YOKOHAMA MATHEMATICAL
JOURNAL VOL. 31, 1983

ON DISTRIBUTIVE AND MODULAR χ-LATTICES

By

JUHANI NIEMINEN

(Received January 30, 1982; Revised September 10, 1982)

1. Introduction. In a poset $(P, \geqq)$, which is a lattice, $a \vee b = lub\{a, b\}$ and $a \wedge b = glb\{a, b\}$ for every two elements $a, b \in P$. In fact, lub picks up a single element from the set $ub\{a, b\}$ of upper bounds of a and b, and moreover, this element is among the minimal elements $mub\{a, b\}$ of $ub\{a, b\}$. Similarly, glb picks up a single element from the set $mlb\{a, b\}$ of maximal lower bounds of a and b. The picking up is impossible by means of rules lub and glb, if $lub\{a, b\}$ and $glb\{a, b\}$ do not exist for every pair $a, b \in P$. In this paper we will consider posets, where $mub\{a, b\} \neq \emptyset \neq mlb\{a, b\}$ for every two elements $a, b \in P$, and in particular, we will consider algebras, where $a \vee b$ is a freely chosen single element of $mub\{a, b\}$ as well as $a \wedge b$ is an anologous element from $mlb\{a, b\}$. When $lub\{a, b\}$ exists, then $mub\{a, b\} = lub\{a, b\}$, and similarly, when $glb\{a, b\}$ exists, then $mlb\{a, b\} = glb\{a, b\}$; thus in those cases the only possible choice for $a \vee b$ is $lub\{a, b\}$ and for $a \wedge b$ the element $glb\{a, b\}$. Hence the algebras considered here reduce to a lattice $(P, \vee, \wedge)$ whenever $(P, \geqq)$ is a lattice.

A poset $(P, \geqq)$, where $mub\{a, b\} \neq \emptyset \neq mlb\{a, b\}$ for every two elements $a, b \in P$, is called a χ-poset, where χ is a choice function choosing a single element from $mub\{a, b\}$ as well as from $mlb\{a, b\}$. The chosen element $\chi(mub\{a, b\})$ is denoted by $a \vee b$ and $\chi(mlb\{a, b\})$ by $a \wedge b$. After the choice χ, the elements $a \vee b$ and $a \wedge b$ are fixed, and because $mub\{a, b\} = mub\{b, a\}$, the choice $a \vee b$ is independent of the order of a and b; this holds also for $a \wedge b$. On the other hand, the choice χ is not assumed to be consequential, i.e. although $mub\{a, b\} = mub\{c, d\}$ for some elements $a, b, c, d \in P$, $c \neq a, b$, $a \vee b$ and $c \vee d$ need not be equal; an analogy holds for $a \wedge b$ and $c \wedge d$. Thus the choice χ depends on elements a and b only. Note that if $(P, \geqq)$ is a finite χ-poset, there is a least element $0 \in P$ as well as a greatest element $1 \in P$. The algebra $(P, \vee, \wedge)$ derived from a χ-poset $(P, \geqq)$ by means of the choice function χ is called a χ-lattice.

The purpose of this paper is to present a few properties of distributive and modular χ-lattices. It seems to be so that the translations of a distributive (modular) χ-lattice imitate the translations of the corresponding lattice. Also ideals and congruence relations are considered.

χ-lattices are introduced in [2] and some results of that paper, necessary here, are listed below. Translations are investigated by Szász e.g. in [3] and [4]. As a general reference we have used Grätzer's book [1].

The following result is proved in [2, Thm. 1]: *If* $(P, \geqq)$ *is a* χ*-poset, then the* χ*-lattice* $(P, \vee, \wedge)$ *is an algebra, where* $a\vee b=\chi(mub\{a, b\})$ *and* $a\wedge b=\chi(mlb\{a,b\})$, *and the operations* $\wedge$ *and* $\vee$ *satisfy the following conditions for all* $a,b,c\in P$:

(1) $a\wedge a=a$; (1′) $a\vee a=a$;

(2) $a\wedge b=b\wedge a$; (2′) $a\vee b=b\vee a$;

(3) $(a\wedge b)\wedge((a\vee b)\vee c)=a\wedge b$; (3′) $(a\vee b)\vee((a\wedge b)\wedge c)=a\vee b$;

(4) *if* $a\wedge c=b\wedge c=c\vee(a\wedge b)=c$, *then* $c=a\wedge b$;

(4′) *if* $a\vee c=b\vee c=c\wedge(a\vee b)=c$, *then* $c=a\vee b$.

