YOKOHAMA MATHEMATICAL
JOURNAL VOL. 28, 1980

THE TEMPERATURE STATE SPACE OF A C^*-DYNAMICAL SYSTEM, I

By

OLA BRATTELI*, GEORGE A. ELLIOTT** and AKITAKA KISHIMOTO***

(Received Aug. 7, 1980)

ABSTRACT Let $(K_\beta)_{\beta\in\mathbf{R}}$ be a family of subsimplexes of a compact, convex, metrizable set K such that $\{(\beta, \omega);\ \beta\in\mathbf{R}, \omega\in K_\beta\}$ is a closed subset of $\mathbf{R}\times K$. We prove that there exist a simple separable C^*-algebra $\mathscr{A}$ with identity, and a strongly continuous one-parameter group γ of $*$-automorphisms of $\mathscr{A}$, such that the set of (γ, β)–KMS states is affinely isomorphic to K_β for each $\beta\in\mathbf{R}$. Furthermore, $(\mathscr{A}, \gamma)$ may be chosen such that the set of ground states, resp. ceiling states, is isomorphic to an arbitrary face in the state space of an arbitrary simple, unital, separable AF algebra. We finally prove that any metrizable simplex is isomorphic to a face in the state space of a certain simple, unital, separable AF algebra.

I Introduction

In [5] a simple C^*-algebra and a continuous one-parameter automorphism group were constructed such that the set of inverse temperatures β at which there exist equilibrium states (i.e. KMS states, or, for $\beta=\pm\infty$ ground or ceiling states) is an arbitrary given closed subset of $[-\infty, \infty]=\mathbf{R}\cup\{\pm\infty\}$. The construction was such that the equilibrium state at β is unique for all β in the closed subset. In [5] a couple of alternative constructions were proposed which allowed certain cases of specified nonuniqueness of the equilibrium states at a fixed temperature, i.e. a specification of the compact, convex set K_β of KMS-states at temperature β. In this paper we will pursue a more systematic investigation in this direction.

We will consider C^*-dynamical systems $(\mathscr{A}, \gamma)$, where $\mathscr{A}$ is a C^*-algebra with identity and $t\in\mathbf{R}\mapsto\gamma_t$ is a strongly continuous one-parameter group of $*$-automorphisms of $\mathscr{A}$. If ω is a state on $\mathscr{A}$ and β is a real number, ω is said to be a (γ, β)-KMS state if

$$\omega(A\gamma_{i\beta}(B))=\omega(BA)$$

for all $A, B\in\mathscr{A}$ such that B is entire analytic for γ. We use the terminology that ω

* Permanent Address: Institute of Mathematics, NTH, Trondheim 7034, Norway.
** Permanent Address: Mathematics Institute, Universitetsparken 5, 2100 Cophenagen Ø, Denmark.
*** Permanent Address: Department of Mathematics, Yokohama City University, Yokohama, 236 Japan.

is a $(\gamma, +\infty)$-KMS state if ω is a ground state, i.e.

$$-i\omega(A^*\delta(A))\geq 0$$

for all A in the domain of the generator δ of γ. The notion of $(\gamma, -\infty)$-KMS state or ceiling state is defined by the converse inequality, [7].

The set K_β of (γ, β)-KMS states is a compact, convex subset of the state space $E_{\mathscr{A}}$ of $\mathscr{A}$ for any $\beta \in \mathbf{R} \cup \{\pm\infty\}$. K_β is a simplex if $\beta \in \mathbf{R}$ and a face in $E_{\mathscr{A}}$ for $\beta = \pm\infty$. $K_{+\infty}$ is not in general a simplex; a certain condition of asymptotic abelianness is necessary and sufficient for this, [3], [4], [7].

The structure of the map

$$\beta \in [-\infty, \infty] \mapsto K_\beta$$

has been analyzed in detail in several models, notably in quantum lattice spin systems, [7]. In this paper we will attack the converse problem, i.e. we will assume that the map is given and then construct a C^*-dynamical system where the equilibrium states define the given map. It is then necessary that K_β is a simplex for each $\beta \in \mathbf{R}$. It is also necessary to impose some continuity on the "field" $\beta \mapsto K_\beta$, and we will do that by requiring that each K_β be contained in a common compact convex set K, in such a way that if (ω_α) is a convergent net in K such that $\omega_\alpha \in K_{\beta_\alpha}$ and β_α converges to some $\beta \in \mathbf{R}$ then $\lim_\alpha \omega_\alpha \in K_\beta$. If $(\mathscr{A}, \gamma)$ is a C^*-dynamical system and $K = E_{\mathscr{A}}$, it is well known that the family of K_β's has this property (see for example [7], Proposition 5.3.23). The only assumption we must impose on the field $\beta \mapsto K_\beta$ which is not fulfilled for general C^*-dynamical systems is a separability assumption; we assume that K is metrizable. We do not know if this assumption can be avoided; we make it because we do not know whether the general groups considered in [10] are dimension groups of (not necessarily separable) AF algebras or not. Even when this is the case we do not know whether automorphisms lift.

The construction is such that the sets of ground and ceiling states can be identified with the state spaces of two arbitrary simple unital AF algebras $\mathscr{A}^{\pm\infty}$. If $K_\beta = \phi$ for all large positive, resp. negative β, the construction can be made such that there are no ground states, resp. ceiling states.

By perturbing the dynamics γ by an inner perturbation we can obtain a C^*-dynamical system $(\mathscr{A}, \gamma^P)$ such that the set of (γ^P, β)-KMS states is still affinely isomorphic to K_β for $\beta \in \mathbf{R}$, but the set of ground states, resp. ceiling states is affinely isomorphic with an arbitrarily given face in the state space of $\mathscr{A}^{+\infty}$, resp. $\mathscr{A}^{-\infty}$. We will show in Section 9 that any metrizable simplex is affinely isomorphic to such a face.

The construction in this paper is a modification of the construction in [5], incorporating and extending the result of [15] (for which we give a new proof). The main theorem is stated in Section 2 and proved in Sections 3–6. The complements on ground and ceiling states are in Sections 7–10.

In Section 2 we also give a necessary and sufficient condition ensuring that a given one-parameter family $(K_\beta)_{\beta\in\mathbf{R}}$ of finite-dimensional simplexes can be embedded in a metrizable compact convex set K in the manner described above.

II KMS-states

Our first main result is the following

Theorem 2.1. *Let K be a compact, convex metrizable set, and let K_β be a closed convex subset of K for each $\beta\in\mathbf{R}$. Assume that*

1. *Each K_β is a simplex.*
2. *If (ω_α) is a convergent net in K such that $\omega_\alpha\in K_{\beta_\alpha}$, and β_α converges to some $\beta\in\mathbf{R}$, it follows that $\lim_\alpha \omega_\alpha\in K_\beta$.*

There exists a C^*-dynamical system $(\mathscr{A},\gamma)$ such that $\mathscr{A}$ is simple with unit, and the set of (γ,β)-KMS states on $\mathscr{A}$ is affinely isomorphic to K_β for each $\beta\in\mathbf{R}$.

Furthermore, $(\mathscr{A},\gamma)$ can be constructed such that $\mathscr{A}$ is separable and amenable and γ is periodic with period 2π.

The proof of Theorem 2.1 goes in two steps.

Step 1. One constructs an approximately finite-dimensional C^*-algebra $\mathscr{B}$, an automorphism α of $\mathscr{B}$ and a projection E in $\mathscr{B}$ such that

1. The set of lower semicontinuous traces ρ on $\mathscr{B}$ such that

$$\rho(E)=1,\quad \rho\circ\alpha=e^{-\beta}\rho$$

is affinely isomorphic to K_β for each $\beta\in\mathbf{R}$.

2. There are no globally α-invariant ideals in $\mathscr{B}$ except for the trivial ones.
3. α transforms each nonzero projection in $\mathscr{B}$ into a non-equivalent projection.

$\mathscr{B}$ will be constructed by constructing its dimension group, in Sections 3–5.

Step 2. Conditions 2 and 3 above imply that the C^*-crossed product $C^*(\mathscr{B},\alpha)$ of $\mathscr{B}$ by α is simple. We let γ be the dual action of α restricted to $\mathscr{A}=EC^*(\mathscr{B},\alpha)E$. Then we establish a one-one correspondence between (γ,β)-KMS states ω and traces ρ on $\mathscr{B}$ such that

$$\rho(E)=1,\quad \rho\circ\alpha=e^{-\beta}\rho\,.$$

This correspondence is given by

$$\omega=\rho\circ\varepsilon$$

where $\varepsilon\colon \mathscr{A}\to E\mathscr{B}E$ is the projection defined by

$$\varepsilon(A)=\frac{1}{2\pi}\int_0^{2\pi}dt\gamma_t(A)\,.$$

This will be done in Section 6.

In this statement of Theorem 2.1 it was assumed that all K_β are contained in a common convex compact subset K in a certain continuous manner. This suggests the problem of deciding when a one-parameter family $\beta\mapsto K_\beta$ of simplexes can be embedded in a metrizable convex compact set K in this manner. In the case that all the K_β's have finite affine dimension we can give a completely satisfactory answer to this question:

Corollary 2.2. *Let $F_0, F_1, F_2,\dots$ be a sequence of subsets of $\mathbf{R}$. Then the following conditions are equivalent.*

1. *The sets $F_1, F_2,\dots$ are all F_σ-sets, F_0 is closed and*

$$F_0\supseteq F_1\supseteq F_2\supseteq\cdots.$$

2. *There exists a C^*-dynamical system $(\mathscr{A}, \gamma)$ such that $\mathscr{A}$ is separable simple amenable with unit, γ is periodic with period 2π, and F_k is the set of β's such that the set of (γ, β)-KMS states has affine dimension greater than or equal to k for $k=0, 1,\dots$.*
3. *There exists a C^*-dynamical system $(\mathscr{A}, \gamma)$ such that $\mathscr{A}$ is separable with unit, and F_k is the set of β's such that the set of (γ, β)-KMS states has affine dimension greater than or equal to k for $k=0, 1,\dots$.*

Proof. We prove $1\Rightarrow2\Rightarrow3\Rightarrow1$.

$1\Rightarrow2$. Let $\xi_1, \xi_2,\dots$ be an orthonormal basis in an infinite-dimensional separable real Hilbert space $\mathscr{H}$, and let

$$K=\{\xi\in\mathscr{H};\, (\xi_i, \xi)\geq0,\ \sum_{i=1}^{\infty}(\xi_i, \xi)\leq1\}\,.$$

Then K is a simplex, equipped with the weak topology. For $k=1, 2,\dots$, let $F_{k,i}$, $i=1, 2,\dots$ be an increasing sequence of closed sets such that $\bigcup_{i\geq1}F_{k,i}=F_k$ and define a function $f_k\colon \mathbf{R}\to[0, 1]$ by

$$f_k(\beta)=1 \qquad \text{if} \quad \beta\in F_{k,1}$$

$$f_k(\beta)=\frac{1}{i} \qquad \text{if} \quad \beta\in F_{k,i}\backslash F_{k,i-1},\quad i=2, 3,\dots$$

$$f_k(\beta)=0 \qquad \text{if} \quad \beta \notin F_k.$$

Then f_k is an upper semicontinuous function of β, for $k=1, 2,\dots$.

We define subsimplexes K_β of K as follows:

$$K_\beta=\phi \qquad \text{if} \quad \beta \notin F_0$$

$$K_\beta=\{\xi;\ \xi=\sum_{i=1}^{\infty} \lambda_i f_i(\beta)\xi_i \text{ where } \lambda_i \geq 0,\ \sum_i \lambda_i \leq 1\} \qquad \text{if} \quad \beta \in F_0.$$

Then each K_β is a subsimplex of K, K_β has affine dimension k if $\beta \in F_k \backslash F_{k+1}$, $k=0$, 1, 2,..., and K_β is affinely isomorphic to K if $\beta \in \bigcap_k F_k$. Furthermore, the set

$$\{(\beta, \omega);\ \beta \in \mathbf{R},\ \omega \in K_\beta\}$$

is a closed subset of K as a consequence of the upper semicontinuity of each f_i. Hence 2 follows by Theorem 2.1.

2⇒3. Trivial.

3⇒1. Use K_β to denote the set of (γ, β)-KMS states, and define

$$F_k=\{\beta;\ \dim K_\beta \geq k\}$$

for $k=0, 1, 2,\dots$. Clearly F_0 is closed and $F_0 \supseteq F_1 \supseteq \cdots$. It remains to show that each F_k is an F_σ-set.

Let $(A_n)_{n\geq 1}$ be a dense sequence of self-adjoint elements in $\mathscr{A}$, and if $n_1,\dots, n_k$ is a finite sequence of natural numbers, define a subset of $\mathbf{R}^k$ by

$$K(\beta;\ n_1,\dots, n_k)=\{(\omega(A_{n_1}),\dots, \omega(A_{n_k}));\ \omega \in K_\beta\}$$

for all $\beta \in \mathbf{R}$. $K(\beta;\ n_1,\dots, n_k)$ is then a compact convex subset of $\mathbf{R}^k$, and $\dim K_\beta \geq k$ if and only if $\dim K(\beta;\ n_1,\dots, n_k) \geq k$ for some $(n_1,\dots, n_k)$. Define

$$\begin{aligned} &K(\beta;\ n_1,\dots, n_k;\ N)\\ &\quad =\{\alpha\omega_1+\beta\omega_2;\ \omega_1, \omega_2 \in K(\beta;\ n_1,\dots, n_k),\ \alpha, \beta \in \mathbf{R},\ \alpha+\beta=1,\ |\alpha|, |\beta| \leq N\}\\ &\qquad\qquad\qquad\qquad \text{for} \quad N=1, 2,\dots. \end{aligned}$$

Then $\bigcup_N K(\beta;\ n_1,\dots, n_k;\ N)$ is the affine subspace of $\mathbf{R}^k$ spanned by $K(\beta;\ n_1,\dots, n_k)$, and $\dim K_\beta \geq k$ if and only if this space is equal to $\mathbf{R}^k$ for some $n_1,\dots, n_k$. Let $\{x_n\}_{n=1}^{k+1}$ be $k+1$ affinely independent points in $\mathbf{R}^k$. Then the set

$$F(n_1,\dots, n_k;\ N;\ n)=\{\beta;\ K(\beta;\ n_1,\dots, n_k;\ N) \ni x_n\}$$

is a closed subset of $\mathbf{R}$, since

$$\{(\beta, x);\ \beta \in \mathbf{R},\ x \in K(\beta;\ n_1,\dots, n_k;\ N)\}$$

is a closed subset of $\mathbf{R}\times\mathbf{R}^k$. Hence

$$F(n_1,\dots, n_k;\ N)=\bigcap_n F(n_1,\dots, n_k;\ N;\ n)$$

is a closed subset of $\mathbf{R}$. But

$$\bigcup_N F(n_1,\dots, n_k;\ N)\equiv F(n_1,\dots, n_k)$$

is exactly the set of β such that $\dim K(\beta;\ n_1,\dots, n_k)\geq k$, and hence one has

$$F_k=\bigcup_{n_1,\dots,n_k} F(n_1,\dots, n_k).$$

It follows that F_k is a F_σ-set.

