in $B(l_2(G, H_u))$.

Lemma 5. *We have the the following commutative diagram*

$$\begin{tikzcd}
A \arrow[r, phantom, "\subset"] \arrow[dr, "\pi_u"'] & A\times_\alpha G \arrow[rr, "\tilde{\pi}_u\times\lambda"] \arrow[dr, "\pi"] & & B(l_2(G, H_u)) \\
& A'' \arrow[r, "\tilde{\rho}"] & (A\times_\alpha G)'' \arrow[ur, "\tau"'] &
\end{tikzcd}$$

in which $\tau(\tilde{\rho}(\mathrm{m}))=\iota(m)$ *for each* $m\in A''$.

Proof. *Since* $\pi_u(A)$ is σ-weakly dense in A'' and since τ, $\tilde{\rho}$ and ι are σ-weakly continuous, it suffices to show that $\tau(\tilde{\rho}(\pi_u(a)))=\iota(\pi_u(a))$for $a\in A$. Indeed, we have $\tau(\tilde{\rho}(\pi_u(a)))=\tau(\pi(a))=(\tilde{\pi}_u\times\lambda)(a)=\iota(\pi_u(a))$ for $a\in A$.

Theorem 6. *If G is a central shift in (A, G, α), then $(A\times_\alpha G)''$ is $*$-isomorphic to $A''\times_{\alpha''} G$.*

Proof. We need only prove that the σ-weakly continuous $*$-epimorphism $\tau\colon (A\times_\alpha G)''\to A''\times_{\alpha''} G$ in Lemma 5 is faithful. Let $z\in(A\times_\alpha G)''$ be a central projection such that $\ker\tau=(A\times_\alpha G)''z$. By the 'relative commutant property' of A'' proved in [3; Proposition 4], we have $z=\tilde{\rho}(m)$ for some m in A''. Now $\iota(m)=\tau(\tilde{\rho}(m))=\tau(z)=0$ implies that $m=0$ as ι is faithful. Therefore $z=0$ and τ is faithful.

Remark. The above conclusion is not true in general as the faithfulness of τ in Lemma 5 would imply that $A\times_\alpha G$ is isomorphic to the reduced C^*-crossed product $(\tilde{\pi}_u\times\lambda)(A\times_\alpha G)$.

We have the following immediate consequences.

Corollary 7. *If G is a central shift in (A, G, α), then $(A \underset{\alpha}{\times} G)''$ is *-isomorphic to $M \otimes B(l_2(\alpha''(G)))$ where $M=\{m \in A'' : \alpha_t''(m)=m \forall t \in G\}$ is the fixed-point algebra under G.*

Proof. By [2; Theorem 2], $A'' \underset{\alpha''}{\times} G$ is isomorphic to $M \otimes B(l_2(\alpha''(G)))$.

Theorem 8. *If G is a central shift in (A, G, α), then $A \underset{\alpha}{\times} G$ is a type* I *C^*-algebra if (and only if) A is a type* I *C^*-lagebra.*

Proof. Since A'' is a type I von *N*eumann algebra, by [2; Theorem 7], the W^*crossed product $A'' \underset{\alpha''}{\times} G$ is a type I von *N*eumann algebra and so is $(A \underset{\alpha}{\times} G)''$. Hence $A \underset{\alpha}{\times} G$ is a type I C^*-algebra.

In the above theorem, if both A and G are separable, then the type I-ness of $A \underset{\alpha}{\times} G$ that implies the action of G induces a *smooth* action on $\hat{A}$ as in [10; Theorem 8.1]. It follows that separable central shifts on separable type I algebras induce smooth actions. It would be of interest to obtain a direct proof of this fact which would then give an alternative proof of Theorem 8 in the separable case since it has been shown in [10, 11] that free and smooth action of a separable group on a separable type I algebra give rise to type I crossed product.

Rieffel [9] has shown that if G is finite in any C^*-dynamical system (A, G, α) in which A is a type I C^*-algebra, then $A \underset{\alpha}{\times} G$ is also a type I C^*-algebra.

In contrast to Proposition 4, the condition in Theorem 8 can not be weakened to a centrally free action as the following example shows.

Example 2. Let $T=\{e^{it} : t \in \boldsymbol{R}\}$ be the unit circle and $C(T)$ the C^*-algebra of continuous functions on T. Consider the rational rotations $\alpha : Q \to \mathrm{Aut}(C(T))$ (modulo 2π):

$$(\alpha_r f)(e^{it}) = f(e^{i(t+r)}) \qquad (f \in C(T)) .$$

Then Q acts on T naturally and it is evident that the stabilizer of each point in T reduces to $\{0\}$. So Q acts centrally freely in $(C(T), Q, \alpha)$ [11; 1.18]. Also it is easily seen that the orbit of Q at each point in T is dense in T. So $C(T) \underset{\alpha}{\times} Q$ is not a type I C^*-algebra by [11; Théorème 7.7].

Finally we remark that if G is finite or compact in a C^*-dynamical system (A, G, α), then by a result of Rieffel [9], $A \underset{\alpha}{\times} G$ is isomorphic to the fixed-point algebra $(A \otimes K(L_2(G)))^{\alpha \otimes \gamma}$ where γ is the action of G on the compact operators $K(L_2(G))$ by conjugation by the action of right translation on $L_2(G)$. Therefore if A is scattered (or type I), then so is $A \underset{\alpha}{\times} G$ since the scatteredness is preserved under C^*-tensor product and is hereditary [3]. This answers a question in [3; Remark 3].

Acknowledgements

This paper was prepared during visits to the C. N. R. S., Centre de Physique Théorique, Marseille and the University of Hong Kong. I wish to thank Professor *B*. Iochum of the former institute and Professor A. J. Ellis of the latter for their hospitality and useful discussions. I have also benefited from illuminating conversations with Professors H. Behncke, H.-J. D. Testard and G. Zeller-Meier.

References

[1] L. Asimow and A. J. Ellis, *Convexity theory and its applications in Functional analysis*, Academic Press, 1980.
[2] M. Choda, *Shift automorphism groups of von Neumann algebras*, Proc. Japan Acad., **50** (1974), 470–475.
[3] C. H. Chu, *Crossed products of scattered C*-algebras*, J. London Math. Soc., **26** (1982), 317–328.
[4] N. A. Friedman, *Introduction to ergodic theory, van Nostrand*, 1970.
[5] T. Huruya, *The second dual of a tensor product of C*-algebras* III, Sci. Reports, Niigata University, **13** (1976), 25–29.
[6] R. R Kallman, *A generalization of free action*, Duke Math. J., **36** (1969), 781–789.
[7] Y. Nakagami and M. Takesaki, *Duality for crossed products of von Neumann algebras*, Lecture Notes in Maths. **731** (Springer, Berlin 1979).
[8] G. K. Pedersen, *C*-algebras and their automorphism groups*, Academic Press, 1979.
[9] M. Rieffel, *Actions of finite groups on C*-algebras*, Math. Scand., **47** (1980), 157–176.
[10] M. Takesaki, *Covariant representations of C*-algebras and their locally compact automorphism groups*, Acta Math., **119** (1967), 273–303.
[11] G. Zeller-Meier, *Produits croisés d'une C*-algebre par un groupe d'automorphismes*, J. Math. Pures Appl., **47** (1968), 101–239.

Goldsmiths' College
University of London
London SE14.