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A NOTE ON NEARLY PARACOMPACTNESS

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Introduction

In this note we give some new characterisations on nearly paracompactness. Let X denote the topological space with no separation axiom and let X_s be its semi regularisation i.e. X with the topology generated by regularly open subsets of X. For any subset A of X, $\overline{A}=\mathrm{cl}\, A$ ($\mathring{A}=\mathrm{int}\, A$) and $\mathrm{cl}_s A$ ($\mathrm{int}_s A$) denote the closures (interiors) of A in X and X_s respectively. A is called regularly open in X iff $A=\mathrm{int}\,\overline{A}$. It is known that X is nearly paracompact i.e. every regularly open cover of X has an open locally finite refinement [8] iff X_s is paracompact. For the definitions of the concepts of star refinement, cushioned refinement and σ-cushioned refinement see [5] and for the barycentric refinement see [3]. Almost regular spaces are defined in [6].

Results

Lemma 1. *A T_1-space X is paracompact iff every open covering has an open cushioned refinement.*

Proof. Direct consequence of Theorem 1.1 of Micheal [5].

Theorem 1. *X with X_s satisfying T_1 axiom is nearly paracompact iff every regularly open cover of X has an open cushioned refinement.*

Proof. Under the sufficiency hypothesis let $\mathcal{G}=(G_\alpha)_{\alpha\in I}$ be an open cover of X_s. Then for each $\alpha\in I$, $G_\alpha=\bigcup_{\beta\in I_\alpha} G_{\alpha\beta}$ where $G_{\alpha\beta}$ is regularly open in X for all $\beta\in I_\alpha$. Let $\mathcal{U}=(U_\mu)_{\mu\in J}$ be an open cushioned refinement of $\{G_{\alpha\beta}:\alpha\in I,\ \beta\in I_\alpha\}$ in X. Since

$$\mathrm{cl}_s \bigcup_{\mu\in J_0} \mathring{\bar{U}}_\mu=\overline{\bigcup_{\mu\in J_0} \mathring{\bar{U}}_\mu}=\overline{\bigcup_{\mu\in J_0} U_\mu}$$

for all $J_0\subseteqq$, $\{\mathring{\bar{U}}J_\mu:\mu\in J\}$ is an open cushioned refinement of $\mathcal{G}$ in X_s. Lemma 1 yields that X_s is paracompact i.e. X is nearly paracompact. Necessity is clear after Lemma 1.

Corollary 1 (Kovacevic, [4]). *T_2 space X is nearly paracompact iff every reg-*

ularly open cover of X has a regularly open star refinement(1).

Proof. An open star refinement is a cushioned refinement [5] and every T_2 space has T_2 semi regularisation.

Theorem 2. *A space X with X_s satisfying T_1 axiom is nearly paracompact iff every regularly open cover of X has an open σ-cushioned refinement.*

Proof. After the Theorem 1.2. of Micheal [5] it is clear.

Corollary 2. *Almost regular X with X_s satisfying T_1 axiom is nearly paracompact iff every regularly open cover of X has an open σ-locally finite refinement.*

Proof. In an almost regular space any regularly open cover $\mathcal{G}$ has a refinement $\mathcal{U}$ such that $\{\overline{U}: U \in \mathcal{U}\}$ refines $\mathcal{G}$. If $\mathcal{V} = \bigcup_{n=1}^{\infty} \mathcal{V}_n$ is σ-locally finite refinement of $\mathcal{U}$ then $\mathcal{V} = \bigcup_{n=1}^{\infty} \mathcal{V}_n$ is a σ-cushioned refinement of $\mathcal{G}$.

Definition 1. X is called (nearly) point paracompact iff every (regularly) open cover $\mathcal{G}$ of X and point $x \in X$, there exists an open refinement $\mathcal{U}$ of $\mathcal{G}$ such that $\mathcal{U}$ is locally finite at x (See [1]).

Definition 2. X is called (nearly) order paracompact iff every (regularly) open cover of X has an open refinement $\mathcal{U} = (U_\alpha)_{\alpha \in I}$ such that for all $\alpha \in I$ the family $\{U_\beta : \beta < \alpha\}$ is locally finite at each point of $\overline{U}_\alpha$ where I is linarly ordered by $<$ (See [2]).

Definition 3. X is called (nearly) weakly order paracompact iff every (regularly) open cover of X has an open refinement $\mathcal{U} = (U_\alpha)_{\alpha \in I}$ such that for all $\alpha \in I$ the family $\{U_\beta : \beta < \alpha\}$ is locally finite at each point of U_α where I is linearly ordered by $<$ (See [7]).