III A Riesz group

Following [10], we will say that an ordered abelian group G is a *Riesz group* if G is *unperforated*, i.e.,

If $g\in G$ and $ng\geq 0$ for some $n\in\mathbf{N}$ then $g\geq 0$,

and if G has the *Riesz interpolation property*, i.e.,

If $g_1, g_2, g_3, g_4\in G$ and

$$g_1, g_2\leq g_3, g_4$$

then there exists a $g_5\in G$ such that

$$g_1, g_2\leq g_5\leq g_3, g_4.$$

The dimension group of an AF algebra was defined in [12], see also [11], and it was proved in [10] that a countable ordered group is the dimension group of a separable AF algebra if and only if it is a Riesz group.

We will construct a certain ordered group G_0, and the dimension group G of $\mathscr{B}$ will be an ordered subgroup of G_0. In the definition of G_0 we will tacitly assume that $K_\beta\neq\phi$ for at least one $\beta\in\mathbf{R}$; the case that $K_\beta=\phi$ for all β can be treated by taking the tensor product of two C^*-dynamical systems $(\mathscr{A}_1, \gamma_1)$, $(\mathscr{A}_2, \gamma_2)$, where $(\mathscr{A}_i, \gamma_i)$ has (γ_i, β)-KMS states only for one $\beta=\beta_i$, and $\beta_1\neq\beta_2$.

In order to proceed with the construction we will make a certain replacement of the sets K_β within K. Let ω_0 be a fixed arbitary element in K, and define affine continuous maps Φ_β: $K\to K$ by

$$\Phi_\beta(\omega)=e^{-|\beta|}\omega+(1-e^{-|\beta|})\omega_0.$$

Define

$$K'_\beta=\Phi_\beta(K_\beta).$$

Then K'_β is affinely isomorphic to K_β, and the family $(K'_\beta)_{\beta\in\mathbf{R}}$ still has the properties 1 and 2 in Theorem 2.1. Furthermore, if (ω_α) is a convergent net in K such that $\omega_\alpha \in K_{\beta_\alpha}$ and $\beta_\alpha \to +\infty$, then it follows from the compactness of K that $\omega_\alpha \to \omega_0$. The same statement is true for $-\infty$. Thus, if we define $K'_{\pm\infty} = \{\omega_0\}$, or $K'_{\pm\infty}$ as any simplex in K containing ω_0, or in the case that $K_\beta = \phi$ for all large positive/negative β, $K'_{+\infty} = \phi / K'_{-\infty} = \phi$, we obtain a family $(K'_\beta)_{\beta\in[-\infty,\infty]}$ of subsimplexes of K which has the properties 1, 2 of Theorem 2.1 when $\mathbf{R}$ is replaced by $[-\infty, \infty]$ (here $[-\infty, +\infty] = \mathbf{R} \cup \{\pm\infty\}$ with the obvious topology). For simplicity of notation we drop the prime and denote this new family by (K_β). In summary, the properties of the family (K_β) needed in the sequel are

1. K_β is a simplex for each $\beta \in [-\infty, +\infty]$.
2. If (ω_α) is a convergent net in K such that $\omega_\alpha \in K_{\beta_\alpha}$, and β_α converges to some $\beta \in [-\infty, \infty]$, it follows that $\lim_\alpha \omega_\alpha \in K_\beta$.

Definition 3.1. *Adopt the assumptions above. Let $\tilde{K}$ be the subset of $[-\infty, +\infty] \times K$ defined by*

$$\tilde{K} = \{(\beta, \omega);\ \beta \in [-\infty, \infty],\ \omega \in K_\beta\}.$$

(Condition 2 implies that $\tilde{K}$ is closed.) Let G_0 be the additive group consisting of real continuous functions f on $\langle -\infty, +\infty\rangle \times K$ such that

(a) *$\omega \mapsto f(\beta, \omega)$ is an affine function for each $\beta \in \mathbf{R}$.*

(b) *There exist (necessarily unique) continuous real affine functions f_n^+, f_n^-, $n \in \mathbf{Z}$ and a constant $N = N(f) > 0$ such that only finitely many of f_n^+, f_n^- are non-zero and*

$$f(\beta, \omega) = \sum_n e^{n\beta} f_n^+(\omega) \qquad \text{for} \quad \beta \geq N(f)$$

$$f(\beta, \omega) = \sum_n e^{n\beta} f_n^-(\omega) \qquad \text{for} \quad \beta \leq -N(f).$$

Write $f > 0$ if and only if there exist integers n, m such that

i. *$f(\beta, \omega) > 0$ for $\beta \in \mathbf{R}$ and $(\beta, \omega) \in \tilde{K}$*

ii. *$f_k^+(\omega) = 0$ for $\omega \in K$ and $k \geq n+1$*

iii. *$f_n^+(\omega) > 0$ for $\omega \in K_{+\infty}$*

iv. *$f_k^-(\omega) = 0$ for $\omega \in K$ and $k \leq m-1$*

v. *$f_m^-(\omega) > 0$ for $\omega \in K_{-\infty}$.*

The most important property of G_0 is the following

Lemma 3.2. *G_0 is a Riesz group.*

Proof. It is easily verified that G_0 is an ordered group, i.e.

$$G_{0+}+G_{0+}\subseteq G_{0+}$$

$$G_{0+}-G_{0+}=G_0$$

$$G_{0+}\cap(-G_{0+})=\{0\}.$$

(The second property follows by adding and subtracting an element of the form $f(\beta, \omega)=C(e^{n\beta}+e^{-n\beta})$ to a given element in G_0, with C and n sufficiently large.) Obviously G_0 is unperforated in the sense that if $f\in G_0$ and $nf\geq 0$ for some $n=2$, $3,\ldots$, then $f\geq 0$. Hence it is sufficient to verify that G_0 has the Riesz interpolation property. We show this in the case that $K_{\pm\infty}\neq\phi$. The other cases can be treated with small modifications in the argument, and are actually simpler.

Assume that $a, b\neq c, d$ in G_0. The case that one of a, b is equal to one of c, d is trivial, so we may assume $a, b>c, d$. Let $A(K)$ denote the real continuous affine functions on K.

We will find an $N\in\mathbf{R}$ and a function f on $[N, +\infty]\times K$ of the form

$$f(\beta, \omega)=\sum_k e^{k\beta}f_k^+(\omega)$$

where the sum is finite and $f_k^+\in A(K)$ such that f lies between a, b and c, d in the sense of Definition 3.1 when $\tilde{K}$ is replaced by $\tilde{K}\cap([N, +\infty]\times K)$. Let n be the largest integer such that one of the four elements

$$a_n^+(\omega)-c_n^+(\omega),\quad a_n^+(\omega)-d_n^+(\omega),\quad b_n^+(\omega)-c_n^+(\omega),\quad b_n^+(\omega)-d_n^+(\omega)$$

is nonzero for some $\omega\in K$; by relabeling we may assume the first of these elements is nonzero. We define

$$f_k^+(\omega)=a_k^+(\omega)=b_k^+(\omega)=c_k^+(\omega)=d_k^+(\omega)$$

for all $k\geq n+1$ and all $\omega\in K$. Since $a>c$, Definition 3.1 implies that

$$a_n^+(\omega)>c_n^+(\omega)$$

for all $\omega\in K_{+\infty}$, and all the other three differences are either strictly positive on $K_{+\infty}$ or zero on K. There are then eight possibilities, which can be written as

1)	+	+	+	+
2)	+	+	+	0
3)	+	+	0	+

4)	+	+	0	0
5)	+	0	+	+
6)	+	0	+	0
7)	+	0	0	+
8)	+	0	0	0

where + means that the difference is strictly positive on $K_{+\infty}$ and 0 that it is zero on K. We treat the possibilities successively:

1) Since $K_{+\infty}$ is a simplex it follows that there exists an $h \in A(K_{+\infty})$ such that

$$a_n^+(\omega),\ b_n^+(\omega) > h(\omega) > c_n^+(\omega),\ d_n^+(\omega)$$

for all $\omega \in K_{+\infty}$, [1], [6]. But $A(K_{+\infty}, K)$, the restrictions to $K_{+\infty}$ of the affine continuous functions on K, is uniformly dense in $A(K_{+\infty})$ (see [1], Cor. I.1.5). It follows that there exists an $f_n^+ \in A(K)$ such that

$$a_n^+(\omega),\ b_n^+(\omega) < f_n^+(\omega) < c_n^+(\omega),\ d_n^+(\omega)$$

for all $\omega \in K_{+\infty}$. Defining $f_k^+(\omega)=0$ for $k \leq n-1$, $\omega \in K$, it follows that

$$f(\beta, \omega) = \sum_k e^{k\beta} f_n^+(\omega)$$

has the right properties for a sufficiently large N.

2) In this case one defines

$$f_n^+(\omega) = b_n^+(\omega) = d_n^+(\omega)$$

and then $a > f > c$ independently of the choice of f_k^+ for $k \leq n-1$. One has now two sub-possibilities.

2.1. $b_{n-1}^+(\omega) > d_{n-1}^+(\omega)$ for $\omega \in K_{+\infty}$. In this case one lets f_{n-1}^+ be a function in $A(K)$ such that

$$b_{n-1}^+(\omega) > f_{n-1}^+(\omega) > d_{n-1}^+(\omega)$$

for all $\omega \in K_{+\infty}$, and sets $f_k^+ = 0$ for $k \leq n-2$.

2.2. $b_{n-1}^+(\omega) = d_{n-1}^+(\omega)$ for $\omega \in K$. In this case one defines

$$f_{n-1}^+(\omega) = b_{n-1}^+(\omega) = d_{n-1}^+(\omega)$$

and then proceeds to compare b_{n-2}^+ and d_{n-2}^+. Here the same two possibilities occur as for b_{n-1}^+ and d_{n-1}^+, and one proceeds inductively until one reaches a k such that $b_{n-k}^+ - d_{n-k}^+$ is strictly positive on $K_{+\infty}$. One then chooses $f_{n-k}^+ \in A(K)$ such that

$$b_{n-k}^+(\omega) > f_{n-k}^+(\omega) > d_{n-k}^+(\omega), \quad \omega \in K_{+\infty}$$

and defines $f_{n-k-1}^+ = f_{n-k-2}^+ = \cdots = 0$.

The cases 3) and 5) are equivalent to 2).

4) In this case one defines

$$f_n^+(\omega) = b_n^+(\omega) = c_n^+(\omega) = d_n^+(\omega)$$

and proceeds in a similar fashion as before; for example if $b_{n-1}^+ - c_{n-1}^+$ is strictly positive on $K_{+\infty}$ and $b_{n-1}^+ - d_{n-1}^+$ is zero on K one defines $f_{n-1}^+(\omega) = b_{n-1}^+(\omega) = d_{n-1}^+(\omega)$ and if $b_{n-2}^+(\omega) - d_{n-2}^+(\omega) > 0$ for $\omega \in K_{+\infty}$ one picks $f_{n-2}^+ \in A(K)$ such that $b_{n-2}^+(\omega) > f_{n-2}^+(\omega) > d_{n-2}^+(\omega)$ for $\omega \in K_{+\infty}$ and defines $f_{n-3}^+ = f_{n-4}^+ = \cdots = 0$.

The case 6) is similar to 4) and cases 7) and 8) are self-contradictory and cannot occur. In connection with this proof, note that the class of Riesz groups is closed under the operation of forming lexicographical direct sums, [13], Theorem 3.10.

Analogously, we can construct a function

$$f(\beta, \omega) = \sum_k e^{k\beta} f_k^-(\omega)$$

such that f is squeezed between a, b and c, d when $\tilde{K}$ is replaced by $\tilde{K} \cap ([-\infty, -N] \times K)$ for some $N \in \mathbf{R}$. For notational simplicity we use the same N in the two cases.

Now, by the reasoning used in Case 1) above, for each $\beta \in [-N, N]$ there exists a function $f_\beta \in A(K)$ such that

$$a(\beta, \omega),\ b(\beta, \omega) > f_\beta(\omega) > c(\beta, \omega),\ d(\beta, \omega)$$

for all $\omega \in K_\beta$. But by continuity and compactness there is then a neighbourhood θ_β around β in $\mathbf{R}$ such that

$$a(\beta', \omega),\ b(\beta', \omega) > f_\beta(\omega) > c(\beta', \omega),\ d(\beta', \omega)$$

whenever $(\beta', \omega) \in \tilde{K} \cap (\theta_\beta \times K)$. We may assume that

$$f_N(\omega) = \sum_k e^{kN} f_k^+(\omega) \quad \text{and} \quad f_{-N}(\omega) = \sum_k e^{-kN} f_k^-(\omega).$$

By compactness, there is a finite set $\{\beta_1, \dots, \beta_n\} \subseteq [-N, N]$ such that $[-N, N] \subseteq \bigcup_{k=1}^n \theta_{\beta_k}$. We may assume that $-N, N \in \{\beta_1, \dots, \beta_n\}$, and that $\pm N \in \theta_{\beta_k}$ if and only if $\beta_k = \pm N$.

Let $(f_k)_{k=1}^n$ be a partition of unity subordinate to $(\theta_{\beta_k})_{k=1}^n$, i.e., the f_k's are non-negative continuous real functions on $[-N, N]$ such that f_k vanishes outside θ_{β_k} for $k = 1, \dots, n$ and

$$\sum_{k=1}^{n} f_k(\beta)=1$$

for all $\beta \in [-N, N]$. Define

$$f(\beta, \omega)=\sum_{k=1}^{n} f_k(\beta) f_{\beta_k}(\omega)$$

for $-N \leq \beta \leq N$, $\omega \in K$. It follows that $f(\beta, \omega)$ coincides with the previously defined $f(\beta, \omega)$ for $\beta = \pm N$, and

$$a(\beta, \omega),\ b(\beta, \omega) < f(\beta, \omega) < c(\beta, \omega),\ d(\beta, \omega)$$

whenever $(\beta, \omega) \in \tilde{K} \cap ([-N, N] \times K)$. Thus, piecing together the definitions of f on $\langle -\infty, -N] \times K$, $[-N, N] \times K$, $[N, +\infty\rangle \times K$ we obtain an element $f \in G_0$ such that

$$a,\ b < f < c,\ d.$$

This ends the proof that G_0 is a Riesz group.

IV An automorphism of the Riesz group

We now define an automorphism α of G_0 by

$$(\alpha f)(\beta, \omega) = e^{-\beta} f(\beta, \omega).$$

One verifies easily that α is an order automorphism of G_0.

Lemma 4.1. *If $(\beta, \omega) \in \tilde{K}$ and $\beta \in \mathbf{R}$, define an additive map $\rho(\beta, \omega)$ from G_0 into $\mathbf{R}$ by*

$$\rho(\beta, \omega)(f) = f(\beta, \omega).$$

It follows that $\rho(\beta, \omega)$ is a positive map such that

$$\rho(\beta, \omega) \circ \alpha = e^{-\beta} \rho(\beta, \omega).$$

Conversely, if ρ is an additive positive map from G_0 into $\mathbf{R}$ such that

$$\rho(1)=1$$

and

$$\rho \circ \alpha = e^{-\beta} \rho$$

for some $\beta \in \mathbf{R}$, then there exists an $\omega \in K_\beta$ such that

$$\rho = \rho(\beta, \omega).$$

Proof. The first part of the Lemma is clear. Assume that ρ is an additive positive map such that

$$\rho\circ\alpha = e^{-\beta_0}\rho$$

and $\rho(1)=1$, where $\beta_0\in\mathbf{R}$.