As easily seen, the absorption laws hold in $(P, \vee, \wedge)$, and the associativity for $\vee$ and for $\wedge$ not; not even the weak associativity of weakly associative lattices does hold. It is also observed that $a\leqq b\Leftrightarrow a\wedge b=a\Leftrightarrow a\vee b=b$. In [2, Thm. 2] it is proved that: *Let* $(P, \vee, \wedge)$ *be a* χ*-lattice. A* χ*-poset* $(P, \geqq)$ *is obtained by putting* $a\wedge b=a\Leftrightarrow a\leqq b$. *Moreover, if* $(P, \geqq)$ *is a* χ*-poset,* $(P, \vee, \wedge)$ *a* χ*-lattice derived form* $(P, \geqq)$, *and* $(P, \preccurlyeq)$*the* χ*-poset derived from* $(P, \vee, \wedge)$, *then* $(P, \geqq)=(P, \preccurlyeq)$. If X denotes a fixed choice on a χ-lattice $(P, \geqq)$, it is observed in [2] that there is a one-to-one correspondence between χ-lattices $(P, \vee, \wedge)$ and pairs $((P, \geqq), X)$.

In [2] it is assumed that if $mub\{a, b\}\neq\emptyset$, then for every $d\in ub\{a, b\}$ there is at least one $c_d\in mub\{a, b\}$ such that $c_d\leqq d$; an analogy holds for $mlb\{a, b\}$. As the referee has pointed out, such assumptions are not necessary for obtaining the results of [2] cited above. In this paper the abovementioned assumptions are not made.

2. Distributivity and modularity. A χ-poset $(P, \geqq)$ is called distributive (modular), if the following conditions (D_1) and (D_2) $((M_1)$ and $(M_2))$ hold for every possible choice χ on $(P, \geqq)$:

(D_1) $(a\vee b)\wedge(a\vee c)\leqq a\vee(b\wedge c)$ for all $a, b, c\in P$;

(D_2) $(a\wedge b)\vee(a\wedge c)\geqq a\wedge(b\vee c)$ for all $a, b, c\in P$;

(M_1) $(a\vee b)\wedge(a\vee c)\leqq a\vee(b\wedge(a\vee c))$ for all a, b ‘$c\in P$;

(M_2) $(a\wedge b)\vee(a\wedge c)\geqq a\wedge(b\vee(a\wedge c))$ for all $a, b, c\in P$.

These conditions are the well known conditions of the distributivity (modularity) in lattices. If we say that a χ-lattice $(P, \vee, \wedge)$ is distributive (modular), it means that the χ-poset $(P, \geqq)$ derived from $(P, \vee, \wedge)$ is distributive (modular), and thus every χ-lattice derived from $(P, \geqq)$ is distributive (modular).

At first we present a connection between the modularity and the distributivity; thereafter we consider, when a distributive (modular) χ-lattice is a lattice.

Lemma 1. *If a* χ*-lattice* $(P, \vee, \wedge)$ *is distributive, then it is also modular.*

Proof. Because $a \leqq a \vee c$, then $a \vee (a \vee c) = a \vee c$, and similarly, $a \wedge c = a \wedge (a \wedge c)$. Thus we can substitute $a \vee c$ by $a \vee (a \vee c)$ in (D_1), and so $(a \vee b) \wedge (a \vee c) = (a \vee b) \wedge (a \vee (a \vee c)) \leqq a \vee (b \wedge (a \vee c))$, whence ($M_1$) holds. The validity of (M_2) is proved analogously.

Lemma 2. *If implication* A (B) *holds for all* $a, b, x, y \in P$ *in a* χ*-lattice* $(P, \vee, \wedge)$ *derived from a* χ*-poset* $(P, \geqq)$, *then* $(P, \geqq)$ *is a join-semilattice (meet-semilattice), where*

(A) *if* $a \leqq x$ *and* $b \leqq y$, *then* $a \vee b \leqq x \vee y$;

(B) *if* $a \leqq x$ *and* $b \leqq y$, *then* $a \wedge b \leqq x \wedge y$.