Lemma 2. *A subset of X is regularly open in X iff it is regularly open in X_s.*

Proof. It is easy to prove that $\overline{U} = \mathrm{cl}_s U$ holds for every open subset U of X. Since $\mathrm{cl}(X \backslash A) = X \backslash \mathrm{int}\, A$ for any subset A of X, one can easily get

$$\mathring{\overline{U}} = X \backslash \overline{X \backslash \overline{U}} = X_s \backslash \mathrm{cl}_s(X_s \backslash \overline{U}) = X_s \backslash (X_s \backslash \mathrm{int}_s \overline{U}) = \mathrm{int}_s \mathrm{cl}_s U .$$

The required necessary and sufficient condition is a consequence of the last equality.

Corollary 3. *X_s is always semi regular.*

Proof. After Lemma 2, regularly open subsets of X_s is a base for X_s.

(1) : Kovacevic proved an equivalent statement: T_2 space X is nearly paracompact iff it is almost full normal i.e. every regularly open covering has a regularly open barycentric refinemnet.

Lemma 3. *X is nearly weakly order paracompact (resp. nearly point paracompact, resp. nearly order paracompact) iff X_s is weakly order paracompact (resp. point paracompact, resp. order paracompact).*

Proof. Only the proof of the first statement will be given. A regularly open cover $\mathcal{U}$ of X_s is a regularly open cover of the nearly weakly order paracompact space X by Lemma 2 and so $\mathcal{U}$ has an order locally finite open refinement $\mathcal{G}=(G_\alpha)_{\alpha\in I}$ in X. For any $\alpha\in I$ let $I(\alpha)=\{\beta\in I:\beta<\alpha\}$. Then for any $x\in X$, there exists an open neighborhood W_x of x and a finite subset $I_x(\alpha)$ of $I(\alpha)$ such that $(G_\alpha\cap W_x)\cap G_\beta=\phi$ and consequently $(\mathring{\bar{G}}_\alpha\cap\mathring{\bar{W}}_x)\cap\mathring{\bar{G}}_\beta=\phi$ for all $\beta\in I(\alpha)\backslash I_x(\alpha)$ i.e. the family $\{\mathring{\bar{G}}_\alpha:\alpha\in I\}$ is an order locally finite open refinement of $\mathcal{U}$ in X_s. Hence X_s is nearly weakly order paracompact and so weakly order paracompact since it is semi regular. Conversely if X_s is weakly order paracompact then it is clear that X is nearly weakly order paracompact.

Theorem 3. *X is nearly paracompact iff it is nearly point paracompact and nearly weakly order paracompact.*

Proof. Direct consequence of Lemma 3, the known fact: "X is nearly paracompact iff X_s is paracompact" and the Theorem of Singal and Arya: "X is paracompact iff X is point paracompact and weakly order paracompact" [9].

Remark. One could get Corollary 2 in T_2 spaces as a consequence of Lemma 3 and Theorem 4 since regularity and point paracompactness are equivalent in such spaces (Corollary 1 of Boyte [1]) and since a T_2 space X with the sufficiency hypothesis of Corollary 2 is nearly weakly order paracompact (See the proof of Theorem 5).

Theorem 4. *T_2 space X is nearly paracompact iff it is nearly point paracompact and nearly order paracompact.*

Proof. In regular T_1 spaces order paracompactness and paracompactness are equivalent [2].

Definition 4. Linearly ordered family $\mathcal{U}=(U_\alpha)_{\alpha\in I}$ is called order cushioned in $\mathcal{G}$ if for all $\alpha\in I$ there exists a $G_{U_\alpha}\in\mathcal{G}$ such that for every $J\subseteq I$

$$\mathrm{cl}_{U_\alpha}\bigcup_{\substack{\beta<\alpha\\ \beta\in J}}(U_\alpha\cap U_\beta)\subseteq\bigcup_{\substack{\beta<\alpha\\ \beta\in J}}G_{U_\beta}$$

holds (See [7]). $\mathcal{U}=\bigcup_{n=1}^{\infty}\mathcal{U}_n$ is called σ-order cushioned in $\mathcal{G}$ if for all $n\in N$, linearly ordered $\mathcal{U}_n$ is order cushioned in $\mathcal{G}$.

Theorem 5. *T_2 space X is nearly paracompact iff every regularly open covering has a regularly open σ-order cushioned refinement.*

Proof. Necessity is clear. Since if X is nearly paracompact then every regularly open cover $\mathcal{G}$ of X has a regularly open, order cushioned refinement, for, if $(U_\alpha)_{\alpha\in I}$ is open cushioned refinement of $\mathcal{G}$ by Theorem 1 then

$$\mathrm{cl}_{\mathring{\bar{U}}}\bigcup_{\substack{\beta<\alpha\\ \beta\in J}}(\mathring{\bar{U}}_\alpha\cap\mathring{\bar{U}}_\beta)=\mathrm{cl}_{\mathring{\bar{U}}_\alpha}\Big(\bigcup_{\substack{\beta<\alpha\\ \beta\in J}}\mathring{\bar{U}}_\beta\cap\mathring{\bar{U}}_\alpha\Big)$$
$$=\mathring{\bar{U}}_\alpha\cap\overline{\bigcup_{\substack{\beta<\alpha\\ \beta\in J}}\mathring{\bar{U}}_\beta}\subseteqq\overline{\bigcup_{\substack{\beta<\alpha\\ \beta\in J}}U_\beta}\subseteqq\bigcup_{\substack{\beta<\alpha\\ \beta\in J}}G_{U_\beta}$$