Observation 1. *If $f\in G_0$ is a function such that $f(\beta, \omega)\geq 0$ for all $(\beta, \omega)\in\tilde{K}$ (with $\beta\in\mathbf{R}$), then*

$$\rho(f)\geq 0\,.$$

Proof. Let $n\in\mathbf{N}$ be so large that $f_k^+=f_k^-=0$ whenever $|k|\geq n$ and define $g\in G_0$ by

$$g(\beta, \omega)=e^{n\beta}+e^{-n\beta}\,.$$

Then $f+\frac{1}{k}g>0$ in G_0 for all $k\in\mathbf{N}$ by Definition 3.1, and hence

$$0\leq\rho\Big(f+\frac{1}{k}g\Big)=\rho(f)+\rho\Big(\frac{1}{k}g\Big)$$

by positivity of ρ. But

$$\rho(g)=\rho\Big(k\frac{1}{k}g\Big)=k\rho\Big(\frac{1}{k}g\Big)$$

by additivity, and hence

$$\rho(f)\geq -\frac{1}{k}\rho(g)$$

for all $k\in\mathbf{N}$. It follows that $\rho(f)\geq 0$.

Observation 2. *If $f\in G_0$ is a function such that $f(\beta, \omega)\geq 0$ for all $(\beta, \omega)\in\tilde{K}$ (with $\beta\in\mathbf{R}$), and $f(\beta, \omega)=0$ either for all $(\beta, \omega)\in\tilde{K}$ such that $\beta<\beta_0+\varepsilon$, or for all $(\beta, \omega)\in\tilde{K}$ such that $\beta>\beta_0-\varepsilon$, where $\varepsilon>0$, then*

$$\rho(f)=0\,.$$

Proof. We consider only the case that $f(\beta, \omega)=0$ for all $(\beta, \omega)\in\tilde{K}$ with $\beta<\beta_0+\varepsilon$. We have

$$(\alpha f)(\beta, \omega)=e^{-\beta}f(\beta, \omega)\leq e^{-(\beta_0+\varepsilon)}f(\beta, \omega)$$

and hence

$$e^{-(\beta_0+\varepsilon)}f(\beta, \omega)-(\alpha f)(\beta, \omega)\geq 0$$

for all $(\beta, \omega) \in \tilde{K}$. By Observation 1 one has

$$e^{-(\beta_0+\varepsilon)}\rho(f) \geq \rho(\alpha f) = e^{-\beta_0}\rho(f) \geq 0$$

and hence $\rho(f)=0$.

Observation 3. *If $f \in G_0$ is a function such that $f(\beta, \omega)=0$ for β in a neighbourhood of β_0 and $(\beta, \omega) \in \tilde{K}$, then*

$$\rho(f)=0.$$

Proof. We can find an $\varepsilon>0$ and $f_1, f_2, g_1, g_2 \in G_0$ such that $f_i(\beta, \omega)$, $g_i(\beta, \omega) \geq 0$ for $(\beta, \omega) \in \tilde{K}$, and $f_1(\beta, \omega)=g_1(\beta, \omega)=0$ for $(\beta, \omega) \in \tilde{K}$ with $\beta<\beta_0+\varepsilon$, and $f_2(\beta, \omega)=g_2(\beta, \omega)=0$ for $(\beta, \omega) \in \tilde{K}$ with $\beta>\beta_0-\varepsilon$, such that

$$-g_1(\beta, \omega)-g_2(\beta, \omega) \leq f(\beta, \omega) \leq f_1(\beta, \omega)+f_2(\beta, \omega)$$

for $(\beta, \omega) \in \tilde{K}$. But then by Observation 1

$$-\rho(g_1)-\rho(g_2) \leq \rho(f) \leq \rho(f_1)+\rho(f_2)$$

and as $\rho(g_i)=\rho(f_i)=0$ by Observation 2 it follows that $\rho(f)=0$.

Observation 4. *If $f \in G_0$ is a function such that $f(\beta_0, \omega)=0$ for all $\omega \in K_{\beta_0}$ then*

$$\rho(f)=0.$$

Proof. There exists for all $\varepsilon>0$ a function $g \in G_0$ such that $g(\beta, \omega)=0$ for β in a neighbourhood of β_0 and

$$g(\beta, \omega)-\varepsilon \leq f(\beta, \omega) \leq g(\beta, \omega)+\varepsilon$$

for all $(\beta, \omega) \in \tilde{K}$. Observation 3 and Observation 1 imply that

$$-\varepsilon \leq \rho(f) \leq \varepsilon$$

and hence $\rho(f)=0$.

Observation 5. *There exists a positive linear functional p on $A(K_{\beta_0})$ such that $p(1)=1$ and*

$$\rho(f)=p(f_{\beta_0})$$

where $f_{\beta_0}(\omega)=f(\beta_0, \omega)$ for $\omega \in K_{\beta_0}$.

It follows from Observation 4 that $\rho(f)$ depends only on the function

$$\omega \in K_{\beta_0} \mapsto f(\beta_0, \omega),$$

i.e.

$$\rho(f)=p(f_{\beta_0})$$

where p is a real additive function on $A(K_{\beta_0}, K)$, the space of restrictions to K_{β_0} of real affine continuous functions on K.

Trivially $p(1)=\rho(1)=1$.

We next argue that p is positive. If $f_{\beta_0}(\omega)>0$ for $\omega \in K_{\beta_0}$ it follows that $f(\beta, \omega) >0$ for (β, ω) in a neighbourhood of $\{\beta_0\} \times K_{\beta_0}$ in $[-\infty, \infty] \times K$. It follows from closedness of $\tilde{K}$ and compactness of K that there exists a neighbourhood $\langle \beta_0-\varepsilon, \beta_0+\varepsilon \rangle$ around β_0 such that $\{\beta\} \times K_\beta$ is contained in the previous neighbourhood for all $\beta \in \langle \beta_0-\varepsilon, \beta_0+\varepsilon \rangle$. Let $\chi: \mathbf{R} \to [0, \infty\rangle$ be a continuous function such that χ is supported in $\langle \beta_0-\varepsilon, \beta_0+\varepsilon \rangle$ and $\chi(\beta_0)=1$, and define

$$g(\beta, \omega)=\chi(\beta)f(\beta, \omega).$$

Then $g \in G_0$, $g(\beta, \omega) \geq 0$ for $(\beta, \omega) \in \tilde{K}$ with $\beta \in \mathbf{R}$ and $g_{\beta_0}=f_{\beta_0}$. It follows from Observation 1 that

$$p(f_{\beta_0})=p(g_{\beta_0})=\rho(g) \geq 0.$$

If $f_{\beta_0}(\omega) \geq 0$ for $\omega \in K_{\beta_0}$ the above reasoning applied to $f+\frac{1}{n}$, $n \in \mathbf{N}$, gives

$$p(f_{\beta_0}) \geq -\frac{1}{n}p(1)=-\frac{1}{n}$$

for all $n \in \mathbf{N}$, and hence $p(f_{\beta_0}) \geq 0$, and p is positive.

By additivity it follows that

$$p(\lambda f_{\beta_0})=\lambda p(f_{\beta_0})$$

for all rational numbers λ, and next it follows from positivity that this relation holds for positive f_{β_0} and all real λ, and hence it holds for all f_{β_0} by additivity. Since $A(K_{\beta_0}, K)$ is uniformly dense in $A(K_{\beta_0})$, p extends uniquely to a positive linear functional on $A(K_{\beta_0})$.

We can now end the proof of Lemma 4.1. If p is the positive linear functional of Observation 5, there exists an $\omega \in K_\beta$ such that

$$p(f_{\beta_0})=f_{\beta_0}(\omega)=f(\beta_0, \omega)$$

(see [1]; one can define ω as the barycentre of any measure μ which defines a state extension of p to $C(K_\beta)$; such extensions exist by the Hahn-Banach Theorem). But this means that

$$\rho(f)=f(\beta_0, \omega).$$

V The dimension group

Up to this point we have not used the metrizability of K, but this is important in constructing a countable subgroup G of G_0 which is "dense" in G_0 in a suitable sense. We first select three subgroups $G^{+\infty}$, G^0, $G^{-\infty}$ of the additive group $A(K)$ in the following manner.

$\underline{G^{+\infty}}$

If $K_{+\infty}=\phi$, put $G^{+\infty}=\{0\}$.
If $K_{+\infty}\neq\phi$, define an ordering $\geq$ on $A(K)$:
$f>0$ if and only if $f(\omega)>0$ for all $\omega\in K_{+\infty}$.
$G^{+\infty}$ is taken to be a subgroup of $A(K)$ such that

1. $G^{+\infty}$ is countable
2. $1\in G^{+\infty}$ (this ensures that $G^{+\infty}$ is an ordered group)
3. $G^{+\infty}$ is a Riesz group
4. If $f, g\in A(K)$ and $f>g$, then there exists a $h\in G^{+\infty}$ such that

$$f>h>g.$$

$\underline{G^{-\infty}}$

$G^{-\infty}$ is chosen as $G^{+\infty}$, with $K_{+\infty}$ replaced by $K_{-\infty}$.

$\underline{G^0}$

G^0 is taken to be a subgroup of $A(K)$ such that

1. G^0 is countable
2. $G^{+\infty}\subseteq G^0$, $G^{-\infty}\subseteq G^0$
3. G^0 is uniformly dense in $A(K)$.

Since K is metrizable and compact, $A(K)$ is separable, and therefore groups $G^{+\infty}$, G^0 and $G^{-\infty}$ with the above properties exist.

We define G as the subgroup of G_0 consisting of functions on $\mathbf{R}\times K$ which are finite sums and differences of elements f of the following three types:

1. $f(\beta, \omega)=g(\beta)e^{n\beta}h(\omega)$ where g is defined from two rational numbers p, q, $p<q$ as follows,

$$g(\beta)=\begin{cases} 0 & \text{for } \beta\leq p \\ \dfrac{\beta-p}{q-p} & \text{for } p\leq\beta\leq q \\ 1 & \text{for } q\leq\beta \end{cases},$$

and $n \in \mathbf{Z}$, and $h \in G^{+\infty}$.

2. $f(\beta, \omega)=g(\beta)e^{n\beta}h(\omega)$ where g is a piecewise linear continuous function with compact support which is differentiable except at a finite number of points $\beta_1,\dots, \beta_k$, with β_i, $g(\beta_i)$ rational, $i=1,\dots, k$, and $n \in \mathbf{Z}$, and $h \in G^0$.

3. $f(\beta, \omega)=g(\beta)e^{n\beta}h(\omega)$ where $\beta \mapsto g(-\beta)$ is as in 1, $n \in \mathbf{Z}$ and $h \in G^{-\infty}$.

The following lemma can now be proved by straightforward but tedious arguments using the techniques of section III. We omit the details.

Lemma 5.1. *G is a subgroup of G_0 with the following properties.*

1. *G is countable.*
2. *G is a Riesz group in the ordering inherited from G_0.*
3. *If a, $b \in G_0$, $a<b$ and a_n^+, $b_n^+ \in G^{+\infty}$, a_n^-, $b_n^- \in G^{-\infty}$ for all $n \in \mathbf{Z}$, then there exists a $c \in G$ with $a<c<b$.*
4. *$\alpha(G)\subseteq G$, $\alpha^{-1}(G)\subseteq G$, i.e. α defines an order automorphism of G by restriction.*
5. *$1 \in G$.*
6. *There exists an element $f \in G$ such that $0<f<1$, $0<\alpha^{-1}(f)<1$ and $f_{-1}^{\pm}>0$ if $K_{+\infty}\neq\phi$, $f_0^->0$ if $K_{-\infty}\neq\phi$.*

We next prove a version of Lemma 4.1 in this context.

Lemma 5.2. *If $(\beta, \omega)\in\tilde{K}$ and $\beta \in \mathbf{R}$, define a map $\rho(\beta, \omega)$ from G into $\mathbf{R}$ by*

$$\rho(\beta, \omega)(f)=f(\beta, \omega)$$

for $f \in G$. It follows that ρ is an additive positive map such that

$$\rho(\beta, \omega)\circ\alpha=e^{-\beta}\rho(\beta, \omega).$$

Conversely, if ρ is an additive positive map from G into $\mathbf{R}$ such that

$$\rho(1)=1$$

and

$$\rho\circ\alpha=e^{-\beta}\rho$$

for some $\beta \in \mathbf{R}$, then there exists an $\omega \in K_\beta$ such that

$$\rho=\rho(\beta, \omega).$$

Proof. The first part follows from Lemma 4.1, and the second part also follows once we can show that ρ extends to a map on G_0 with the same properties. If $a \in G_0$, we first show that

$$\inf\{\rho(c);\ c \in G,\ c\geq a\}=\sup\{\rho(d);\ d \in G,\ a\geq d\}.$$

We first choose an n_0 such that $a_k^{\pm}=0$ whenever $|k|\geq n_0$, and then choose a sequence $b_n\in G$ such that $(b_n)_{n_0}^+>0$, $(b_n)_{-n_0}^->0$, $b_n>0$, and $b_n>2b_{n+1}$ for all n. Then

$$a-b_n<a<a+b_n$$

for all n. Since $a\pm b_n$ has larger leading terms than a at $\pm\infty$, one can find elements d_n, $c_n\in G$ such that

$$a-b_n<d_n<a<c_n<a+b_n$$

and then

$$c_n-d_n<2b_n.$$

Hence

$$\rho(c_n)-\rho(d_n)\leq 2\rho(b_n)\leq\frac{2}{2}\rho(b_{n-1})\leq\cdots\leq 2^{2-n}\rho(b_1).$$

Letting $n\to\infty$ we obtain

$$\inf\{\rho(c);\ c\in G,\ c\geq a\}=\sup\{\rho(d);\ d\in G,\ a\geq d\}$$

and we set $\rho(a)$ equal to these numbers. Since α is an order automorphism of G_0 it follows immediately that the extended ρ still has the property

$$\rho\circ\alpha=e^{-\beta}\rho.$$

VI Construction of the dynamical system

Let $\mathscr{B}$ denote the AF algebra, unique up to $*$-isomorphism (see [12]) whose dimension range is the positive part of the dimension group G defined just before Lemma 5.1. By [12], the automorphism α of G is induced by an automorphism α of $\mathscr{B}$. We choose a particular such automorphism in the following manner. Fix a projection $E\in\mathscr{B}$ in the equivalence class corresponding to $1\in G$. Let $f\in G$ be an element with the properties in Lemma 5.1, 6., i.e. $0<f<1$, $0<\alpha^{-1}(f)<1$, $f_{-1}^{\pm}>0$ if $K_{+\infty}\neq\phi$, $f_0^->0$ if $K_{-\infty}\neq\phi$. Since $f<1$ there exists a projection $F\in\mathscr{B}$ corresponding to f such that $F\leq E$ in the usual ordering of self-adjoint operators. Since $\alpha^{-1}(f)<1$ one has $\alpha^{-1}(F)<E$ in the Murray-von Neumann ordering, independently of the choice of α. Hence, by modifying α by an automorphism implemented by a unitary operator in the multiplier algebra of $\mathscr{B}$, we may require that

$$\alpha^{-1}(F)\leq E$$

in the usual ordering of self-adjoint operators. In the sequel we will assume that α has been chosen in this way.