Proof. Let $a, b \in P$. If $x \in ub\{a, b\}$, then since $a \leqq x$ and $b \leqq x$, it follows from (A) that $a \vee b \leqq x \vee x = x$. Hence, $a \vee b = lub\{a, b\}$, and $(P, \geqq)$ is a join-semilattice. The second assertion is proved dually.

Theorem 1. *Let* $(P, \vee, \wedge)$ *be a modular* χ*-lattice derived from a* χ*-poset* $(P, \geqq)$. *If* (M_1) *is an equality, then* $(P, \geqq)$ *is a meet-semilattice, and if* (M_2) *is an equality, then* $(P, \geqq)$ *is a join-semilattice.*

Proof. We will show that the equality (M_2) implies the condition (A) of Lemma 2, from which the assertion follows. The proof of the second assertion is proved analogously and hence omitted.

Let $x \leqq a$ and $y \leqq b$. Thus $x, y \leqq a \vee b$, and consequently, $x = x \wedge (a \vee b)$ and $y = y \wedge (a \vee b)$. Now $x \vee y = (x \wedge (a \vee b)) \vee (y \wedge (a \vee b)) = (a \vee b) \wedge (x \vee (y \wedge (a \vee b))) = (a \vee b) \wedge (x \vee y)$, whence $x \vee y \leqq a \vee b$.

A distributive χ-poset $(P, \geqq)$, which is not a lattice, is given in Figure 1, and a modular one in Figure 2. Let a choice χ be defined as follows in $(P, \geqq)$

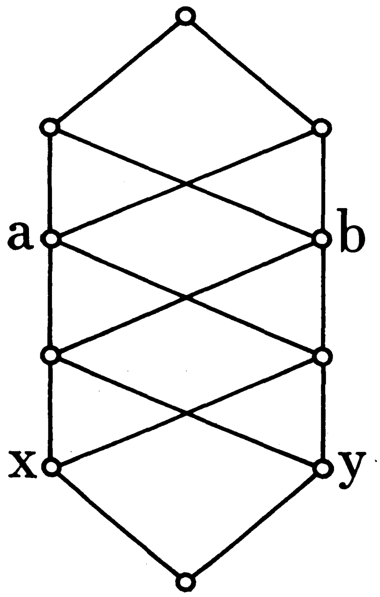

Figure 1

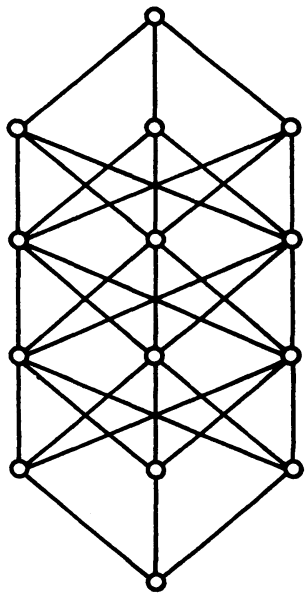

Figure 2

of Figure 1: $\chi(mlb\{a, b\})=x$ and $\chi(mub\{x, y\})=a$. Then $(y\vee a)\wedge(y\vee b)=a\wedge b=x<y\vee(a\wedge b)=y\vee x=a$. Similarly, $(b\wedge x)\vee(b\wedge y)=x\vee y=a>b\wedge(x\vee y)=b\vee a=x$.

The converse inequalities of (D_1), (D_2), (M_1) and (M_2) are characteristic for lattices as proved in the following theorem.

Theorem 2. *Let $(P, \geqq)$ be a χ-poset. If (M_1') holds for a χ-lattice $(P, \vee, \wedge)$ derived from $(P, \geqq)$, then $(P, \geqq)$ is a join-semilattice, and if (M_2') holds, then $(P, \geqq)$ is a meet-semilattice, where*

(M_1') $(a\vee b)\wedge(a\vee c)\geqq a\vee(b\wedge(a\vee c))$ *for all* $a, b, c\in P$;

(M_2') $(a\wedge b)\vee(a\wedge c)\leqq a\wedge(b\vee(a\wedge c))$ *for all* $a, b, c\in P$.