holds by using the Lemma 1 of MacCandless [2] at the third step. Before proving the sufficiency, note that every regular open covering of the space X with the sufficiency hypothesis has a regularly open order cushioned refinement. Now let X be a T_2 space with the sufficiency hypothesis and $x_0\notin K=\bar{K}$. Then for every point $x\in K$, there exists disjoint neighborhoods $U_{x_0}(x)$ and V_x of x_0 and x respectively so $\mathring{\bar{U}}_{x_0}(x)\cap\mathring{\bar{V}}_x=\phi$ holds. Let $\mathcal{U}=(U_\alpha)_{\alpha\in I}$ be a regularly open order cushioned refinement of $\mathcal{V}=\{X\backslash K\}\cup\{\mathring{\bar{V}}_x : x\in K\}$. Then $\mathcal{W}=\{W_\alpha=(U_\alpha\backslash\bigcup_{\beta<\alpha}U_\beta) : \alpha\in I\}$ is a cushioned refinement of $\mathcal{V}$. For, if $J\subseteqq I$, $x\in\mathrm{cl}\bigcup_{\alpha\in J}W_\alpha$, G_x be any neighborhood of x and $x\in U_{\alpha_0}\in\mathcal{U}$ then

$$\phi\neq U_{\alpha_0}\cap G_x\cap\bigcup_{\alpha\in J}W_\alpha=U_{\alpha_0}\cap G_x\cap\bigcup_{\substack{\alpha<\alpha_0\\ \alpha\in J}}W_\alpha\subseteqq G_x\cap\bigcup_{\substack{\alpha<\alpha_0\\ \alpha\in J}}W_\alpha$$

which yields

$$x\in U_{\alpha_0}\cap\overline{\bigcup_{\substack{\alpha<\alpha_0\\ \alpha\in J}}W_\alpha}\subseteqq U_{\alpha_0}\cap\overline{\bigcup_{\substack{\alpha<\alpha_0\\ \alpha\in J}}U_\alpha}=\mathrm{cl}_{U_{\alpha_0}}\bigcup_{\substack{\alpha<\alpha_0\\ \alpha\in J}}(U_{\alpha_0}\cap U_\alpha)$$
$$\subseteqq\bigcup_{\substack{\alpha<\alpha_0\\ \alpha\in J}}V_{U_\alpha}\subseteqq\bigcup_{\alpha\in J}V_{U_\alpha}.$$

If $W_0=\cup\{W_\alpha\in\mathcal{W} : V_{W_\alpha}=X\backslash K\}$, then W_0 and $X\backslash\overline{W}_0$ are disjoint open neighborhoods of x_0 and K respectively since for at least one W_α which contains x_0, $V_{W_\alpha}=X\backslash K$, holds otherwise one gets the contradiction

$$x_0\in\overline{\bigcup_{\alpha\in I_0}W_\alpha}\subseteqq\bigcup_{\alpha\in I_0}V_{W_\alpha}=\bigcup_{\alpha\in I_0}\mathring{\bar{V}}_{x_\alpha}\subseteqq X\backslash x_0$$

where $I_0=\{\alpha\in I : x_0\in W_\alpha\}$. Hence X is almost regular i.e. X_s is regular. Without restricted the generality any open cover $\mathcal{G}$ of X_s could be taken regularly open. There exists a regularly open order cushioned refinement $\mathcal{U}$ of $\mathcal{G}$ in X by Lemma 2. If cl_{s,U_α} denotes the closure operation of the relative topology on U_α determined by the space X_s,

$$\mathrm{cl}_{s,U_\alpha}\bigcup_{\substack{\beta<\alpha\\ \beta\in J}}(U_\alpha\cap U_\beta)=U_\alpha\cap\mathrm{cl}_s\bigcup_{\substack{\beta<\alpha\\ \beta\in J}}U_\beta=U_\alpha\cap\overline{\bigcup_{\substack{\beta<\alpha\\ \beta\in J}}U_\beta}$$

$$=\mathrm{cl}_{U_\alpha}\bigcup_{\substack{\beta<\alpha\\ \beta\in J}}(U_\alpha\cap U_\beta)\subseteqq\bigcup_{\substack{\beta<\alpha\\ \beta\in J}}G_{U_\beta}\subseteqq\bigcup_{\beta\in J}G_{U_\beta}$$

yields that $\mathcal{U}$ is an open order cushioned refinement of $\mathcal{G}$ in X_s i.e. X_s is paracompact by Theorem 3 of [7].

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