We next analyze the ideal structure in $\mathscr{B}$. Ideals in $\mathscr{B}$ are in one-one correspondence with ideals in G, i.e. subgroups $H \subseteq G$ such that $H = H_+ - H_+$ ($H_+ = G_+ \cap H$), and if $h \in H_+$, $g \in G_+$ and $h > g$ then $g \in H_+$. The ideals in G can be classified by two elements n and m in $\mathbf{Z} \cup \{\pm\infty\}$, and will be denoted by $I_{n,m}$. We define $I_{n,m}$ as the set of $f \in G$ such that

$$\begin{aligned} f_k^- &= 0 \qquad \text{for all} \quad k \leq -n-1 \\ f_k^+ &= 0 \qquad \text{for all} \quad k \geq m+1 . \end{aligned}$$

Alternatively, if $n, m \in \mathbf{Z}$, $I_{n,m}$ may be defined as the set of $f \in G$ such that $-\lambda f_{n,m} < f < \lambda f_{n,m}$ for some $\lambda > 0$. Here $f_{n,m}$ is defined by

$$f_{n,m}(\beta, \omega) = \begin{cases} e^{-n\beta} & \text{for} \quad -\infty < \beta \leq 0 \\ e^{m\beta} & \text{for} \quad 0 \leq \beta < +\infty . \end{cases}$$

In particular $I_{n,m} = \{0\}$ if $n = -\infty$ or $m = -\infty$ and $I_{+\infty,+\infty} = G$. $I_{n,m}$ is increasing in both n and m. The function $n \mapsto I_{n,m}$ is strictly increasing if $m \neq -\infty$ and $K_{-\infty} \neq \phi$; in all other cases $n \mapsto I_{n,m}$ is constant. The function $m \mapsto I_{n,m}$ is strictly increasing if $n \neq -\infty$ and $K_\infty \neq \phi$; in all other cases $m \mapsto I_{n,m}$ is constant. Note that α maps $I_{n,m}$ into $I_{n+1,m-1}$, and hence there are no α-invariant ideals in G and thus no α-invariant ideals in $\mathscr{B}$.

If $f \in G$ and $f \neq 0$, then $\alpha f \neq f$. It follows that α transforms each projection in $\mathscr{B}$ into an inequivalent projection and hence α is properly outer in the sense of [14]. Since $\mathscr{B}$ has no α-invariant ideals it follows from [14] that the C^*-crossed product $C^*(\mathscr{B}, \alpha)$ of $\mathscr{B}$ by α is simple. $\mathscr{B}$ is canonically embedded in $C^*(\mathscr{B}, \alpha)$ and in particular the projection E corresponding to $1 \in G$ is contained in $C^*(\mathscr{B}, \alpha)$. We cut down the crossed product with E, and set

$$\mathscr{A} = EC^*(\mathscr{B}, \alpha)E .$$

Let U be a fixed unitary multiplier of $C^*(\mathscr{B}, \alpha)$ such that

$$\alpha(B) = U B U^*$$

for $B \in \mathscr{B}$, and $C^*(\mathscr{B}, \alpha)$ is the closed linear span of elements of the form AU^n, $A \in \mathscr{B}$, $n \in \mathbf{Z}$. We will denote both the dual automorphism group of α on $C^*(\mathscr{B}, \alpha)$, and its restriction to $\mathscr{A}$, by γ. In fact we will lift γ from $\hat{\mathbf{Z}} = \mathbf{T}$ to $\mathbf{R}$, so γ is defined by

$$\gamma_t(AU^n) = e^{int} AU^n$$

for $A \in \mathscr{B}$, $n \in \mathbf{Z}$, $t \in \mathbf{R}$. Since $\gamma_t(E) = E$, γ defines a strongly continuous one-parameter group of $*$-automorphisms of $\mathscr{A}$.

Lemma 6.1. *If $A \in \mathscr{B}$ and $n \in \mathbf{Z}$, then $AU^n \in \mathscr{A}$ if and only if*

$$EA = A\alpha^n(E) = A .$$

$\mathscr{A}$ is the closed linear span of such elements.

Proof. We have that $AU^n \in \mathscr{A}$ if and only if

$$EAU^n = AU^nE = AU^n$$

i.e.

$$(EA)U^n = A\alpha^n(E)U^n = AU^n$$

i.e.

$$EA = A\alpha^n(E) = A .$$

$C^*(\mathscr{B}, \alpha)$ is the closure of elements of the form $\sum_n B_nU^n$, where $B_n \in \mathscr{B}$ and the sum is finite. Hence $\mathscr{A}$ is the closure of elements of the form

$$\begin{aligned} E(\sum_n B_nU^n)E &= \sum_n EB_n\alpha^n(E)U^n \\ &= \sum_n A_nU^n \end{aligned}$$

where $A_n = EB_n\alpha^n(E)$. But then

$$EA_n = A_n\alpha^n(E) = A_n .$$

Lemma 6.2. *For all $\beta \in \mathbf{R}$ there is an affine isomorphism between K_β and the set of lower semicontinuous traces ρ on $\mathscr{B}$ with the properties*

$$\rho\circ\alpha = e^{-\beta}\rho \quad \textit{and} \quad \rho(E) = 1 .$$

If these traces are equipped with the weak-topology defined by their restrictions to $E\mathscr{B}E$ this isomorphism is a homeomorphism.*

Proof. By [12] there is a one-one correspondence between positive additive functionals ρ on G_+ and lower semicontinuous traces ρ on $\mathscr{B}$, given by

$$\rho([P]) = \rho(P)$$

where P is a projection in $\mathscr{B}$ and $[P]$ its representative in G_+. It follows that there is a one-one correspondence between positive additive functionals ρ on G_+ such that

$$\rho\circ\alpha = e^{-\beta}\rho \quad \text{and} \quad \rho(1) = 1$$

and lower semicontinuous traces ρ on $\mathscr{B}$ such that

$$\rho\circ\alpha=e^{-\beta}\rho \quad \text{and} \quad \rho(E)=1\,.$$

But by Lemma 5.2 there is a one-one correspondence between elements $\omega\in K_\beta$ and additive positive maps ρ on G_+ such that $\rho\circ\alpha=e^{-\beta}\rho$ and $\rho(1)=1$. This correspondence is given by

$$\rho(f)=f(\beta,\,\omega)$$

and as $f(\beta,\,\cdot)\in A(K)$, this correspondence is affine. Since the correspondence between positive functionals and traces clearly is affine, there is an affine one-one correspondence between elements $\omega\in K_\beta$ and traces ρ on $\mathscr{B}$ with $\rho\circ\alpha=e^{-\beta}\rho$ and $\rho(E)=1$, given by

$$\rho(F)=[F](\beta,\,\omega)$$

where $F\in\mathscr{B}$ is a projection, and $[F]$ its representative in G. If one now equips these traces on $\mathscr{B}$ with the weak*-topology defined by their restrictions to $E\mathscr{B}E=\mathscr{B}\cap\mathscr{A}$, one sees that the mapping $\omega\mapsto\rho(F)$ is continuous for projections $F\in E\mathscr{B}E$, and as $E\mathscr{B}E$ is approximately finite-dimensional, it follows that the map $\omega\mapsto\rho$ is continuous. Since K_β is compact this map is a homeomorphism.

Let ε be the canonical projection from $\mathscr{A}$ onto its fixed point algebra $\mathscr{A}^\gamma=\mathscr{A}\cap\mathscr{B}=E\mathscr{B}E$ under γ, i.e.

$$\varepsilon(A)=\frac{1}{2\pi}\int_0^{2\pi}dt\gamma_t(A)\,.$$

Lemma 6.3. *For all $\beta\in\mathbf{R}$ there is an affine isomorphism between the set of* $(\gamma,\,\beta)$-KMS *states ω on $\mathscr{A}$ and the set of lower semicontinuous traces ρ on $\mathscr{B}$ with the properties*

$$\rho\circ\alpha=e^{-\beta}\rho \quad \textit{and} \quad \rho(E)=1\,.$$

This isomorphism is given by

$$\omega=\rho\circ\varepsilon\,.$$

Proof. Assume first that ρ is given, and define ω by

$$\omega=\rho\circ\varepsilon\,.$$

We will show that ω is a γ-KMS state at value β. It is enough to show that for elements $X,\,Y\in\mathscr{A}$ of the form $X=AU^n$, $Y=BU^m$, $A,\,B\in\mathscr{B}$, one has that

$$\omega(Y\gamma_{i\beta}(X))=\omega(XY),$$

[7]. But

$$Y\gamma_{i\beta}(X)=BU^m e^{-n\beta} AU^n=e^{-n\beta} B\alpha^m(A)U^{m+n},$$
$$XY=A\alpha^n(B)U^{n+m},$$

Thus $\omega(Y\gamma_{i\beta}(X))=\omega(XY)=0$ unless $m=-n$ and then

$$\begin{aligned}\omega(Y\gamma_{i\beta}(X))&=e^{-n\beta}\rho(B\alpha^{-n}(A))\\&=\rho(\alpha^n(B)A)\\&=\rho(A\alpha^n(B))\\&=\omega(XY).\end{aligned}$$

Suppose conversely that ω is a γ-KMS state at value β. By γ-invariance ω has the form

$$\omega=\rho\circ\varepsilon$$

where ρ is a state on $\mathscr{A}\cap\mathscr{B}$. The KMS condition implies that ρ is a trace, [7]. If $X=AU$, $Y=BU^{-1}$ are elements in $\mathscr{A}$, i.e.,

$$EA=A\alpha(E)=A$$
$$EB=B\alpha^{-1}(E)=B$$

by Lemma 6.1, then the KMS condition,

$$\omega(Y\gamma_{i\beta}(X))=\omega(XY),$$

implies

$$e^{-\beta}\rho(B\alpha^{-1}(A))=\rho(A\alpha(B)).$$

Putting $B=E\alpha^{-1}(E)$ in this relation gives

$$e^{-\beta}\rho(E\alpha^{-1}(EA))=\rho(A\alpha(E)E)$$

and combining this with the above relations for A we obtain

$$e^{-\beta}\rho(E\alpha^{-1}(A)E)=\rho(EAE).$$

In particular this relation is valid for all $A\in(\mathscr{A}\cap\mathscr{B})\cap\alpha(\mathscr{A}\cap\mathscr{B})$. If $F\in\mathscr{B}$ is the projection defined in the beginning of this section, then

$$F\le E,\quad \alpha^{-1}(F)\le E$$

and hence

$$F \in (\mathscr{A} \cap \mathscr{B}) \cap \alpha(\mathscr{A} \cap \mathscr{B}).$$

It follows that

$$e^{-\beta}\, \rho(E\alpha^{-1}(P)E) = \rho(EPE)$$

for all projections $P \in \mathscr{B}$ such that $P \leq F$.

Since $E\mathscr{B}E$ is a hereditary subalgebra of $\mathscr{B}$, two projections in $E\mathscr{B}E$ are equivalent if and only if they are equivalent in $\mathscr{B}$, and hence by [12], Corollary 5.7, the dimension group of $E\mathscr{B}E$ coincides with the ideal of G generated by $[E] = 1 \in G$, i.e., $I_{0,0}$.

Let ρ also denote the positive additive functional determined by ρ on $I_{0,0}$. Then it follows that $\rho(1) = 1$ and

$$e^{-\beta}\, \rho(\alpha^{-1}(g)) = \rho(g)$$

whenever $g \in G$ is a function such that

$$-nf < g < nf$$

for some $n \in \mathbf{N}$, i.e., whenever $g \in I_{0,-1}$. It follows by a slight extension of Lemma 5.2 that ρ is the restriction of $\rho(\beta, \omega)$ for some $\omega \in K_\beta$. (The proof of 5.2 shows that ρ agrees with some $\rho(\beta, \omega)$ on $I_{-2,-1}$, and it follows that this holds also on $I_{0,-3} = \alpha^2 I_{-2,-1}$ and on $I_{-3,0} = \alpha^{-1} I_{-2,-1}$, and hence on $I_{0,0} = I_{0,-3} + I_{-3,0}$.) Thus the trace ρ on $E\mathscr{B}E$ extends to the trace ρ on $\mathscr{B}$ corresponding to $\omega \in K_\beta$ given by Lemma 6.2, and this ends the proof of the affine correspondence between ω and ρ.

Proof of Theorem 2.1. Lemmas 6.2 and 6.3 immediately imply that K_β is affinely isomorphic with the set of (γ, β)-KMS states on $\mathscr{A}$ for each $\beta \in \mathbf{R}$. $\mathscr{A}$ is separable since $\mathscr{B}$ is, and $\mathscr{A}$ is amenable by [18]. γ is periodic with period 2π by construction.

VII Ground states

We now analyze the ground states (and ceiling states) of the C^*-dynamical system we have constructed. Let $G^{\pm\infty}$ be the subgroups of $A(K)$ introduced in the beginning of Section 5. Then $G^{+\infty}$ is identified with the quotient of the ideal $I_{0,0}$ generated by 1 in G by the ideal $I_{0,-1}$,

$$G^{+\infty} \cong I_{0,0}/I_{0,-1},$$

and correspondingly

$$G^{-\infty} \cong I_{0,0}/I_{-1,0}.$$

Let $\mathscr{A}^{\pm\infty}$ be the unital AF algebra with dimension group $G^{\pm\infty}$ and dimension range the closed interval between 0 and 1. Then

$$\mathscr{A}^{+\infty} \cong (\mathscr{B} \cap \mathscr{A})/(EI_{0,-1}E)$$
$$\mathscr{A}^{-\infty} \cong (\mathscr{B} \cap \mathscr{A})/(EI_{-1,0}E)$$

where we use $I_{n,m}$ also to denote the ideal in $\mathscr{B}$ defined by the ideal $I_{n,m}$ in G. Since $G^{\pm\infty}$ contains no nontrivial ideals, the C^*-algebras $\mathscr{A}^{\pm\infty}$ are simple. We have $\mathscr{A}^{\pm\infty} \neq \{0\}$ if and only if $K_{\pm\infty} \neq \phi$.

Proposition 7.1. *If $K_{+\infty} = \phi$, the C^*-dynamical system $(\mathscr{A}, \gamma)$ has no ground states. If $K_{+\infty} \neq \phi$, the set of ground states of $(\mathscr{A}, \gamma)$ is identified with the set of all states on $\mathscr{A}^{+\infty}$. The same statements hold for $-\infty$ and ceiling states.*

Proof. We use the notation

$$I^{+\infty} = EI_{0,-1}E, \quad I^{-\infty} = EI_{-1,0}E$$

for the two maximal ideals in $\mathscr{B} \cap \mathscr{A} = E\mathscr{B}E$. Assume first that ρ is a state on $\mathscr{A}^{+\infty} = E\mathscr{B}E/I^{+\infty}$. Then ρ lifts uniquely to a state ρ on $E\mathscr{B}E$ and we define

$$\omega = \rho \circ \varepsilon$$

where $\varepsilon \colon \mathscr{A} \to E\mathscr{B}E$ is the projection defined prior to Lemma 6.3. To show that ω is a ground state it is enough to demonstrate for each $X, Y \in \{AU^n;\, A \in \mathscr{B},\, AU^n \in \mathscr{A}\}$ that the function

$$z \longmapsto \omega(Y\gamma_z(X))$$

is bounded in the upper half plane, [7]. But if $X = AU^n$, $Y = BU^m$, then

$$\omega(Y\gamma_z(X)) = e^{inz}\, \omega(B\alpha^m(A)U^{m+n})$$

and hence $\omega(Y\gamma_z(X)) = 0$ unless $m = -n$. When $m = -n$ one has

$$\omega(Y\gamma_z(X)) = e^{inz}\, \rho(B\alpha^{-n}(A)).$$

To finish the argument we have to show that

$$\rho(B\alpha^{-n}(A)) = 0$$

when $n \leq -1$. But we have

$$EA = A\alpha^n(E) = A$$

by Lemma 6.1, and hence

$$\alpha^{-n}(A)=\alpha^{-n}(E)\alpha^{-n}(A)=\alpha^{-n}(A)E,$$
$$\alpha^{-n}(A)\in I_{-n,n}\cap I_{0,0}.$$

It follows that when $n\leq -1$,

$$\alpha^{-n}(A)\in I_{0,n},$$
$$B\alpha^{-n}(A)\in I^{+\infty},$$
$$\rho(B\alpha^{-n}(A))=0.$$

Conversely, if ω is a ground state then ω is γ-invariant and hence

$$\omega=\rho\circ\varepsilon$$

for some state ρ on $E\mathscr{B}E$. But reversing the argument above we deduce that

$$\rho(EB\alpha^{-n}(E)AE)=0$$

for all $A, B\in\mathscr{B}$ whenever $n\leq -1$. It follows that ρ annihilates $EI_{1,-1}E=EI_{0,-}\ E=I^{+\infty}$ and hence ρ lifts to a state on $\mathscr{A}^{+\infty}$.