Proof. It suffices to show that (M_1') implies (A). If $a\leqq x$ and $b\leqq y$, then $a\vee(x\vee y)=x\vee y$ and $b\wedge(a\vee(x\vee y))=b$, since $a\leqq x\leqq x\vee y$ and $b\leqq y\leqq x\vee y\leqq a\vee(x\vee y)$. Hence, by (M_1') we have $a\vee b=a\vee(b\wedge(a\vee(x\vee y)))\leqq(a\vee b)\wedge(a\vee(x\vee y))=(a\vee b)\wedge(x\vee y)\leqq x\vee y$. The second assertion is proved analogously.

In the following we give some necessary conditions for distributive and modular χ-posets.

Lemma 3. *If a χ-poset $(P, \geqq)$*

(i) *contains three non-comparable elements a, b, c having a common minimal upper bound and a common maximal lower bound, it is non-distributive;*

(ii) *contains five disjoint elements a, b, c, d, e such that $a<b<d<e$, $a<c<e$, $a\in mlb\{c, d\}\cap mlb\{c, b\}$, and $e\in mub\{c, d\}\cap mub\{c, b\}$, it is non-modular.*

Proof. The system of (i) is the well known diamond of five elements, where $d\in mub\{a, b\}\cap mub\{a, c\}\cap mub\{b, c\}$ and $e\in mlb\{a, b\}\cap mlb\{a, c\}\cap mlb\{b, c\}$. Because (D_1) and (D_2) must hold for every choice χ, by choosing $d=a\vee b=a\vee c=b\vee c$ and $e=a\wedge b=a\wedge c=c\wedge b$, the non-distributivity of $(P, \geqq)$ is now seen.

The system of (ii) is the well known pentagon, by means of which the non-validity of (M_1) and (M_2) is easily shown.

3. Ideals and translations on χ-lattices. A non-empty subset $I\subset P$ of a χ-lattice $H=(P, \vee, \wedge)$ is called an ideal of H, if $i\in I$ and $a\in P\Rightarrow i\wedge a\in I$, and if $i, j\in I\Rightarrow i\vee j\in I$. When $x<i\in I$, then $x=i\wedge x\in I$. A dual ideal of H is defined dually.

The settheoretical intersection of two ideals I and J of H is clearly an ideal of H. Thus the ideals of H constitute a complete lattice $\mathfrak{I}(H)$, where $I\wedge J=I\cap J$ and $I\vee J=\cap\{K|I, J\subset K$ and K is an ideal of $H\}$. Note that $(a]=\{x\,|\,x\leqq a$ in $H\}$ need not be an ideal of H: For example, in the χ-poset $(P, \geqq)$ of Figure 1 we can determine a χ-lattice H such that $x\vee y=b$. Then $x, y\in(a]$ and $x\vee y\notin(a]$.

Theorem 3. *Let $H=(P, \vee, \wedge)$ be a χ-lattice derived from a χ-poset $(P, \geqq)$. If $(a]$ is an ideal of H for every $a \in P$, then $(P, \geqq)$ is a join-semilattice, and if $[a)=\{x \mid x \geqq a$ in $H\}$ is a dual ideal of H for every $a \in P$, then $(P, \geqq)$ is a meet-semilattice.*

Proof. It suffices to show that (A) of Lemma 2 holds. If $a \leqq x$ and $b \leqq y$, then since $a, b \in (x \vee y]$ and since $(x \vee y]$ is an ideal, we have $a \vee b \in (x \vee y]$, whence $a \vee b \leqq x \vee y$. The second assertion is proved analogously.

Following Szász [3; 4] we say that a mapping $\lambda : P \to P$ is a meettranslation on a χ-lattice $(P, \vee, \wedge)=H$ if $\lambda(x \wedge y)=\lambda(x) \wedge y$ for every two elements $x, y \in P$. A jointranslation is defined dually. We will call the meettranslation H briefly a translation.