The same reasoning applies to the $-\infty$ case.

By perturbing the dynamics γ by a bounded perturbation one has great liberty in choosing the set of ground states, by using the techniques from [3]. If $P=P^*\in\mathscr{A}$, the perturbed group γ^P is defined by

$$\gamma_t^P(A)=\gamma_t(A)+\sum_{n\geq 1}i^n\int_0^t dt_1\int_0^{t_1}dt_2\cdots\int_0^{t_{n-1}}dt_n[\gamma_{t_n}(P),[\cdots[\gamma_{t_1}(P),\gamma_t(A)]\cdots];$$

see [7], Proposition 5.4.1. In particular, if $\gamma_t(P)=P$ for all t one has

$$\gamma_t^P(A)=\gamma_t(e^{itP}A\,e^{-itP})=e^{itP}\gamma_t(A)\,e^{-itP}$$

for $A\in\mathscr{A}$, $t\in\mathbf{R}$.

From now on we will assume that the field $\beta\mapsto K_\beta$ has been chosen such that $K_{+\infty}$ is either empty or consists of one point, and the same for $K_{-\infty}$. This is possible by the remarks prior to definition 3.1. We will now choose the "dense" subgroup G of G_0 slightly more carefully than in the introduction to Section 5. If $G^{\pm\infty}$ are given countable archimedian totally ordered abelian groups without minimal positive elements, then $G^{\pm\infty}$ can be embedded as subgroups of the additive ordered group $\mathbf{R}$ in such a way that given elements $g^{\pm\infty}$ map into $1\in\mathbf{R}$, [13]. These are dense in $\mathbf{R}$, because otherwise they would contain minimal positive elements. If $K_{+\infty}$, resp. $K_{-\infty}$, is empty we will put $G^{+\infty}=\{0\}$, resp. $G^{-\infty}=\{0\}$.

We now choose the groups $G^{\pm\infty}$ in the beginning of Section 5 as the constant functions on K with value in $G^{\pm\infty}\subseteq\mathbf{R}$ as defined above. Since $K_{\pm\infty}$ consists of at most one point in this case, one easily verifies that the conditions 1 to 4 in the specifications of $G^{\pm\infty}$ in Section 5 are fulfilled, and one can define G as before.

We define a subgroup H of G as the sums and differences of elements f of the following three types:

1. $f(\beta, \omega)=g(\beta)\, e^{n\beta} h$ where $n\in\mathbf{Z}$, $h\in G^{+\infty}$, and g is defined from two rational numbers p, q, $p<q$, as follows

$$g(\beta)=\begin{cases} 0 & \text{for } \beta\le p \\ \dfrac{\beta-p}{q-p} & \text{for } p\le\beta\le q \\ 1 & \text{for } q\le\beta. \end{cases}$$

2. $f(\beta, \omega)=g(\beta)\, e^{n\beta} h$ where $n\in\mathbf{Z}$, h is in the subgroup of $\mathbf{R}$ generated by $G^{+\infty}$ and $G^{-\infty}$, and g is a piecewise linear continuous function with compact support which is differentiable except at a finite number of points $\beta_1,\dots,\beta_k$, and β_i, $g(\beta_i)$ are all rational.

3. $f(\beta, \omega)=g(\beta)\, e^{n\beta} h$ where $\beta\mapsto g(-\beta)$ is as in 1, $n\in\mathbf{Z}$ and $h\in G^{-\infty}$.

Note that H is a Riesz subgroup of G in the inherited ordering. The crucial properties of H which will be needed in the sequel are

1. If $f\in H$, then $f(\beta, \omega)$ only depends on β.
2. There exist elements f, $g\in H$ such that

$$f>0,\quad g>0,\quad f+g<1,$$
$$f_0^+=1,\quad g_0^-=1,\quad f_{+1}^->0,\quad g_{-1}^{\pm}>0.$$

The last properties ensure that the ideal in H generated by f, resp. g, is just $I_{-1,0}\cap H$, resp. $I_{0,-1}\cap H$. If we use the identifications $G^{+\infty}=I_{0,0}/I_{0,-1}$, $G^{-\infty}=I_{0,0}/I_{-1,0}$, it is not hard to prove

3. $I_{0,-1}\cap(I_{-1,0}\cap H)=I_{-1,-1}\cap H$ is an ordered group, and

$$(I_{-1,0}\cap H)/I_{-1,-1}\cap H=G^{+\infty}.$$

Likewise, $I_{-1,0}\cap(I_{0,-1}\cap H)=I_{-1,-1}\cap H$ is an ordered group and

$$(I_{0,-1}\cap H)/(I_{-1,-1}\cap H)=G^{-\infty}.$$

Theorem 7.2. *Let K be a compact, convex metrizable set, and let K_β be a closed convex subset of K for each $\beta\in\mathbf{R}$.*

Assume that

1. *Each K_β is a simplex.*
2. *If (ω_α) is a convergent net in K such that $\omega_\alpha\in K_{\beta_\alpha}$, and β_α converges to some $\beta\in\mathbf{R}$, it follows that $\lim_\alpha \omega_\alpha\in K_\beta$.*

Let $G^{\pm\infty}$ be non-zero, countable, archimedian totally ordered abelian groups without minimal positive elements, and let $\mathscr{A}^{\pm\infty}$ be unital AF algebras corresponding to intervals $[0, g^{\pm\infty}]$ in $G^{\pm\infty}$. Let $F_{\pm\infty}$ be arbitrary non-empty closed faces in the state spaces of $\mathscr{A}^{\pm\infty}$, and let $(\mathscr{A}, \gamma)$ be the C^*-dynamical system mentioned before the Theorem.

It follows that there exists a self-adjoint operator $P \in \mathscr{A}$ such that $\gamma_t(P) = P$ for all $t \in \mathbf{R}$, the set of (γ^P, β)-KMS states is affinely isomorphic to K_β when $\beta \in \mathbf{R}$, and the set of $(\gamma^P, \pm\infty)$-KMS states is affinely isomorphic to $F_{\pm\infty}$.

In the case that $K_\beta = \phi$ for large positive β the statements above remain true with $F_{+\infty}$ replaced by ϕ; the same is true for $-\infty$, or both $\pm\infty$.

Remark. This theorem can be extended to the case where $G^{\pm\infty}$ are just countable, simple (with respect to the order) Riesz groups, i.e. to the case where $\mathscr{A}^{\pm\infty}$ are general simple unital separable AF algebras, but to avoid an impenetrably complicated proof we defer a discussion of this extension to Section 10. We will also show in Section 9 that there exists a simple, separable, unital AF algebra $\mathscr{A}$ such that any metrizable simplex is affinely isomorphic to a face in the state space of $\mathscr{A}$. This means in particular that $F_{\pm\infty}$ can be taken to be arbitrary metrizable simplices in the theorem above.

Proof. We will assume that we are in the $K_{\pm\infty} \neq \phi$ cases; the other cases can be treated by minor modifications in the arguments. Let $\eta_\pm : E\mathscr{B}E \to E\mathscr{B}E/I^{\pm\infty} = \mathscr{A}^{\pm\infty}$ denote the quotient maps. Since $F_{\pm\infty}$ are faces in $\mathscr{A}^{\pm\infty}$ there exist positive elements $P^\pm \in \mathscr{A}^{\pm\infty}$ such that

$$F_{\pm\infty} = \{\omega \in E_{\mathscr{A}^{\pm\infty}};\ \omega(P^\pm) = 0\};$$

see e.g. [3].

Now, choose elements f, g in the subgroup H of G with the following properties:

$$f > 0, \quad g > 0, \quad f + g < 1, \quad f_0^+ = 1, \quad g_0^- = 1, \quad f_{+1}^- > 0, \quad g_{-1}^+ > 0.$$

The existence of f, g follows from property 2. Let E_+, E_- be projections in $\mathscr{B}$ such that $[E_+] = f$, $[E_-] = g$, $E_+E_- = 0$ and $E_+ + E_- \leq E$, where [ ] denote representative in G. Then $\eta_\pm(E_\pm) = 1$ and $\eta_\mp(E_\pm) = 0$. We now apply a lemma, Lemma 8.1, which will be proved in the next section. This lemma together with property 3 of H implies that there exist AF subalgebras $\mathscr{C}_+ \subseteq E_+\mathscr{B}E_+$, $\mathscr{C}_- \subseteq E_-\mathscr{B}E_-$ such that $\eta_\pm(\mathscr{C}_\pm) = \eta_\pm(E_\pm\mathscr{B}E_\pm) = \eta_\pm(E\mathscr{B}E)$ and the equivalence classes of all projections in $\mathscr{C}_\pm$ are contained in H. Let $P_1 \in \mathscr{C}_+$, $P_2 \in \mathscr{C}_-$ be positive elements such that $\eta_+(P_1) = P^+$, $\eta_-(P_2) = P^-$ and define

$$P=P_1-P_2.$$

Then $\eta_+(P)=P^+$, $\eta_-(P)=-P^-$ and $P\in E\mathscr{B}E$, so $\gamma_t(P)=P$ for all $t\in\mathbf{R}$. By multiplying P by a positive number we may also assume that

$$\|P\|<1/2.$$

We first prove that the set of (γ^P, β)-KMS states is affinely isomorphic to K_β in the case $\beta\in\mathbf{R}$. It is known that there is a homeomorphism between the set of (γ, β)-KMS states ω and the set of (γ^P, β)-KMS states ω^P which maps the extremal elements onto the extremal elements, [2], [8]; see also [7], Theorem 5.4.4 and Corollary 5.4.5. In our case, where $\gamma_t(P)=P$ for all t, this map has the simple form

$$\omega^P(A)=\omega(e^{-\beta P}A)/\omega(e^{-\beta P})$$

where $A\in\mathscr{A}$. But we will show in a moment that $K(\beta)=\omega(e^{-\beta P})$ is independent of the particular (γ, β)-KMS state ω chosen in our case. Hence

$$\omega^P(A)=\omega(e^{-\beta P}A)/K(\beta)$$

and this defines an affine isomorphism between the (γ, β)-KMS states and the (γ^P, β)-KMS states. Thus the (γ^P, β)-KMS state space is affinely isomorphic to K_β by Theorem 2.1.

To show that $K(\beta)$ is independent of ω, we use that $P=P_1-P_2$ where $P_1\in\mathscr{C}_+$, $P_2\in\mathscr{C}_-$. If Q is a projection in $\mathscr{C}_+$, then $[Q]\in H_+$ is a function f independent of ω, and hence $\omega(Q)=f(\beta)$ is independent of ω. Since $e^{-\beta P_1}$ can be approximated uniformly by linear combinations of projections in $\mathscr{C}_+$, it follows that $\omega(e^{-\beta P_1})$ is independent of the (γ, β)-KMS state ω. Similarly $\omega(e^{\beta P_2})$ is independent of ω. But $P_1P_2=P_1E_+E_-P_2=0$ and hence

$$e^{-\beta P}=e^{-\beta P_1}+e^{\beta P_2}-1$$

and thus $\omega(e^{-\beta P})=K(\beta)$ is independent of ω.

We now show that the ground states ω for $(\mathscr{A}, \gamma^P)$ correspond in a one-one affine fashion to states ψ on $\mathscr{A}^{+\infty}$ with $\psi(P^+)=0$, i.e. states $\psi\in F_+$.

Assume first that ψ is a state on $\mathscr{A}^{+\infty}$ with $\psi(P^+)=0$, lift ψ to a state ρ on $E\mathscr{B}E$ by setting

$$\rho=\psi\circ\eta_+,$$

and finally extend ρ to a state ω on $\mathscr{A}$ by setting

$$\omega=\rho\circ\varepsilon=\psi\circ\eta_+\circ\varepsilon.$$

Since $P^+ \geq 0$ and $\psi(P^+)=0$, ψ is a ground state for the automorphism group $Ad\, e^{itP^+}$ on $\mathscr{A}^{+\infty}$. It follows that

$$\begin{aligned}\omega(X e^{-itP}) &= \rho(\varepsilon(X e^{-itP})) \\ &= \rho(\varepsilon(X) e^{-itP}) \\ &= \psi(\eta_+(\varepsilon(X)) e^{-itP^+}) \\ &= \psi(\eta_+(\varepsilon(X))) \\ &= \omega(X)\end{aligned}$$

for all $X \in \mathscr{A}$ and $t \in \mathbf{R}$, where the second-last equality follows from

$$\left|\frac{d}{dt}\psi(A e^{-itP^+})\right| = |\psi(A e^{-itP^+}P^+)| \leq \psi(A e^{-itP^+}P^+ e^{itP^+}A^*)^{1/2}\psi(P^+)^{1/2} = 0\,.$$

Hence, if $X=AU^n$, $Y=BU^m$ are elements in $\mathscr{A}$ it follows that

$$\begin{aligned}\omega(Y\gamma_z^P(X)) &= \omega(Y e^{izP}\gamma_z(X) e^{-izP}) \\ &= \omega(Y e^{izP}\gamma_z(X)) \\ &= e^{inz}\,\omega(B\alpha^m(e^{izP}A)U^{m+n}),\end{aligned}$$

and hence $\omega(Y\gamma_z^P(X))=0$ unless $m=-n$. When $m=-n$ one has

$$\omega(Y\gamma_z^P(X)) = e^{inz}\rho(B\alpha^{-n}(e^{izP}A)) = e^{inz}\rho(EB\alpha^{-n}(E)\alpha^{-n}(e^{izP})\alpha^{-n}(E)\alpha^{-n}(A)E).$$

By the reasoning in the proof of Proposition 7.1, the right side vanishes whenever $n \leq -1$. When $n=0$, $E\mathscr{B}E=B$, $E e^{izP}E = e^{izP}$, $EAE=A$ are all in $E\mathscr{B}E$, and hence the equation above takes the form

$$\begin{aligned}\omega(Y\gamma_z^P(X)) &= \psi(\eta_+(B)\eta_+(e^{izP})\eta_+(A)) \\ &= \psi(\eta_+(B) e^{izP^+}\eta_+(A))\,.\end{aligned}$$

As $P^+ \geq 0$ this function has a bounded analytic extension to the upper half plane. When n is strictly positive and $Imz \geq 0$, one has the estimate

$$\begin{aligned}|\omega(Y\gamma_z^P(X))| &\leq e^{-nImz}\,\|B\alpha^{-n}(e^{izP}A)\| \\ &\leq e^{-nImz}\,\|B\|\,\|e^{izP}\|\,\|A\| \\ &\leq e^{(\|P\|-n)Imz}\,\|B\|\,\|A\|\end{aligned}$$

and this is bounded in the upper half plane because of the estimate $\|P\|<1/2$. It follows that ω is a ground state for $(\mathscr{A}, \gamma^P)$.

Assume conversely that ω is a ground state for $(\mathscr{A}, \gamma^P)$. We will show that ω also is a ground state of $(\mathscr{A}, \gamma)$, in the form of a general lemma.

Lemma 7.3. *Let $(\mathscr{A}, \gamma)$ be a C^*-dynamical system where γ is periodic with* *period 2π, and let $P=P^*\in\mathscr{A}$ be an element such that $\|P\|<1/2$ and $\gamma_t(P)=P$ for all $t\in\mathbf{R}$. Then any ground state for $(\mathscr{A}, \gamma^P)$ is also a ground state for $(\mathscr{A}, \gamma)$.*

Proof. Assume that ω is a ground state for $(\mathscr{A}, \gamma^P)$. It is enough to show that the analytic function

$$z \longmapsto \omega(B\gamma_z(A))$$

is bounded in the upper half plane when $B\in\mathscr{A}$ and $A\in\mathscr{A}^\gamma(\{n\})$, i.e., A is an element in $\mathscr{A}$ such that $\gamma_t(A)=e^{int}A$; see [7], Proposition 5.3.19 and [6], Section 3.2.3. But

$$\omega(B\gamma_z(A))=e^{inz}\,\omega(BA)$$

and hence it suffices to show that

$$\omega(BA)=0$$

when $n\leq -1$. But as ω is a $(\mathscr{A}, \gamma^P)$-ground state, the function

$$\begin{aligned}\omega(B\gamma_z^P(A)) &= \omega(B\,e^{izP}\gamma_z(A)\,e^{-izP})\\ &= e^{inz}\,\omega(B\,e^{izP}\,A\,e^{-izP})\end{aligned}$$

is bounded in absolute value in the upper half plane:

$$|e^{inz}\,\omega(B\,e^{izP}A\,e^{-izP})|\leq\|B\|\,\|A\|$$

for $Imz\geq 0$. But $e^{izP}\,A\,e^{-izP}\in\mathscr{A}^\gamma(\{n\})$ by γ-invariance of P, and hence, replacing A by $e^{-izP}\,A\,e^{izP}$ in the relation above, we obtain

$$\begin{aligned}|e^{inz}\,\omega(BA)| &\leq \|B\|\,\|e^{-izP}\,A\,e^{izP}\|\\ &\leq \|B\|\,\|A\|\,e^{2\,Imz\,\|P\|}\end{aligned}$$

for $Imz\geq 0$. Setting $z=is$ where $s\geq 0$ one deduces that

$$|\omega(BA)|\leq\|B\|\,\|A\|\,e^{s(n+2\|P\|)}.$$

Since $\|P\|<1/2$ it follows that $\omega(BA)=0$ when $n\leq -1$, and the lemma is proved.

Resuming the proof of Theorem 7.2, since by Lemma 7.3 ω is also a ground state for $(\mathscr{A}, \gamma)$, by Proposition 7.1 ω annihilates the ideal $I^{+\infty}$ in $E\mathscr{B}E$; i.e. $\omega|E\mathscr{B}E\equiv\rho$ induces a state ψ on $E\mathscr{B}E/I^{+\infty}=\mathscr{A}^{+\infty}$. Since the restriction ρ is a $\gamma_t^P|_{E\mathscr{B}E}=Ad\,e^{itP}$ ground state, ψ is an $Ad\,\eta_+(e^{itP})=Ad\,e^{itP^+}$ ground state, and in particular $\psi(P^+)=0$, i.e. $\psi\in F_{+\infty}$. We conclude that

$$\omega=\psi\circ\eta_+\circ\varepsilon$$

establishes a one-one affine correspondence between $(\mathscr{A}, \gamma^P)$-ground states ω and states $\psi \in F_{+\infty}$.

The reasoning for ceiling states is analogous.

Remark 7.4. The proof of Theorem 7.2 can be greatly simplified (i.e. one does not need the subgroup H and Lemma 8.1) in the case that all K_β are Bauer simplexes, i.e. simplexes where the sets of extremal points are closed. In this case one can let P be any element in $E\mathscr{B}E$ such that $\eta_+(P)=P^+$, $\eta_-(P)=-P^-$ and $\|P\|<1/2$. This is because the map

$$\omega \longmapsto \tilde{\omega}^P$$

where $\tilde{\omega}^P(A)=\omega(e^{-\beta P}A)$ is an affine isomorphism between the set of positive (γ, β)-KMS functionals and the set of positive (γ^P, β)-KMS functionals. Hence the map

$$\omega \longmapsto \bar{\omega}(A)=\tilde{\omega}^P(A)/\omega(e^{-\beta P})$$

is a one-one continuous map from the extremal (γ, β)-KMS states ω to the extremal (γ^P, β)-KMS states $\bar{\omega}$. Since the set K_β of (γ, β)-KMS states is a simplex, there exists for each $\omega \in K_\beta$ a unique maximal measure $\mu_\omega \in M_1^+(K_\beta)$ with barycenter ω, where $M_1^+(K_\beta)$ denotes the probability measures on K_β, [1], [6]. μ_ω is supported by the extremal points $\mathscr{E}(K_\beta)$ in K_β since K_β is metrizable, and hence we may define

$$\bar{\omega}=\int_{\mathscr{E}(K_\beta)} \bar{\rho}\, d\mu_\omega(\rho).$$

This is an affine map from K_β into the set of (γ^P, β)-KMS states. But as K_β is a Bauer simplex, the map $\omega \mapsto \mu_\omega$ is continuous when $M_1^+(K_\beta)$ is equipped with the weak topology, [1], Theorem II, 4.1, and as

$$\rho \longmapsto \rho(e^{-\beta P}\ \cdot)/\rho(e^{-\beta P})=\bar{\rho}$$

is a continuous function it follows that $\omega \mapsto \bar{\omega}$ is continuous; hence it is a homeomorphism by the compactness of K_β.

VIII A Lemma on *AF* algebras

In this section we will prove the lemma needed in the proof of Theorem 7.2. We use freely results and methods from [9] during the proof.

Lemma 8.1. *Let G be the dimension group of a separable AF algebra $\mathscr{B}$ with identity* 1, *let I denote both an ideal in G and the corresponding ideal in $\mathscr{B}$. If E is a projection in $\mathscr{B}$, resp. $\mathscr{B}/I$, let $[E]$ denote its Murray-von Neumann equivalence class in $\mathscr{B}$, resp. $\mathscr{B}/I$. Assume that H is a Riesz subgroup of G such that*

$H \cap I$ is an ordered group, $[1] \in H$, and $H/I = G/I$. Then there exists an AF subalgebra $\mathscr{C} \subseteq \mathscr{B}$ such that the equivalence classes in $\mathscr{B}$ of all the projections in $\mathscr{C}$ belong to H and $\mathscr{C}/I = \mathscr{B}/I$.

Proof. Let η denote both the quotient maps $G \to G/I$ and $\mathscr{B} \to \mathscr{B}/I$.

Observation 1. *If V is a unitary in $\mathscr{B}/I$ there exists a unitary $U \in \mathscr{B}$ such that $V = \eta(U)$.*

This is because V has the form $V = e^{iH_1} e^{iH_2}$ where H_1, H_2 are self-adjoint elements in $\mathscr{B}/I$ (If V is a finite-dimensional subalgebra this is clear with $H_2 = 0$, if not we can approximate V by an element in a finite-dimensional subalgebra, and taking the polar decomposition we find a unitary e^{iH_1} in the finite-dimensional algebra such that $\|e^{-iH_1} V - 1\| = \|V - e^{iH_1}\| < 2$, and hence $e^{-iH_1} V = e^{iH_2}$ by spectral theory). H_1 and H_2 can be lifted to self-adjoint elements K_1 and K_2 in $\mathscr{B}$, and then $U = e^{iK_1} e^{iK_2}$ has the desired property.

Observation 2. *If E is a projection in $\mathscr{B}/I$ and $g \in G$ is such that $0 \leq g \leq [1]$ and $\eta(g) = [E]$, then there exists a projection $F \in \mathscr{B}$ such that $[F] = g$ and $\eta(F) = E$.*

This is because if $F' \in \mathscr{B}$ is any projection with $[F'] = g$, then $[\eta(F')] = \eta[F'] = [E]$, and hence there exists a unitary $V \in \mathscr{B}/I$ with $E = V\eta(F')V^*$. By Observation 1 there is a unitary $U \in \mathscr{B}$ with $\eta(U) = V$, and we may define $F = UF'U^*$.

Observation 3. *If E_1, E_2 are mutually orthogonal projections in $\mathscr{B}/I$, F is a projection in $\mathscr{B}$ and h_1, h_2 are positive elements in G such that $E_1 + E_2 \leq \eta(F)$,*

$$\eta(h_i) = [E_i], \ i = 1, 2 \quad \text{and} \quad h_1 + h_2 \leq [F]$$

then there exist mutually orthogonal projections F_1, F_2 in $\mathscr{B}$ such that $[F_i] = h_i$,

$$\eta(F_i) = E_i, \ i = 1, 2 \quad \text{and} \quad F_1 + F_2 \leq F.$$

We first apply Observation 2 with $\mathscr{B}$ replaced by $F\mathscr{B}F$ to get F_1, and next we replace $\mathscr{B}$ by $(F - F_1)\mathscr{B}(F - F_1)$ to get F_2.

Observation 4. *If F_1, F_2 are equivalent projections in $\mathscr{B}$ and V is a partial isometry in $\mathscr{B}/I$ such that*

$$\eta(F_1) = V^*V, \quad \eta(F_2) = VV^*$$

then there exists a partial isometry $U \in \mathscr{B}$ such that

$$F_1 = U^*U, \quad F_2 = UU^* \quad \text{and} \quad \eta(U) = V.$$

Let U' be a partial isometry in $\mathscr{B}$ such that $F_1 = U'^*U'$, $F_2 = U'U'^*$ and define

$V'=\eta(U')$. Then V' is a partial isometry with $\eta(F_1)$, $\eta(F_2)$ as initial and final projection, respectively, and hence VV'^* is a partial isometry with $\eta(F_2)$ as both initial and final projection. If we replace $\mathscr{B}$ by $F_2\mathscr{B}F_2$ in Observation 1 we get a partial isometry U'' in $\mathscr{B}$ with

$$U''^*U''=U''U''^*=F_2 \quad \text{and} \quad \eta(U'')=VV'^*.$$

Put $U=U''U'$. Then

$$U^*U=U'^*F_2U'=F_1$$
$$UU^*=U''F_2U''^*=F_2,$$

and

$$\eta(U)=VV'^*V'=V\eta(F_1)=V.$$

Observation 5. *Let $g_1, g_2, g \in G/I$ and $h \in H$ be positive elements such that*

$$g_1+g_2=g \quad \textit{and} \quad \eta(h)=g.$$

It follows that there exist positive elements $h_1, h_2 \in H$ such that

$$h_1+h_2 \le h \quad \textit{and} \quad \eta(h_1)=g_1,\ \eta(h_2)=g_2.$$

This is a consequence of the Riesz decomposition property for H, [10]. Since $H/I=G/I$ there exist positive elements h_1', h_2' in H such that $\eta(h_i')=g_i$ for $i=1, 2$. But then

$$h_1'+h_2'=h+c$$

where $c \in H \cap I$, and as $H \cap I$ is a ordered group there exist positive elements $c_+, c_- \in H \cap I$ with $c=c_+-c_-$. It follows that

$$h_1'+h_2'+c_-=h+c_+.$$

But by the Riesz decomposition property there exist positive elements a_{ij}, $i=1, 2, 3$, $j=1, 2$, in H such that

$$h_1'=a_{11}+a_{12}, \quad h=a_{11}+a_{21}+a_{31},$$
$$h_2'=a_{21}+a_{22}, \quad c_+=a_{12}+a_{22}+a_{32},$$
$$c_-=a_{31}+a_{32}.$$

Define $h_1=a_{11}$, $h_2=a_{21}$. Since $a_{12}, a_{22} \le c_+$ one has $a_{12}, a_{22} \in I$ and hence

$$\eta(h_i)=\eta(h_i')=g_i$$

for $i=1, 2$. Also

$$h_1+h_2+a_{31}=h$$

so

$$h_1+h_2\leq h .$$

Observation 6. *Let h_1, h_2 be positive elements in H such that $\eta(h_1)=\eta(h_2)$. It follows that there exists a positive element h in H such that*

$$h\leq h_1, \ h\leq h_2 \quad \textit{and} \quad \eta(h)=\eta(h_1)=\eta(h_2) .$$

This is again a consequence of the Riesz decomposition property. One has

$$h_1+c_-=h_2+c_+$$

for suitable positive elements $c_\pm \in H\cap I$, and then there exist positive elements a_{ij}, $i, j=1, 2$ in H such that

$$h_1=a_{11}+a_{12}, \quad h_2=a_{11}+a_{21} ,$$

$$c_-=a_{21}+a_{22}, \quad c_+=a_{12}+a_{22} .$$

The last two relations imply that $a_{21}, a_{12}\in I$, and hence $h=a_{11}$ has the correct properties.

Observation 7. *Let $\mathscr{A}$, $\mathscr{D}$ be finite-dimensional C^*-algebras such that $\mathscr{D}\subseteq\mathscr{A}$ and $\mathscr{A}$ and $\mathscr{D}$ has a common identity. Then there exists a finite increasing sequence*

$$\mathscr{D}=\mathscr{D}_1\subset\mathscr{D}_2\subset\cdots\subset\mathscr{D}_n=\mathscr{A}$$

of finite-dimensional C^*-algebras such that the diagram, [9], *of the embedding of $\mathscr{D}_k$ into $\mathscr{D}_{k+1}$ has one of the two forms*

$$\begin{array}{ccccccc} & X & & X & X & \cdots & X \\ \diagup & & \diagdown & | & | & & | \\ X & & X & X & X & \cdots & X \end{array}$$

or

$$\begin{array}{ccccccc} X & & X & X & X & \cdots & X \\ & \diagdown \ \diagup & & | & | & & | \\ & X & & X & X & \cdots & X \end{array} .$$

We construct $\mathscr{D}_1$, $\mathscr{D}_2$,... inductively. Put $\mathscr{D}_1=\mathscr{D}$ and assume $\mathscr{D}_1,\ldots, \mathscr{D}_k$ have been constructed. Since $\mathscr{D}_k\subseteq\mathscr{A}$ we have

$$\mathscr{D}_k' \cap \mathscr{D}_k \subseteq \mathscr{D}_k' \cap \mathscr{A}$$

($\mathscr{D}_k' \cap \mathscr{E}$ = the relative commutant of $\mathscr{D}_k$ in $\mathscr{E}$). There are two possibilities:

1. $\mathscr{D}_k' \cap \mathscr{D}_k \neq \mathscr{D}_k' \cap \mathscr{A}$.