Lemma 4. *Let λ be a translation on a χ-lattice H. Then*

(i) $x \leqq y \Rightarrow \lambda(x) \leqq \lambda(y)$;

(ii) $\lambda(x) \leqq x$;

(iii) $\lambda(\lambda(x))=\lambda(x)$;

(iv) $\lambda(x \wedge y)=\lambda(x) \wedge \lambda(y)$;

(v) *the fixelements $x=\lambda(x)$ of λ constitute an ideal of H.*

Proof. (i): If $x \leqq y$, then since $y \wedge x=x$, we have $\lambda(x)=\lambda(x \wedge y)=\lambda(y) \wedge x \leqq \lambda(y)$.

(ii): $\lambda(x)=\lambda(x \wedge x)=\lambda(x) \wedge x \Rightarrow \lambda(x) \leqq x$.

(iii): $\lambda(\lambda(x))=\lambda(\lambda(x \wedge x))=\lambda(x \wedge \lambda(x))=\lambda(x) \wedge \lambda(x)=\lambda(x)$.

(iv): $\lambda(x \wedge y)=\lambda(\lambda(x \wedge y))=\lambda(x \wedge \lambda(y))=\lambda(x) \wedge \lambda(y)$.

(v): Let J be the set of fixelements of λ. Assume that $k \leqq j \in J$. Then $k=k \wedge j=k \wedge \lambda(j)=\lambda(k \wedge j)=\lambda(k)$. Moreover, when $i, j \in J$, then $\lambda(i \vee j) \leqq i \vee j$, and on the other hand, $i=\lambda(i) \leqq \lambda(i \vee j)$ and $j=\lambda(j) \leqq \lambda(i \vee j)$. Because $i, j \leqq \lambda(i \vee j) \leqq i \vee j$, we conclude that $i \vee j=\lambda(i \vee j)$. Thus J is an ideal of H.

Theorem 4. *If λ is a translation on a χ-lattice H satisfying* (D_2), *then λ is an endomorphism on H.*

Proof. As seen in Lemma 4: (iv), λ is a meet-endomorphism, and so it remains to show that it is a join-endomorphism on H, too. $\lambda(x) \vee \lambda(y)=(\lambda(x \vee y) \wedge x) \vee(\lambda(x \vee y) \wedge y) \geqq \lambda(x \vee y) \wedge(x \vee y)=\lambda(x \vee y)$. Because $\lambda(x) \vee \lambda(y) \geqq \lambda(x \vee y) \geqq \lambda(x), \lambda(y)$, we can conclude that $\lambda(x \vee y)=\lambda(x) \vee \lambda(y)$, and the theorem follows.

In the case of lattices [3], the property of Thm. 4 characterized the distributivity of lattices. According to the lack of special translations on χ-lattices we were unable to characterize the distributivity of χ-lattices by means of translations. Following Szász [3] we can also prove.

Theorem 5. *If λ is a translation on a χ-lattice H satisfying* (M_2), *then $\lambda(x) \vee \lambda(y)=\lambda(x \vee y)$ for $y \in P$ and $x=\lambda(x)$.*

Proof. $\lambda(x)\vee\lambda(y)=(\lambda(x\vee y)\wedge x)\vee(\lambda(x\vee y)\wedge y)\geqq\lambda(x\vee y)\wedge(y\vee(\lambda(x\vee y)\wedge x))=\lambda(x\vee y)\wedge(y\vee\lambda(x))=\lambda(x\vee y)\wedge(y\vee x)=\lambda(x\vee y)$. As above, we can now conclude that $\lambda(x\vee y)=\lambda(x)\vee\lambda(y)$.

As easily seen, the dual statements of Thms. 4 and 5 can be proved by substituting (D_2) by (D_1) and (M_2) by (M_1). One can also prove that the ideal of fixelements of a translation determines the translation uniquely on a χ-lattice.

Theorem 6. *Let λ be a translation on a χ-lattice H and I_λ the ideal of its fixelements. Then*

(i) $I_\lambda\frown(x]=(y_x]$;

(ii) $I_\lambda\frown(x\wedge y]=(I_\lambda\frown(x])\frown(y]=(I_\lambda\frown(y])\frown(x]$.

Conversely, if I is an ideal of H satisfying (i) *and* (ii), *then I is the set of fixelements of a translation on H.*

Proof. Let I_λ be the ideal of fixelements of a translation λ on H.