In this case, pick a minimal projection P in $\mathscr{D}_k' \cap \mathscr{A}$ such that $P \notin \mathscr{D}_k' \cap \mathscr{D}_k$, and let $\mathscr{D}_{k+1}$ be the C^*-algebra generated by $\mathscr{D}_k$ and P. The embedding of $\mathscr{D}_k$ into $\mathscr{D}_{k+1}$ then has the form

$$\begin{array}{ccccccc} & X & & X & X & \cdots & X \\ \diagup & & \diagdown & | & | & & | \\ X & & X & X & X & \cdots & X\,. \end{array}$$

2. $\mathscr{D}_k' \cap \mathscr{D}_k = \mathscr{D}_k' \cap \mathscr{A}$.

In this case one has

$$\mathscr{D}_k' \cap \mathscr{A} = \mathscr{D}_k' \cap \mathscr{D}_k \supseteq \mathscr{A}' \cap \mathscr{A}$$

and there are two possibilities.

2a. $\mathscr{A}' \cap \mathscr{A} \neq \mathscr{D}_k' \cap \mathscr{A} = \mathscr{D}_k' \cap \mathscr{D}_k$.

In this case each factor in $\mathscr{D}_k$ is embedded in just one factor in $\mathscr{A}$ and with multiplicity one. Let P_1 be a minimal projection in $\mathscr{D}_k \cap \mathscr{D}_k'$ such that $P_1 \notin \mathscr{A}' \cap \mathscr{A}$. Let P_2 be another minimal projection in $\mathscr{D}_k \cap \mathscr{D}_k'$ such that $P = P_1 + P_2$ is dominated by a minimal projection in $\mathscr{A}' \cap \mathscr{A}$, and let $\mathscr{D}_{k+1}$ be the C^*-algebra generated by $\mathscr{D}_k$ and $P\mathscr{A}P$. The embedding of $\mathscr{D}_k$ into $\mathscr{D}_{k+1}$ then has the form

$$\begin{array}{ccccccc} X & & X & X & X & \cdots & X \\ \diagdown & & \diagup & | & | & & | \\ & X & & X & X & \cdots & X\,. \end{array}$$

2b. $\mathscr{A}' \cap \mathscr{A} = \mathscr{D}_k' \cap \mathscr{A} = \mathscr{D}_k' \cap \mathscr{D}_k$.

In this case one must have $\mathscr{D}_k = \mathscr{A}$ and the induction is finished.

Since $\mathscr{A}$ is finite-dimensional the induction must end after a finite number of steps.

Observation 8. *If $\mathscr{A}$ is a finite-dimensional subalgebra of $\mathscr{B}/I$ with $\eta(1) \in \mathscr{A}$, and $\mathscr{C}_1$ is a finite-dimensional subalgebra of $\mathscr{B}$ such that $[E] \in H$ for all projections $E \in \mathscr{C}_1$ and $\eta(\mathscr{C}_1) \subseteq \mathscr{A}$, there exists a finite-dimensional subalgebra $\mathscr{C}_2$ of $\mathscr{B}$ such that $\mathscr{C}_1 \subseteq \mathscr{C}_2$, $[E] \in H$ for all $E \in \mathscr{C}_2$ and $\mathscr{A} = \eta(\mathscr{C}_2)$.*

By applying Observation 7 on the pair $\mathscr{D} = \eta(\mathscr{C}_1)$ and $\mathscr{A}$ and an induction argument, we may assume that the embedding of $\eta(\mathscr{C}_1)$ in $\mathscr{A}$ has one of the forms

1.
$$\begin{tikzcd}[column sep=small]
& X \arrow[dl, dash] \arrow[dr, dash] & & X \arrow[d, dash] & X \arrow[d, dash] & \cdots & X \arrow[d, dash] \\
X & & X & X & X & \cdots & X
\end{tikzcd}$$

or

2.
$$\begin{tikzcd}[column sep=small]
X \arrow[dr, dash] & & X \arrow[dl, dash] & X \arrow[d, dash] & X \arrow[d, dash] & \cdots & X \arrow[d, dash] \\
& X & & X & X & \cdots & X ,
\end{tikzcd}$$

We treat the two cases separately.

Case 1. Let P be the minimal projection in $\mathscr{C}_1 \cap \mathscr{C}_1'$ such that $\eta(P)$ is the sum of two minimal projections P_1 and P_2 in $\mathscr{A} \cap \mathscr{A}'$. Let F be a minimal projection in $P\mathscr{C}_1 P$, and define

$$E_1 = \eta(F)P_1, \quad E_2 = \eta(F)P_2 .$$

Then E_1, E_2 are minimal projections in the factors $\mathscr{A}P_1$, $\mathscr{A}P_2$, respectively. Observation 5, applied to $g_1 = [E_1]$, $g_2 = [E_2]$, $g = [\eta(F)]$ and $h = [F]$ and Observation 3 imply the existence of projections F_1 and F_2 in $\mathscr{B}$ such that $[F_1]$, $[F_2] \in H$, $F_1 + F_2 \leq F$ and $\eta(F_i) = E_i$ for $i = 1, 2$. Define

$$F_3 = F - F_1 - F_2 .$$

Then $[F_3] \in H - H = H$. Let $\mathscr{C}_2$ be the C^*-algebra generated by F_1, F_2, F_3 and $\mathscr{C}$. Then $\mathscr{C}_2$ is a direct sum of the three (two if $F_3 = 0$) factors $\mathscr{C}_1 F_1 \mathscr{C}_1$, $\mathscr{C}_1 F_2 \mathscr{C}_1$, $\mathscr{C}_1 F_3 \mathscr{C}_1$ and the algebra $\mathscr{C}_1(1-P)$. These factors have the same dimension as $\mathscr{C}_1 P$, hence $\mathscr{C}_2$ is finite-dimensional. Also $[Q] \in H$ for all projections $Q \in \mathscr{C}_2$, and $\eta(F_1) = E_1$, $\eta(F_2) = E_2$, $\eta(F_3) = 0$, hence $\eta(\mathscr{C}_2) = \mathscr{A}$.

Case 2. Let P_1, P_2 be the two distinct minimal projections in $\mathscr{C}_1 \cap \mathscr{C}_1'$ such that $\eta(P_i) \neq 0$ for $i = 1, 2$ and $\eta(P_1) + \eta(P_2) = P$ is a minimal projection in $\mathscr{A} \cap \mathscr{A}'$. Let $F_1 \leq P_1$, $F_2 \leq P_2$ be minimal projections in $\mathscr{C}_1$, and define $E_1 = \eta(F_1)$, $E_2 = \eta(F_2)$. Then E_1, E_2 are minimal projections in the factor $\mathscr{A}P$ in $\mathscr{A}$, and hence there is a partial isometry V in $\mathscr{A}$ with

$$V^*V = E_1, \quad VV^* = E_2 .$$

One has $[F_1]$, $[F_2] \in H$, $\eta[F_1] = [E_1] = [E_2] = \eta[F_2]$, and thus Observation 6 implies the existence of a positive $h \in H$ such that $h \leq [F_1]$, $h \leq [F_2]$ and $\eta(h) = \eta[F_1] = \eta[F_2]$. By Observation 3 there exist projections Q_1, Q_2 in $\mathscr{B}$ such that $[Q_1] = [Q_2] = h$ and

$$Q_1 \leq F_1, \quad Q_2 \leq F_2 .$$

Define $Q_3=F_1-Q_1$, $Q_4=F_2-Q_2$. Then $[Q_i]\in H-H=H$ for $i=3, 4$. By Observation 4 there exists a partial isometry $U\in\mathscr{B}$ such that

$$U^*U=Q_1,\quad UU^*=Q_2 \quad\text{and}\quad \eta(U)=V.$$

Let $\mathscr{C}_2$ be the C^*-algebra generated by U and $\mathscr{C}_1$. Then $Q_i\in\mathscr{C}_2$ for all i. The algebra $\mathscr{C}_2$ is the direct sum of three (the two last of them may be zero) factors and $(1-P_1-P_2)\mathscr{C}_1(1-P_1-P_2)$. The factors are $\mathscr{C}_1Q_1\mathscr{C}_1+\mathscr{C}_1U\mathscr{C}_1+\mathscr{C}_1U^*\mathscr{C}_1+\mathscr{C}_1Q_2\mathscr{C}_1$, $\mathscr{C}_1Q_3\mathscr{C}_1$, and $\mathscr{C}_1Q_4\mathscr{C}_1$. The equivalence classes of all projections in these factors are in H, and as $\eta(Q_1)=E_1$, $\eta(U)=V$, $\eta(Q_3)=\eta(Q_4)=0$ one has $\eta(\mathscr{C}_2)=\mathscr{A}$.

Lemma 8.1 follows by considering an increasing sequence $\mathscr{A}_n$ of finite-dimensional subalgebras of $\mathscr{B}/I$ such that $\bigcup_n\mathscr{A}_n$ is dense in $\mathscr{B}/I$. Using Observation 8 recursively we find an increasing sequence $\mathscr{C}_n$ of finite-dimensional C^*-subalgebras of $\mathscr{B}$ such that $\eta(\mathscr{C}_n)=\mathscr{A}_n$. Defining $\mathscr{C}=\overline{\bigcup_n\mathscr{C}_n}$ one has that $\eta(\mathscr{C})=\mathscr{B}/I$ and since each projection in $\mathscr{C}$ is equivalent to a projection in some $\mathscr{C}_n$, it follows that $[F]\in H$ for all projections $F\in\mathscr{C}$.

Remark that the assumption in Lemma 8.1 that $H\cap I$ is ordered, i.e. $(H\cap I)_+-(H\cap I)_+\subset H\cap I$, does not follow from the other assumptions. This assumption implies furthermore that $H\cap I$ is a Riesz group (by the decomposition property).

IX Representation of simplexes as faces in the state spaces of *AF* algebras

In this section we will prove the result alluded to in the remark after Theorem 7.2, i.e.,

Proposition 9.1. *There exists a simple, separable, unital AF algebra $\mathscr{A}$ such that any metrizable simplex is affinely isomorphic to a face in the state space of $\mathscr{A}$.*

Several results of a related nature have been proved by Batty, [3], [4]. It follows from [3], Corollary 3.4, that if $\mathscr{A}$ is a simple, separable, infinite-dimensional, unital AF algebra, then the state space of $\mathscr{A}$ contains a face isomorphic to the simplex $K_{\mu\infty}(\overline{\mathbf{N}})$, where μ is a probability measure on the one-point compactification $\overline{\mathbf{N}}$ of $\mathbf{N}$, and $K_{\mu\infty}(\overline{\mathbf{N}})$ is the state space of the order unit space

$$\left\{f\in C(\overline{\mathbf{N}}); f(\infty)=\int f d\mu\right\}.$$

Batty exhibited a canonical affine isomorphism between the set $E_{\mathscr{A}}^G$ of invariant states of a C^*-dynamical system $(\mathscr{A}, G, \alpha)$ and a face in the state space of the crossed product $C^*(\mathscr{A}, \alpha)$ (where G is equipped with the discrete topology). If $f\in L^1(\mathscr{A}, G)\subseteq C^*(\mathscr{A}, \alpha)$, this correspondence is given by $\omega\in E_{\mathscr{A}}^G\mapsto\tilde{\omega}$, where

$$\tilde{\omega}(f)=\sum_{g\in G}\omega(f(g)).$$

If α is the Bernoulli shift on the CAR algebra $\mathscr{A}=\bigotimes_{\mathbf{Z}} M_2$, one can show that $E_{\mathscr{A}}^{\mathbf{Z}}$ is a simplex where the set of extremal points is dense; see e.g. [6], Example 4.3.26. It is known that a metrizable simplex with this property is unique up to affine isomorphism, and any metrizable simplex is isomorphic with a face in this simplex, which is called the Poulsen simplex, [16]. Hence, any metrizable simplex is isomorphic with a face in the state space of $C^*(\mathscr{A}, \alpha)$ for this particular example, which is due to Batty, [4]. $C^*(\mathscr{A}, \alpha)$ is simple by [14], but by [5] is not AF.

Our proof of Proposition 9.1 will be based on Batty's method. Let K be the Poulsen simplex, and choose a countable dense subgroup H of the additive group $A(K)$, equipped with the strict order, such that H contains the constant function 1. Then H is a Riesz group. Let $\mathscr{A}$ be the unital AF algebra with dimension group H, and such that the identity of $\mathscr{A}$ corresponds to $1\in H$, [10], [12]. It follows that $\mathscr{A}$ is simple, and the trace state space of $\mathscr{A}$ is affinely isomorphic to K.

If G is the group of inner automorphisms of $\mathscr{A}$, then $E_{\mathscr{A}}^{G}$ is just the trace state space of $\mathscr{A}$, and hence the Poulsen simplex is a face in the state space of the C^*-crossed product $\mathscr{A}\times G$. This crossed product is neither simple nor AF however. It can be made AF by replacing G by a large locally finite subgroup, but in order to make the crossed product simple G would have to act as outer automorphisms on $\mathscr{A}$. We will remedy this defect by replacing $\mathscr{A}$ by $\mathscr{B}=\mathscr{A}\otimes\mathscr{F}$, where $\mathscr{F}$ is a suitable UHF algebra, and then finding a group G of outer automorphisms of $\mathscr{B}$ such that $E_{\mathscr{B}}^{G}$ consists of just the trace states of $\mathscr{B}$.

Let $\mathscr{A}_n$ be an increasing sequence of finite-dimensional C^*-subalgebras of $\mathscr{A}$, each containing the unit of $\mathscr{A}$, and with dense union in $\mathscr{A}$. Each $\mathscr{A}_n$ has the central decomposition

$$\mathscr{A}_n=\sum_k M_{(n,k)}$$

where each $M_{(n,k)}$ is a full matrix algebra of order $[n, k]$.

We define an increasing sequence $\mathscr{B}_n$ of finite-dimensional C^*-algebras as follows:

$$\begin{aligned}
&\mathscr{B}_1=\mathscr{A}_1\\
&\mathscr{B}_2=\mathscr{A}_2\otimes M_{\eta(1)}\\
&\mathscr{B}_3=\mathscr{A}_3\otimes M_{\eta(1)}\otimes M_{\eta(2)}\\
&\quad\cdots\\
&\mathscr{B}_n=\mathscr{A}_n\otimes M_{\eta(1)}\otimes\cdots\otimes M_{\eta(n-1)}
\end{aligned}$$

where $\eta(n)$, $n=1, 2,\dots$ are integers defined inductively by

$$\eta(1)=\sum_k [1, k]$$

and

$$\eta(n)=\eta(1)\eta(2)\cdots\eta(n-1)\sum_k [n, k]$$

for $n=2, 3,\dots$. The embedding of $\mathscr{B}_n$ in $\mathscr{B}_{n+1}$ is the canonical one determined from the embedding of $\mathscr{A}_n$ into $\mathscr{A}_{n+1}$. We choose matrix elements for the subfactors $M_{(n,k)}$ of $\mathscr{A}_n$ inductively in such a way that these matrix elements are sums of matrix elements for $\mathscr{A}_{n+1}$; see [9].

It is clear that $\mathscr{B}=\mathscr{A}\otimes\mathscr{F}$, where $\mathscr{F}$ is the UHF algebra $M_{\eta(1)\eta(2)\cdots}$.

We next define an increasing sequence $G_1\subseteq G_2\subseteq G_3\cdots$ of finite groups acting on $\mathscr{B}$. Note that the central decomposition of $\mathscr{B}_n$ has the form

$$\mathscr{B}_n\cong\bigoplus_k M_{\eta(1)\cdots\eta(n-1)[n,k]}.$$

We define

$$G_n=\times_k S_{\eta(1)\cdots\eta(n-1)[n,k]}$$

where S_k is the permutation group on k elements. One has a representation of G_n in the unitary group in $\mathscr{B}_n$ such that an element of the form $g=\times_k g_k$ is represented by a unitary operator of the form $U_g^{(n)}=\oplus U_{g_k}^{(n,k)}$, where $U^{(n,k)}$ is a representation of $S_{\eta(1)\cdots\eta(n-1)[n,k]}$ by permutation matrices; i.e. each $U_g^{(n,k)}$ is a matrix with entries just 0 and 1, and 1 occurs exactly once in each row and in each column of the matrix. One has a representation of G_n in the automorphism group of $\mathscr{B}_n$ by $g\mapsto Ad\, U_g^{(n)}$.

We next describe the embedding of G_{n-1} into G_n. The embedding of

$$\mathscr{B}_{n-1}=\mathscr{A}_{n-1}\otimes M_{\eta(1)}\otimes\cdots\otimes M_{\eta(n-2)}$$

into

$$\mathscr{B}_n=\mathscr{A}_n\otimes M_{\eta(1)}\otimes\cdots\otimes M_{\eta(n-2)}\otimes M_{\eta(n-1)}$$

is such that the matrices in $\mathscr{B}_{n-1}$ are represented by matrices of the form

$$B\otimes 1$$

in $\mathscr{B}_n$, where 1 is the identity in $M_{\eta(n-1)}$. But

$$\eta(n-1)=\eta(1)\cdots\eta(n-2)(\sum_k [n-1, k])$$

is just the sum of the orders of the factors in the central decomposition of $\mathscr{B}_{n-1}$. Hence, if $g \in G_{n-1}$, then one can string the unitaries in the decomposition

$$U_g^{(n-1)} = \bigoplus_k U_{gk}^{(n-1,k)}$$

along the diagonal in $M_{\eta(n-1)}$, and by setting all other matrix elements equal to zero one obtains a permutation matrix $V_g^{(n)}$ in $M_{\eta(n-1)}$, and $g \mapsto V_g^{(n)}$ is a faithful representation of G_{n-1}. Also, the embedding of

$$\mathscr{B}_{n-1} = \mathscr{A}_{n-1} \otimes M_{\eta(1)} \otimes \cdots \otimes M_{\eta(n-2)}$$

into

$$\mathscr{A}_n \otimes M_{\eta(1)} \otimes \cdots \otimes M_{\eta(n-2)}$$

maps each central sum of permutation matrices into a central sum of permutation matrices, and hence $U_g^{(n-1)}$ is mapped into a unitary $W_g^{(n)}$ which is a central sum of permutation matrices. Thus

$$W_g^{(n)} \otimes V_g^{(n)}$$

is a central sum of permutation matrices and hence there exists a unique $h = \varphi_{n,n-1}(g) \in G_n$ such that

$$U_h^{(n)} = W_g^{(n)} \otimes V_g^{(n)} .$$

The injective morphism $\varphi_{n,n-1}\colon G_{n-1} \to G_n$ defines the embedding of G_{n-1} in G_n. We define G as the inductive limit of the G_n, and write simply $G = \bigcup_n G_n$. Since G_n acts on all the algebras $\mathscr{B}_n, \mathscr{B}_{n+1}, \ldots$ in a consistent fashion, we get an action of G_n on $\bigcup_m \mathscr{B}_m$ and since this is an action by isometries it extends by closure to an action of G_n on $\mathscr{B} = \overline{\bigcup_m \mathscr{B}_m}$. This defines an action α of G on $\mathscr{B}$.

We prove Proposition 9.1 via a series of observations.

Observation 1. *The action α is outer, and hence $C^*(\mathscr{B}, \alpha)$ is simple.*

Proof. Assume that $g \in G_n$, $g \neq 1$, and consider the decomposition

$$\begin{aligned}\mathscr{B} &= (\mathscr{A} \otimes M_{\eta(1)} \otimes \cdots \otimes M_{\eta(n)}) \otimes M_{\eta(n+1)} \otimes M_{\eta(n+2)} \otimes \cdots \\ &= \mathscr{C}_n \otimes M_{\eta(n+1)} \otimes M_{\eta(n+2)} \otimes \cdots\end{aligned}$$

of $\mathscr{B}$. It follows from the construction of α that α_g leaves all of the factors in this tensor product decomposition invariant, and on each factor α_g is implemented by a nontrivial permutation matrix. If α_g were inner, it would be approximible in the uniform norm by an inner automorphism implemented by a unitary operator in one of the finite sub-tensor-products in the decomposition of $\mathscr{B}$, and hence $\|(\alpha_g-1)|_{M_{n(n+k)}}\|\to 0$ as $k\to\infty$. As this is not the case, α_g is outer (see [17] for the details of a similar argument).

It follows from [14] that $C^*(\mathscr{B}, \alpha)$ is simple.

Observation 2. *$C^*(\mathscr{B}, \alpha)$ is an AF algebra.*

Proof. Since all the $\mathscr{B}_n$'s are finite-dimensional and all the G_n's are finite, and $\alpha|_{G_n}$ leaves $\mathscr{B}_n$ invariant, the embeddings $\mathscr{B}_1\subseteq\mathscr{B}_2\subseteq\mathscr{B}_3\subseteq\cdots$ and $G_1\subseteq G_2\subseteq G_3\subseteq\cdots$ define embeddings $C^*(\mathscr{B}_1, \alpha|_{G_1})\subseteq C^*(\mathscr{B}_2, \alpha|_{G_2})\subseteq C^*(\mathscr{B}_3, \alpha|_{G_3})\subseteq\cdots$, and it is easily proved via the regular representations of the L^1-algebras that the inductive limit of these algebras is identified with $C^*(\mathscr{B}, \alpha)$. This proves that $C^*(\mathscr{B}, \alpha)$ is an AF algebra.

Observation 3. *A state ω on $\mathscr{B}$ is α-invariant if and only if ω is a trace.*

Proof. If ω is α-invariant, then $\omega|_{\mathscr{B}_n}$ is invariant under the action of G_n. But since G_n acts ergodically on each factor in the central decomposition of $\mathscr{B}_n$, it follows that the restriction of ω to each such factor is a scalar multiple of the trace on the factor. It follows that $\omega|_{\mathscr{B}_n}$ is a trace, and hence ω is a trace.

Conversely, if ω is a trace, then $\omega|_{\mathscr{B}_n}$ is a trace and since G_n acts on $\mathscr{B}_n$ by inner automorphisms, it follows that $\omega|_{\mathscr{B}_n}$ is G_n-invariant. As $\mathscr{B}=\overline{\bigcup_n \mathscr{B}_n}$, $G=\bigcup_n G_n$, it follows that ω is G-invariant.

Observation 4. *The trace state space of $\mathscr{B}$ is affinely isomorphic to the trace state space of $\mathscr{A}$, and hence it is affinely isomorphic to the Poulsen simplex.*

Proof. Since $\mathscr{B}=\mathscr{A}\otimes\mathscr{F}$, and $\mathscr{F}$ has the Dixmier property (i.e., if $A\in\mathscr{F}$, there exists a sequence of convex combinations $\sum_k \lambda_k U_k A U_k^*$ of unitary translates of A converging in norm to $\tau(A)1$, where τ is the unique trace state on $\mathscr{F}$) it follows that any trace state ω on $\mathscr{B}$ has the form

$$\omega=\varphi\otimes\tau$$

where φ is a trace state on $\mathscr{A}$. This establishes a one-one affine correspondence between φ and ω.

Alternatively, one could prove this from the fact that the dimension group of $\mathscr{B}$ is the tensor product of the dimension groups of $\mathscr{A}$ and $\mathscr{F}$,

$$K_0(\mathscr{B})=K_0(\mathscr{A})\otimes K_0(\mathscr{F}),$$

and $K_0(\mathscr{F})$ is a subgroup of $\mathbf{Q}$, hence there is a one-one correspondence between additive positive functionals on $K_0(\mathscr{B})$ and $K_0(\mathscr{A})$.

We conclude from Observations 1 and 2 that $C^*(\mathscr{B}, \alpha)$ is an AF algebra, and Observations 3 and 4 combined with Batty's result mentioned earlier imply that the Poulsen simplex K is affinely isomorphic to a face in the state space of $C^*(\mathscr{B}, \alpha)$. Since any metrizable simplex is affinely isomorphic to a face in K, this ends the proof of Proposition 9.1, with the $\mathscr{A}$ of that Proposition equal to $C^*(\mathscr{B}, \alpha)$.

X Remarks on ground and ceiling states

In Theorem 7.2 the groups $G^{\pm\infty}$ were assumed to be countable, archimedian, totally ordered abelian groups without minimal positive elements. The theorem can be extended to the case where $G^{\pm\infty}$ are just countable, simple (with respect to the order) Riesz groups, i.e. to the case where $\mathscr{A}^{\pm\infty}$ are general simple unital separable AF algebras. In order to make this extension the Riesz group G_0 in Section 3 and the dimension group G in Section 5 have to be defined more carefully. This will be done in detail in a more general setting in a forthcoming paper, but the construction can be outlined as follows: we may assume, by the remarks prior to Definition 3.1, that $K_{\pm\infty}$ consists of at most one point ω_0. Assume that $K_{\pm\infty}\neq\phi$, and $G^{\pm\infty}\neq\{0\}$. Let $g_\pm\in G^{\pm\infty}$ correspond to the unit in $\mathscr{A}^{\pm\infty}$, and let $\omega_\pm$ be an additive positive functional on $G^{\pm\infty}$ such that $\omega_\pm(g_\pm)=1$. We replace G_0 in Definition 3.1 by a certain subgroup G_0' of

$$(\bigoplus_{\mathbf{Z}} G^{-\infty})\oplus G_0\oplus(\bigoplus_{\mathbf{Z}} G^{+\infty})$$

where $\bigoplus$, $\oplus$ denote direct sum ($=$restricted direct product); i.e., G_0' consists of triples

$$((g_n^-)_{n\in\mathbf{Z}}, f, (g_n^+)_{n\in\mathbf{Z}})$$

where $g_n^\pm\in G^{\pm\infty}$, $f\in G_0$, only a finite number of the $g_n^\pm$ are nonzero, the functions $f_n^\pm$ in Definition 3.1 are all independent of ω (as we may assume when $K_{\pm\infty}$ consists of one point), and the compatibility conditions

$$f_n^-=\omega_-(g_n^-),\quad f_n^+=\omega_+(g_n^+)$$

are fulfilled. The order on G_0' is defined by: $((g_k^-)_{k\in\mathbf{Z}}, f, (g_k^+)_{k\in\mathbf{Z}})>0$ if and only if there exist integers n, m such that

i. $f(\beta, \omega)>0$ for $\beta\in\mathbf{R}$ and $(\beta, \omega)\in\tilde{K}$
ii. $g_k^+=0$ for $k\geq n+1$
iii. $g_n^+>0$

iv. $g_k^- = 0$ for $k \leq m-1$

v. $g_m^- > 0$.

One proves as in Lemma 3.2 that G_0' is a Riesz group, and defines an automorphism α of G_0' by

$$\alpha((g_n^-), f, (g_n^+)) = (\alpha(g_n^-), \alpha f, \alpha(g_n^+))$$

where

$$\alpha((g_n^-)_{n\in\mathbf{Z}}) = (g_{n+1}^-)_{n\in\mathbf{Z}}$$
$$\alpha f(\beta, \omega) = e^{-\beta} f(\beta, \omega)$$
$$\alpha((g_n^+)_{n\in\mathbf{Z}}) = (g_{n+1}^+)_{n\in\mathbf{Z}}\,.$$

One then chooses a suitable countable subgroup G' of G_0' as in Section 5, and proceeds to prove the generalized version of Theorem 7.2 with minor modifications in the argument. This justifies the penultimate statement of the abstract of this paper.

Acknowledgements

It is a pleasure for one of us (O. B.) to thank M. Takesaki at University of California, Los Angeles and D. Kastler and M. Mebkhout at Université d'Aix-Marseille II, as well as the Centre de Physique Théorique, Marseille, for their hospitality while part of this work was done.

References

[1] Alfsen, E., *Compact convex sets and boundary integrals,* Springer Verlag, Berlin-Heidelberg-New York, 1971.

[2] Araki, H., *Relative Hamiltonian for faithful normal states of a von Neumann algebra,* Publ. RIMS Kyoto Univ. **9** (1973), 165–209.

[3] Batty, C. J. K., *Simplexes of states of C*-algebras,* J. Operator Theory **4** (1980), 3–23.

[4] Batty, C. J. K., *Abelian faces of state spaces of C*-algebras,* Comm. Math. Phys. **75** (1980), 43–50.

[5] Bratteli, O., G. A. Elliott and R. H. Herman, *On the possible temperatures of a dynamical system,* Comm. Math. Phys. **74** (1980), 281–295.

[6] Bratteli, O. and D. W. Robinson, *Operator algebras and quantum statistical mechanics I,* Springer Verlag, Berlin-Heidelberg-New York, 1979.

[7] Bratteli, O. and D. W. Robinson, *Operator algebras and quantum statistical mechanics II,* Springer Verlag, Berlin-Heidelberg-New York , 1981.

[8] Bratteli, O., A. Kishimoto and D. W. Robinson, *Stability properties and the KMS-condition,* Comm. Math. Phys. **61** (1978), 209–238.

[9] Bratteli, O., *Inductive limits of finite dimensional C*-algebras,* Trans. Amer. Math. Soc. **171** (1972), 195–234.

[10] Effros, E. G., D. E. Handelman and C. L. Shen, *Dimension groups and their affine represen-*

tations, Amer. J. Math. **102** (1980), 385–407.

[11] Effros, E. G. and J. Rosenberg, *C^*-algebras with approximately inner flip*, Pacific J. Math. **77** (1978), 417–443.

[12] Elliott, G. A., *On the classification of inductive limits of sequences of semisimple finite-dimensional algebras*, J. Algebra **38** (1976), 29–44.

[13] Elliott, G. A., *On totally ordered groups, and K_0*, in Ring theory Waterloo 1978, Lecture Notes in Math. 734, Springer Verlag, Berlin-Heidelberg-New York, 1979.

[14] Elliott, G. A., *Some simple C^*-algebras constructed as crossed products with discrete outer automorphism groups*, Publ. RIMS Kyoto. Univ. **16** (1980), 299–311.

[15] Goodearl, K., *Algebraic representations of Choquet simplexes*, J. Pure Appl. Algebra **11** (1977), 111–130.

[16] Lindenstrauss, J., G. Olsen and Y. Sternfeld, *The Poulsen simplex*, Ann. Inst. Fourier **28** (1978), 91–114.

[17] Lance, E. C., *Inner automorphisms of UHF algebras*, J. London Math. Soc. **43** (1968), 681–688.

[18] Rosenberg, J., *Amenability of crossed products of C^*-algebras*, Comm. Math. Phys. **57** (1977), 187–191.

Department of Pure Mathematics
The University of New South Wales
Kensington, N. S. W. 2033
Australia