(i): Because $\lambda(x)\leqq x$, $\lambda(x)\in I_\lambda\frown(x]$, and because $y\leqq x$ implies $\lambda(y)\leqq\lambda(x)$, $t\leqq\lambda(x)$ for element $t\in I_\lambda\frown(x]$. Hence $I_\lambda\frown(x]=(\lambda(x)]$.

(ii): $I_\lambda\frown(x\wedge y]=(\lambda(x\wedge y)]=(I_\lambda\frown(x])\frown(I_\lambda\frown(y])=(I_\lambda\frown(x])\frown(y]=(I_\lambda\frown(y])\frown(x]$ because of $\lambda(x\wedge y)=\lambda(x)\wedge\lambda(y)$ and the properties of $\frown$.

The converse proof is now obvious.

4. On congruences on a χ-lattice. Let θ be a congruence relation on a χ-lattice H. As usually, the congruences on H constitute a lattice $C(H)$ with ω as the least element ($\langle x, y\rangle\in\omega\Leftrightarrow x=y$) and τ as the greatest element ($\langle x, y\rangle\in\tau$ for every two elements $x, y\in P$). Further, $\langle x, y\rangle\in\theta\wedge\phi\Leftrightarrow\langle x, y\rangle\in\theta$ and $\langle x, y\rangle\in\phi$. Moreover, $\langle x, y\rangle\in\theta\vee\phi\Leftrightarrow$ there is a sequence $x=z_0, z_1, \cdots, z_n=y$ of elements such that $\langle z_{i-1}, z_i\rangle\in\theta$ or $\langle z_{i-1}, z_i\rangle\in\phi$ for every i, $i=1, \cdots, n$.

Before proving the distributivity of the lattice $C(H)$ of all congruences on a χ-lattice H we need a lemma. It is an analogy of [1, Lemma 2.3.8].

Lemma 5. *Let H be a χ-lattice. A reflexive binary relation on H is a congruence on H if and only if the following three properties are satisfied for $x, y, z, t\in P$:*

(i) $\langle x, y\rangle\in\theta\Leftrightarrow\langle x\wedge y, x\vee y\rangle\in\theta$;

(ii) $x\leqq y\leqq z$, $\langle x, y\rangle, \langle y, z\rangle\in\theta\Rightarrow\langle x, z\rangle\in\theta$;

(iii) $x\leqq y$ *and* $\langle x, y\rangle\in\theta\Rightarrow\langle x\wedge t, y\wedge t\rangle, \langle x\vee t, y\vee t\rangle\in\theta$.

Proof. If θ is a congruence on H, then it obviously satisfies the conditions of the lemma. Hence we prove the converse only. At first we prove that if $b, c\in[a, d]=\{x\,|\,a\leqq x\leqq d\}$ and if $\langle a, d\rangle\in\theta$, then $\langle b, c\rangle\in\theta$. According to (iii) we obtain, $\langle b, d\rangle, \langle a, b\rangle\in\theta$. By using (iii) again we obtain $\langle b\wedge c, c\rangle, \langle c, c\vee b\rangle\in\theta$. Because $b\wedge c\leqq c\leqq b\vee c$, (ii) implies now $\langle b\wedge c, b\vee c\rangle\in\theta$, and by (i), $\langle b, c\rangle\in\theta$, According to (i) θ is symmetric. To prove the transitivity of θ, we assume that $\langle x, y\rangle, \langle y, z\rangle \in \theta$. Then by (i), $\langle x\wedge y, x\vee y\rangle \in \theta$, and by (iii), $\langle (y\vee z)\vee (x\wedge y), (x\vee y)\vee (y\vee z)\rangle = \langle y\vee z, (x\vee y)\vee (y\vee z)\rangle \in \theta$. Similarly, $\langle y\wedge z, (x\wedge y)\wedge (y\wedge z)\rangle \in \theta$. Because $\langle (x\wedge y)\wedge (y\wedge z), y\wedge z\rangle, \langle y\wedge z, y\vee z\rangle, \langle y\vee z, (x\vee y)\vee (y\vee z)\rangle \in \theta$ and $(x\wedge y)\wedge (y\wedge z) \leqq y\wedge z \leqq y\vee z \leqq (x\vee y)\vee (y\vee z)$, we obtain by applying (ii) twice that $\langle (x\wedge y)\wedge (y\wedge z), (x\vee y)\vee (y\vee z)\rangle \in \theta$. Because $x, z\in [(x\wedge y)\wedge (y\wedge z), (x\vee y)\vee (y\vee z)]$, the proof of the preceding paragraph implies that $\langle x, z\rangle \in \theta$.

Next we prove the assertion: if $\langle x, y\rangle \in \theta$, then $\langle x\vee t, y\vee t\rangle \in \theta$. Because $x, y\in [x\wedge y, x\vee y]$, (i) and the proof of the first paragraph imply that $\langle x, x\vee y\rangle$, $\langle y, x\vee y\rangle \in \theta$. Now, according to (iii), $\langle x\vee t, (x\vee y)\vee t\rangle, \langle y\vee t, (x\vee y)\vee t\rangle \in \theta$, and by applying the transitivity proved above, we obtain $\langle x\vee t, y\vee t\rangle \in \theta$. Now we are able to prove the substitution property of θ for $\vee$: Let $\langle x_0, y_0\rangle, \langle x_1, y_1\rangle \in \theta$. Then $\langle x_0\vee x_1, x_0\vee y_1\rangle, \langle x_0\vee y_1, y_0\vee y_1\rangle \in \theta$, and according to the transitivity, $\langle x_0\vee x_1, y_0\vee y_1\rangle \in \theta$. The substitution property for $\wedge$ is proved similarly. This completes the proof.

Theorem 7. *The lattice $C(H)$ of all congruences on a χ-lattice H is distributive.*

Proof. Let $\phi, \theta, \psi \in C(H)$. In order to show the distributivity of $C(H)$ it is sufficient to prove that $\phi\wedge(\theta\vee\psi) \leqq (\phi\wedge\theta)\vee(\phi\wedge\psi)$. Thus we assume that $\langle a, b\rangle \in \phi\wedge(\theta\vee\psi)$ and show that $\langle a, b\rangle \in (\phi\wedge\theta)\vee(\phi\wedge\psi)$. According to Lemma 5 we may assume further that $a\leqq b$. The relation $\langle a, b\rangle \in \phi\wedge(\theta\vee\psi)$ implies $\langle a, b\rangle \in \phi$ and $\langle a, b\rangle \in \theta\vee\psi$, and the latter relation implies the existence of a sequence $a=z_0, z_1, \cdots, z_{n-1}, z_n=b$ of elements of H such that $\langle z_i, z_{i+1}\rangle \in \theta$ or $\langle z_i, z_{i+1}\rangle \in \psi$, $0\leqq i<n$. Because θ and ψ are congruences on H, we obtain a new sequence $a=(a\vee a)\wedge b=z_0, (z_1\vee a)\wedge b, \cdots, (z_{n-1}\vee a)\wedge b, (z_n\vee a)\wedge b=(b\vee a)\wedge b=b$ of elements of H such that $\langle (z_i\vee a)\wedge b, (z_{i+1}\vee a)\wedge b\rangle$ belongs to θ or to ψ, $0\leqq i<n$. Now $a\leqq z_i\vee a$, whence $a\wedge(z_i\vee a)=a$, and thus we obtain from $\langle a, b\rangle \in \phi$ the relation $\langle a\wedge(z_i\vee a), (z_i\vee a)\wedge b\rangle = \langle a, (z_i\vee a)\wedge b\rangle \in \phi$ for every i, $i=0, \cdots, n$. According to the transitivity of ϕ we obtain further that $\langle (z_i\vee a)\wedge b, (z_{i+1}\vee a)\wedge b\rangle \in \phi$ for $0\leqq i<n$. But then $\langle (z_i\vee a)\wedge b, (z_{i+1}\vee a)\wedge b\rangle$ belongs to $\phi\wedge\theta$ or to $\phi\wedge\psi$ for $0\leqq i<n$, where $(z_0\vee a)\wedge b=a$ and $(z_n\vee a)\wedge b=b$. Accordingly, $\langle a, b\rangle \in ((\phi\wedge\theta)\vee(\phi\wedge\psi)$, and the theorem follows.

Acknowledgements. The author likes to express his sincere gratitude to the referee for his many kind comments and suggestions for correcting and improving this paper.

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Department of Mathematics
Faculty of Technology
University of Oulu
90570 Oulu 57
Finland